Math 248B. $\mathrm{GL}_2(\mathbf{Z})$-action and modular forms

1. Motivation

In class we saw that the Weierstrass elliptic curve $\mathscr{E} \to \mathbf{C}-\mathbf{R}$ *equipped with* its associated H_1-trivialization $\Psi : \mathbf{Z}^2 \times (\mathbf{C}-\mathbf{R}) \simeq \underline{\mathrm{H}}_1(\mathscr{E}/(\mathbf{C}-\mathbf{R}))$ is universal for pairs (E, ψ) consisting of an elliptic curve $f : (E, e) \to M$ over a (varying) complex manifold M and an H_1-trivilaization $\psi : \mathbf{Z}^2 \times M \simeq \underline{\mathrm{H}}_1(E/M) := \mathrm{R}^1 f_*(\mathbf{Z})^\vee$.

To make this universality more precise, we defined the contravariant functor F from the category of complex manifolds to the category of sets by assigning to each complex manifold M the set $F(M)$ of isomorphism classes of pairs (E, ψ) over M, with the functoriality $F(h) : F(M) \to F(M')$ for $h : M' \to M$ defined via base change along h. We proved in class that $\mathbf{C}-\mathbf{R}$ represents the functor F via the object $(\mathscr{E}, \Psi) \in F(\mathbf{C}-\mathbf{R})$. That is, the natural transformation $\mathrm{Hom}(\cdot, \mathbf{C}-\mathbf{R}) \to F$ defined by the object $(\mathscr{E}, \Psi) \in F(\mathbf{C}-\mathbf{R})$ is an isomorphism of functors. In more concrete terms, given any M and any (E, ψ) over M, there is a *unique* cartesian diagram of elliptic curves

$$\begin{array}{ccc} E & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow \\ M & \xrightarrow[h]{} & \mathbf{C}-\mathbf{R} \end{array}$$

such that the top horizontal map carries ψ over to Ψ. (We also saw in class that, due to the rigidity of H_1-trivializations of elliptic curves over complex manifolds, in this cartesian diagram the top horizontal arrow is uniquely determined by the bottom arrow h and the compatibility requirement with ψ and Ψ. In this sense, we can often focus attention primarily on h, but it is nonetheless important not to forget about the top horizontal map.)

In this handout, we use F to interpret certain classical constructions involving the representing object $(\mathscr{E}, \Psi)$ for F. In particular, the classical action of $\mathrm{GL}_2(\mathbf{Z})$ on $\mathbf{C}-\mathbf{R}$ will be related to an action of this group on the functor F, and upon working out this effect on the universal elliptic curve $\mathscr{E}$ (not just on the base space $\mathbf{C}-\mathbf{R}$!) we will see how the automorphy factor arising for weight-k modular forms is encoded in terms of our interpretation of the Weierstrass family (equipped with Ψ!) as a representing object for F. In the final section we explain Deligne's generalization to the case of $\mathrm{GL}_2(\mathbf{R})$ acting on $\mathbf{C}-\mathbf{R}$.

The importance of interpreting classical explicit formulas and group actions more conceptually in terms of the functor F and variants on it is that this will pave the way for proofs of consistency between the analytic and algebraic approaches to the theory of modular forms.

2. The $\mathrm{GL}_2(\mathbf{Z})$-action

Define an action of $\Gamma := \mathrm{GL}_2(\mathbf{Z})$ on F as follows. For any complex manifold M, the action of Γ on $F(M)$ is defined to be $\gamma.(E, \psi) = (E, \psi \circ \gamma^t)$, where $(E, e) \to M$ is an elliptic curve over M and $\psi : \mathbf{Z}^2 \times M \simeq \underline{\mathrm{H}}_1(E/M)$ is an isomorphism of M-groups. The intervention of matrix-transpose in this definition makes it a left action of Γ on $F(M)$, and it is easy to check that for any $h : M' \to M$ the base change map $F(h) : F(M) \to F(M')$ is Γ-equivariant. Thus, we have defined a left action of Γ on the functor F.

