Math 248B. Covariance of double duality

Let $f : (E, e) \to M$ be an elliptic curve over a complex manifold, so we have a "duality" isomorphism

$$\alpha_{E/M} : E \simeq \widehat{E} := \mathrm{Pic}^0_{E/M,e} = \mathrm{R}^1 f_*(\mathscr{O}_E)/\mathrm{R}^1 f_*(\mathbf{Z}(1))$$

whose formation commutes with base change on M. This is a funny isomorphism, since the two sides have opposite variances as functors in the elliptic curve.

Our aim in this handout is to prove that by iterating the construction twice, covariant functoriality is obtained. That is, we claim that the outside part of the following diagram of M-maps

$$\begin{array}{ccccc} E' & \xrightarrow{\alpha_{E'/M}} & \widehat{E'} & \xrightarrow{\alpha_{\widehat{E'}/M}} & E'^{\wedge\wedge} \\ \downarrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \widehat{\varphi}} & & \downarrow{\scriptstyle \varphi^{\wedge\wedge}} \\ E & \xrightarrow[\alpha_{E/M}]{} & \widehat{E} & \xrightarrow[\alpha_{\widehat{E}/M}]{} & E^{\wedge\wedge} \end{array}$$

commutes. (The smaller squares will generally *not* commute, but these failures will "cancel out", as we will see below.) It suffices to check commutativity on fibers over M, so since all constructions involved do naturally commute with base change we are reduced to the classical case (i.e., M a point). But note that this is not the kind of question which is ever really addressed directly in the classical case, so there is some work to do.

If $\varphi = 0$ then everything is clear, so we may and do assume that φ is an isogeny, say with degree $n > 0$. Then the left square commutes up to a factor of n in the sense that

$$\widehat{\varphi} \circ \alpha_{E/M} \circ \varphi = n\alpha_{E'/M}.$$

Indeed, this expresses the fact in the classical theory usually stated as "$\widehat{\varphi} \circ \varphi = [\deg(\varphi)]_{E'/M}$". Applying this fact to $\widehat{\varphi}$ in place of φ, we also have

$$\varphi^{\wedge\wedge} \circ \alpha_{\widehat{E'}/M} \circ \widehat{\varphi} = n\alpha_{\widehat{E}/M}.$$

Our goal is to prove

$$\varphi^{\wedge\wedge} \circ \alpha_{\widehat{E'}/M} \circ \alpha_{E'/M} \stackrel{?}{=} \alpha_{\widehat{E}/M} \circ \alpha_{E/M} \circ \varphi,$$

and it is harmless to multiply both sides by n (why?). Hence, it is the same to prove that

$$\varphi^{\wedge\wedge} \circ \alpha_{\widehat{E'}/M} \circ (n\alpha_{E'/M}) \stackrel{?}{=} (n\alpha_{\widehat{E}/M}) \circ \alpha_{E/M} \circ \varphi.$$

Upon plugging in our two initial identities, this becomes

$$\varphi^{\wedge\wedge} \circ \alpha_{\widehat{E'}/M} \circ (\widehat{\varphi} \circ \alpha_{E/M} \circ \varphi) \stackrel{?}{=} (\varphi^{\wedge\wedge} \circ \alpha_{\widehat{E'}/M} \circ \widehat{\varphi}) \circ \alpha_{E/M} \circ \varphi.$$

This equality of maps is clear, so we are done.