Using the isomorphism $\mathrm{Hom}(\cdot, \mathbf{C}-\mathbf{R}) \simeq F$ defined by the object $(\mathscr{E}, \Psi)$ over $\mathbf{C}-\mathbf{R}$, Yoneda's Lemma yields a left action of Γ on $\mathbf{C}-\mathbf{R}$ as well as a lift of this to a left action on $\mathscr{E}$ over the action on $\mathbf{C}-\mathbf{R}$. To be precise, for any $\gamma \in \Gamma$ the pair $(\mathscr{E}, \Psi \circ \gamma^t)$ over $\mathbf{C}-\mathbf{R}$ fits into a *unique* cartesian diagram of elliptic curves

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{[\gamma]_{\mathscr{E}}} & \mathscr{E} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{C}-\mathbf{R} & \xrightarrow[{[\gamma]}]{} & \mathbf{C}-\mathbf{R} \end{array}$$

in the sense that the map $[\gamma]_{\mathscr{E}}$ over $[\gamma]$ carries $\Psi \circ \gamma^t$ on the source over to Ψ on the target. For $\gamma = 1$ we must have the identity on the top and bottom by uniqueness, and in general the action property on F implies that both $[\gamma]$ and $[\gamma]_{\mathscr{E}}$ are multiplicative in γ. Hence, both are isomorphisms (with $[\gamma^{-1}]$ inverse to $[\gamma]$, and similarly on $\mathscr{E}$).

To make this explicit, $[\gamma]$ is a holomorphic automorphism of $\mathbf{C} - \mathbf{R}$ such that for all $\tau \in \mathbf{C} - \mathbf{R}$ the fibral map

$$[\gamma]_{\mathscr{E},\tau} : \mathbf{C}/\Lambda_\tau = \mathscr{E}_\tau \to \mathscr{E}_{[\gamma](\tau)} = \mathbf{C}/\Lambda_{[\gamma](\tau)}$$

induced by $[\gamma]_{\mathscr{E}}$ carries the ordered basis of Λ_τ corresponding to $\Psi_\tau \circ \gamma^t$ over to the ordered basis $([\gamma](\tau), 1)$ of $\Lambda_{[\gamma](\tau)}$ corresponding to $\Psi_{[\gamma](\tau)}$. We compute $[\gamma]$ and $[\gamma]_{\mathscr{E}}$ from these properties as follows. Writing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the first member of the ordered basis of $\Lambda_\tau \subset \mathbf{C}$ corresponding to $\Psi_\tau \circ \gamma^t$ is the complex number

$$(\Psi_\tau \circ \gamma^t)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \Psi_\tau\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = a\Psi_\tau\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + b\Psi_\tau\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = a\tau + b,$$

and similarly the second member of the ordered basis is $c\tau + d$. In other words, the point $[\gamma](\tau) \in \mathbf{C} - \mathbf{R}$ has the property that there is an isomorphism $[\gamma]_{\mathscr{E},\tau} : \mathbf{C}/\Lambda_\tau \simeq \mathbf{C}/\Lambda_{[\gamma](\tau)}$ whose effect on the homology lattices carries $a\tau + b$ to $[\gamma](\tau)$ and carries $c\tau + d$ to 1.

An isomorphism of elliptic curves $\mathbf{C}/L \simeq \mathbf{C}/L'$ is necessarily induced by multiplication by some $\lambda \in \mathbf{C}^\times$ such that $\lambda(L) = L'$. Thus, $[\gamma]_{\mathscr{E},\tau}$ is induced by multiplication by a complex number λ_τ such that

$$\lambda_\tau \cdot (a\tau + b) = [\gamma](\tau), \quad \lambda_\tau \cdot (c\tau + d) = 1.$$

This says that $\lambda_\tau = (c\tau + d)^{-1}$ and $[\gamma](\tau) = (a\tau + b)/(c\tau + d)$. Summarizing these calculations, we have proved:

Proposition 2.1. *Choose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. The automorphism $[\gamma]$ of $\mathbf{C} - \mathbf{R}$ is $\tau \mapsto (a\tau + b)/(c\tau + d)$, and the automorphism $[\gamma]_{\mathscr{E}}$ of $\mathscr{E} = (\mathbf{C} \times (\mathbf{C} - \mathbf{R}))/\Lambda$ over $[\gamma]$ is induced by the line bundle automorphism of $\mathbf{C} \times (\mathbf{C} - \mathbf{R})$ over $[\gamma]$ defined by $(z, \tau) \mapsto (z/(c\tau + d), (a\tau + b)/(c\tau + d))$.*

Example 2.2. Consider the element $\gamma = -1 \in \Gamma$. The automorphism $[\gamma]$ of $\mathbf{C} - \mathbf{R}$ is the identity. This can be seen from the explicit formula for $[\gamma]$, but it also follows conceptually from the fact that for any (E, ψ) over any M, we have $\psi \circ \gamma^t = -\psi$ and the pair $(E, \psi \circ \gamma^t) = (E, -\psi)$ is isomorphic to (E, ψ) via negation on E (so the effect on the functor F of isomorphism classes is trivial, and hence likewise on the moduli space $\mathbf{C} - \mathbf{R}$). But observe that the automorphism $[\gamma]_{\mathscr{E}}$ of $\mathscr{E}$ (over the identity on $\mathbf{C} - \mathbf{R}$) is *not* the identity map: it is negation. Indeed, this can be seen from the explicit formula, and we also just saw it via considerations with the functor F.

The upshot of this example is that although the Γ-action "upstairs" at the level of H_1-trivialized elliptic curves is determined by its effect "downstairs" on the base of the family of such elliptic curves, triviality of an action on the base does not imply triviality upstairs when the desired effect on ψ is *nontrivial.* This reminds us not to forget that a moduli space is always equipped with a universal structure over it. Any explicit calculation with a moduli space that involve its universal property must use the chosen universal structure that defines the "moduli space" property.

The reader will observe that the explicit formula for $[\gamma]$ recovers the classical action of $\mathrm{GL}_2(\mathbf{Z})$ on $\mathbf{C} - \mathbf{R}$ (inducing the classical action on $\mathrm{SL}_2(\mathbf{Z})$ on each connected component $\mathfrak{h}_{\pm i}$ of $\mathbf{C} - \mathbf{R}$), and the explicit formula for $[\gamma]_{\mathscr{E}}$ over $[\gamma]$ on universal elliptic curve encodes scaling by $1/(c\tau + d)$. The latter is reminiscent of the automorphy factor in the definition of a modular form. In the next section we will use relative 1-forms to make the link with automorphy factors more substantial.

3. Relative 1-forms and automorphy factors

Consider an elliptic curve $f : (E, e) \to M$ over a complex manifold M. (We will ultimately be interested in the universal family $\mathscr{E}$ over $\mathbf{C} - \mathbf{R}$, but at the outset it is clearer not to focus on this special case.) In HW3, Exercise 3, it is shown that the $\mathscr{O}_M$-module $\omega_{E/M} := f_*(\Omega^1_{E/M})$ is an invertible sheaf whose formation naturally commutes with any base change. In particular, if we work locally on M to arrange that this invertible sheaf admits a trivializing global section $\omega \in \Gamma(M, \omega_{E/M}) = \Gamma(E, \Omega^1_{E/M})$ (as we saw always occurs when there is a global Weierstrass model), then the specialization $\omega_m \in \Gamma(E_m, \Omega^1_{E_m})$ on each fibral elliptic curve is a nonzero 1-form. Loosely speaking, such an ω amounts to a holomorphically varying choice of nonzero holomorphic 1-form on the fibers E_m (at least locally on M).

As a special case, for the Weierstrass family $\mathscr{E} \to \mathbf{C} - \mathbf{R}$, in Exercise 3 of HW3 we exhibited an explicit such global section of $\Omega^1_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$ (inducing a nonzero holomorphic 1-form on every fiber): the descent of the global relative 1-form $\mathrm{d}z$ on the covering space $\mathbf{C} \times (\mathbf{C} - \mathbf{R})$ over $\mathscr{E}$. Adopting a standard abuse of notation, we will denote this descent in $\Gamma(\mathscr{E}, \Omega^1_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}) = \Gamma(\mathbf{C} - \mathbf{R}, \omega_{\mathscr{E}/(\mathbf{C}-\mathbf{R})})$ as $\mathrm{d}z$ also when the context makes the intended meaning clear. Viewing $(\mathrm{d}z)^{\otimes k}$ as a trivializing section of $(\Omega^1_{\mathscr{E}/(\mathbf{C}-\mathbf{R})})^{\otimes k}$, we have $(\Omega^1_{\mathscr{E}/(\mathbf{C}-\mathbf{R})})^{\otimes k} = \mathscr{O}_{\mathscr{E}}(\mathrm{d}z)^{\otimes k}$ and so

$$f_*((\Omega^1_{\mathscr{E}/(\mathbf{C}-\mathbf{R})})^{\otimes k}) = f_*(\mathscr{O}_{\mathscr{E}})(\mathrm{d}z)^{\otimes k} = \mathscr{O}_M(\mathrm{d}z)^{\otimes k}.$$

This tensor-power notation is consistent with viewing $(\mathrm{d}z)^{\otimes k}$ as a trivializing section of $\omega^{\otimes k}_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$, via the following result.

Lemma 3.1. *For any $k \geq 1$ and elliptic curve $E \to M$, the natural map of line bundles $\omega^{\otimes k}_{E/M} \to f_*((\Omega^1_{E/M})^{\otimes k})$ is an isomorphism. In particular, there is an open covering $\{U_i\}$ of M such that for $E_i = E|_{U_i}$ the line bundle $(\Omega^1_{E_i/U_i})^{\otimes k}$ has a trivialization section of the form $\omega^{\otimes k}$ where $\omega \in \Gamma(U_i, \omega_{E/M}) = \Gamma(E_i, \Omega^1_{E/M})$.*

The natural map in this lemma is a higher version of the natural map $f_*(\mathscr{F}) \otimes_{\mathscr{O}_M} f_*(\mathscr{G}) \to f_*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})$ for any morphism of ringed spaces $f : X \to M$ and any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$. (This natural map relating tensor products and pushforwards is usually not an isomorphism; the elliptic curve case with Ω^1's is quite special in this respect.)

Proof. The problem is intrinsic to the abstract $\mathscr{O}_E$-module $\Omega^1_{E/M}$, and by working locally on M (to trivialize $\omega_{E/M}$) we can reduce to the case when $\Omega^1_{E/M}$ is globally free on E. That is, $\Omega^1_{E/M} \simeq \mathscr{O}_E$. We may then express our problem in terms of $\mathscr{O}_E$: we seek to prove that the natural map $(f_*\mathscr{O}_E)^{\otimes k} \to f_*(\mathscr{O}_E^{\otimes k})$ is an isomorphism. But the natural map $\mathscr{O}_M \to f_*(\mathscr{O}_E)$ is an isomorphism (since f is a proper submersion with connected fibers), and $\mathscr{O}_E^{\otimes k} \simeq \mathscr{O}_E$ via multiplication, so the problem becomes that of showing the multiplication map $\mathscr{O}_M^{\otimes k} \to f_*(\mathscr{O}_E)$ is an isomorphism. This is easy to verify, due to the natural isomorphism $\mathscr{O}_M \simeq f_*(\mathscr{O}_E)$. ∎

Now consider the action of $\Gamma = \mathrm{GL}_2(\mathbf{Z})$ on the universal elliptic curve $\mathscr{E} \to \mathbf{C} - \mathbf{R}$. That is, for each $\gamma \in \Gamma$ we have an automorphism $[\gamma]$ of $\mathbf{C} - \mathbf{R}$ over which we have an automorphism $[\gamma]_{\mathscr{E}}$ of $\mathscr{E}$, with both actions defined in terms of the functor F which is represented by the object $(\mathscr{E}, \Psi) \in F(\mathbf{C} - \mathbf{R})$. This was all made explicit in Proposition 2.1. Pullback of relative 1-forms along the action $[\gamma]_{\mathscr{E}}$ over $[\gamma]$ defines an isomorphism $\Omega^1_{\mathscr{E}/(\mathbf{C}-\mathbf{R})} \to [\gamma]_*(\Omega^1_{\mathscr{E}/(\mathbf{C}-\mathbf{R})})$, so by passing to kth tensor powers (for $k \geq 1$) and pushing forward to $\mathbf{C} - \mathbf{R}$ we may apply Lemma 3.1 to get an isomorphism

$$\omega^{\otimes k}_{\mathscr{E}/(\mathbf{C}-\mathbf{R})} \simeq [\gamma]_*(\omega^{\otimes k}_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}). \tag{3.1}$$

But $\omega^{\otimes k}_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$ is trivialized by $(\mathrm{d}z)^{\otimes k}$, so it makes sense to ask for an explicit description of the line bundle isomorphism (3.1) over the moduli space $\mathbf{C} - \mathbf{R}$ in terms of holomorphic coefficient functions relative to the global frame $(\mathrm{d}z)^{\otimes k}$. This yields something familiar:

Proposition 3.2. *For any open set $U \subset \mathbf{C} - \mathbf{R}$, the isomorphism $\omega^{\otimes k}_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}(U) \simeq \omega^{\otimes k}_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}([\gamma]^{-1}(U))$ induced by (3.1) on U-sections is*

$$f \cdot (\mathrm{d}z)^{\otimes k} \mapsto (f|_k\gamma) \cdot (\mathrm{d}z)^{\otimes k}$$

where $(f|_k\gamma) : \tau \mapsto (c\tau + d)^{-k} f((a\tau + b)/(c\tau + d)) = j(\gamma, \tau)^{-k} f([\gamma](\tau))$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma$ and $j(\gamma, \tau) := c\tau + d$.

In particular, when the line bundle $\omega^{\otimes k}_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$ on $\mathbf{C} - \mathbf{R}$ is globally trivialized by $(\mathrm{d}z)^{\otimes k}$, the effect of $[\gamma]_{\mathscr{E}}$ on global sections is the classical operator $f \mapsto f|_k\gamma$ on global holomorphic functions on $\mathbf{C} - \mathbf{R}$.

Note that since this proposition concerns the pullback action on (tensor powers of) relative 1-forms in a family of elliptic curves (not 1-forms on the base of the family!), and such pullback is contravariant with respect to composition, it follows by pure thought (along with the fact that $[\gamma]_{\mathscr{E}}$ defines a left action of Γ on $\mathscr{E}$) that the induced operations in this proposition must be *right* actions. Of course, the right-action property of $f \mapsto f|_k\gamma$ can easily be verified by direct computation with the explicit formula, as is done classically, but it is satisfying that we can also see this property of the action by thinking without explicitly computing.

This proposition is an essential ingredient in the "geometrization" of the classical theory of modular forms, as we will see later. After reading the proof below, the interested reader may wish to reflect on how the result is related to "explanations" (in the books by Koblitz, Lang, Silverman, et al.) for the source of the $(\cdot)|_k\gamma$ operation in terms of "moduli" of elliptic curves.

Proof. We simply compute on fibers. For $u \in U$, the u-fiber of $f \cdot (\mathrm{d}z)^{\otimes k} \in \Gamma(U, \omega_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}^{\otimes k})$ corresponds to $f(u)(\mathrm{d}z)^{\otimes k} \in \Gamma(\mathscr{E}_u, (\Omega^1_{\mathscr{E}_u})^{\otimes k})$ via the u-fiber of the isomorphism in Lemma 3.1 applied to $\mathscr{E} \to \mathbf{C} - \mathbf{R}$. For any $\tau \in \mathbf{C} - \mathbf{R}$ (such as $\tau \in [\gamma]^{-1}(U)$), the fibral map

$$[\gamma]_{\mathscr{E},\tau} : \mathbf{C}/\Lambda_\tau \to \mathbf{C}/\Lambda_{[\gamma](\tau)}$$

is induced by multiplication by $(c\tau + d)^{-1}$ on $\mathbf{C}$, so the pullback on global 1-forms on these fibral elliptic curves is determined by carrying (the descent of) $\mathrm{d}z$ back to (the descent of) $(c\tau + d)^{-1}\mathrm{d}z$. Hence, on global sections of the tensor power $(\Omega^1)^{\otimes k}$ on these fibral elliptic curves, $(\mathrm{d}z)^{\otimes k}$ is pulled back to $(c\tau + d)^{-k}(\mathrm{d}z)^{\otimes k}$.

Choose $\tau_0 \in [\gamma]^{-1}(U)$. Since $[\gamma]_{\mathscr{E}}$ sits over $[\gamma]$ in a cartesian square, the $[\gamma]_{\mathscr{E}}$-pullback of $f \cdot (\mathrm{d}z)^{\otimes k}$ has τ_0-fiber equal to the pullback under $[\gamma]_{\mathscr{E},\tau_0}$ of the $[\gamma](\tau_0)$-fiber

$$f([\gamma](\tau_0))(\mathrm{d}z)^{\otimes k} \in \Gamma(\mathscr{E}_{[\gamma](\tau_0)}, (\Omega^1)^{\otimes k})$$

of $f \cdot (\mathrm{d}z)^{\otimes k}$. Taking $\tau = \tau_0$ above, this pullback is $(c\tau_0 + d)^{-k} f([\gamma](\tau_0))(\mathrm{d}z)^{\otimes k}$. ∎

4. The $\mathrm{GL}_2(\mathbf{R})$-action via variations of complex structure

Classically, the holomorphic left action of $\mathrm{GL}_2(\mathbf{Z})$ on $\mathbf{C} - \mathbf{R}$ is induced by a real-analytic left action of $\mathrm{GL}_2(\mathbf{R})$. In this final section, we give a moduli-theoretic interpretation for this $\mathrm{GL}_2(\mathbf{R})$-action on $\mathbf{C} - \mathbf{R}$ as well as for the "weight-k" operator $(\cdot)|_k g$. (Restricting to $\mathrm{SL}_2(\mathbf{R})$ will give similar results over each connected component $\mathfrak{h}_{\pm i}$ of $\mathbf{C} - \mathbf{R}$.)

The viewpoint we will take, due to Deligne, is to think of the injections $\Lambda_\tau \hookrightarrow \mathbf{C}$ not as varying lattices in a fixed complex line but rather as varying complex structures on a fixed $\mathbf{R}$-vector space $\mathbf{R}^2 = \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{Z}^2$. The viewpoint of variations of complex structure arises naturally in many higher-dimensional problems.

Definition 4.1. Let M be a complex manifold, and W_0 a finite-dimensional $\mathbf{R}$-vector space of even dimension $2d > 0$. A (holomorphic) *variation of complex structure on* W_0 *parameterized by* M is a pair (V, φ) consisting of a rank-d holomorphic vector bundle V and an isomorphism $\varphi : W_0 \times M \simeq V$ of real-analytic vector bundles carrying the set W_0 of constant sections into the set $V(M)$ of holomorphic sections.

An *isomorphism* between such variations (V, φ) and (V', φ') is a holomorphic vector bundle isomorphism $V \simeq V'$ that carries φ to φ' (i.e., $V(M) \simeq V'(M)$ restricts to the identity map on W_0).

As in the case of $\underline{\mathrm{H}}_1$-trivialized elliptic curves, it is easy to check that if two pairs (V, φ) and (V', φ') are isomorphic then the isomorphism between them is uniquely determined.

The reason for the name is that for each $m \in M$ the m-fiber W_0 thereby acquires a specified $\mathbf{C}$-vector space structure that generally varies as we change m. This $\mathbf{C}$-linear structure on the fibers makes the real-analytic $\mathbf{R}$-vector bundle $W_0 \times M \to M$ into a holomorphic vector bundle over the complex manifold M, and if two variations of complex structure (V, φ) and (V', φ') on W_0 are isomorphic then the resulting $\mathbf{C}$-vector space structures on the m-fiber W_0 coincide for each $m \in M$. The converse is false:

Example 4.2. Choose $i \in \sqrt{-1}$ and consider a real-analytic map $f : M \to \mathbf{C} - \mathbf{R}$. Define $\varphi_f : \mathbf{R}^2 \times M \to V := \mathbf{C} \times M$ via $(a, b, m) \mapsto (f(m)(a + bi), m)$. This is a variation of complex structure on $\mathbf{R}^2$ parameterized by M such that each fiber $\mathbf{R}^2$ acquires the $\mathbf{C}$-linear structure via the basis $\{1, i\}$ of $\mathbf{C}$ (which has nothing to do with f). If (V, φ_f) is isomorphic to (V, φ_1) then the resulting holomorphic automorphism of V carrying

φ_1 to φ_f must be multiplication by $f(m)$ on the m-fibers and hence f must be holomorphic. Thus, by choose non-holomorphic f we get the desired counterexample.

Here are some basic examples of interest.

Example 4.3. A variation of complex structure on $\mathbf{R}^{2d}$ is a pair $(V,(s_1,\dots,s_{2d}))$ where $V \to M$ is a rank-d holomorphic vector bundles and $s_1,\dots,s_{2d} \in V(M)$ are holomorphic sections such that $\{s_j(m)\}$ is an $\mathbf{R}$-basis of V_m for all $m \in M$.

Example 4.4. Let $j : \mathbf{Z}^{2d} \times M \hookrightarrow V$ be a relative lattice in a rank-d holomorphic vector bundle V over M. Then $\mathbf{R}^{2d} \times M \simeq V$ as real-analytic vector bundles of rank $2d$, so this defines a variation of complex structure on $\mathbf{R}^{2d}$ parameterized by M. In particular, any $\underline{\mathrm{H}}_1$-trivialized elliptic curve $E \to M$ defines a variation of complex structure on $\mathbf{R}^2$ parameterized by M. (Explicitly, $\mathrm{Tan}_0(E_m)$ is a $\mathbf{C}$-line structure on $\mathrm{H}_1(E_m,\mathbf{R}) = \mathbf{R}^2$.) The Weierstrass construction over $M = \mathbf{C} - \mathbf{R}$ is an example of this.

It is easy to check that isomorphism pairs (E,ψ) and (E',ψ') over M define isomorphic variations of complex structure on $\mathbf{R}^2$ parameterized by M.

Example 4.5. An isomorphism class of variations of complex structure on $\mathbf{R}^2$ parameterized by M is an $\mathscr{O}^\times(M)$-homothety class of holomorphic line bundle structures on the rank-2 real-analytic vector bundle $\mathbf{R}^2 \times M$. The resulting fibral complex structures define a collection of $\mathbf{R}$-algebra maps $\mathbf{C} \to \mathrm{End}_{\mathbf{R}}(\mathbf{R}^2) = \mathrm{Mat}_2(\mathbf{R})$ parameterized by the points $m \in M$.

In the special case of the Weierstrass construction over $M = \mathbf{C} - \mathbf{R}$, for $\tau \in \mathbf{C} - \mathbf{R}$ the resulting complex structure on $\mathbf{R}^2$ as the τ-fiber corresponds to the unique complex structure for which $\tau(0,1) = (1,0)$. Explicitly, if $X^2 + \alpha X + \beta \in \mathbf{R}[X]$ is the minimal polynomial of τ over $\mathbf{R}$ then

$$\tau(1,0) = \tau^2(0,1) = (-\alpha\tau - \beta)(0,1) = (-\alpha,-\beta),$$

so the $\mathbf{R}$-algebra map $\mathbf{R}[\tau] = \mathbf{C} \to \mathrm{End}_{\mathbf{R}}(\mathbf{R}^2) = \mathrm{Mat}_2(\mathbf{R})$ is characterized by $\tau \mapsto \left(\begin{smallmatrix} -\alpha & 1 \\ -\beta & 0 \end{smallmatrix}\right)$. Hence, writing $\tau = a + bi$ (so $\alpha = -2a$ and $\beta = a^2 + b^2$), the complex structure associated to $\tau = a + bi$ is characterized by

$$i = (1/b)(\tau - a) \mapsto \begin{pmatrix} a/b & 1/b \\ -b(1+(a/b)^2) & -a/b \end{pmatrix}.$$

In particular,

$$(0,1) \wedge i(0,1) = (0,1) \wedge (1/b, -a/b) = -(1/b)\cdot(1,0)\wedge(0,1),$$

so i-orientation of $\mathbf{R}^2$ relative to the complex structure associated to $\tau \in \mathbf{C} - \mathbf{R}$ is the standard orientation if and only if the imaginary part b of τ relative to i (i.e., $b := (\tau - \overline{\tau})/2i$) is *negative*. (More generally, for any nonzero $v = (x,y) \in \mathbf{R}^2$, $v \wedge iv = -(1/b)((bx)^2 + (y+ax)^2)(1,0)\wedge(0,1)$ with $(bx)^2 + (y+ax)^2 > 0$ since $b \neq 0$ and $v \neq 0$, confirming explicitly that the orientation class of $v \wedge iv$ is independent of v, as we know it must be.)

Definition 4.6. For any complex manifold M, let $F'(M)$ denote the set of isomorphism classes of complex structures on $\mathbf{R}^2$ parameterized by M. This is a contravariant functor on M via pullback of holomorphic line bundles along holomorphic maps $M' \to M$.

We have explained above that there is a natural transformation $F \to F'$ assigning to any (E,ψ) over M the resulting variation of complex structure on $\mathbf{R}^2 = \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{Z}^2$ parameterized by M.

Proposition 4.7. *The natural transformation $F \to F'$ is an isomorphism, so the variation of complex structure on $\mathbf{R}^2$ parameterized by $\mathbf{C} - \mathbf{R}$ arising from the Weierstrass construction $(\mathscr{E},\Psi)$ over $\mathbf{C} - \mathbf{R}$ is the universal variation.*

Proof. Since a lattice inclusion of $\mathbf{Z}^2$ into a $\mathbf{C}$-line L amounts to an $\mathbf{R}$-linear isomorphism $\mathbf{R}^2 \simeq L$, a variation of complex structure on $\mathbf{R}^2$ parameterized by M is the same as an isomorphism class of pairs consisting of a holomorphic line bundle $V \to M$ equipped with an M-lattice $\mathbf{Z}^2 \times M \to V$. But this latter data in turn is the same as the resulting isomorphism class of the $\underline{\mathrm{H}}_1$-trivialized pair $(V/(\mathbf{Z}^2 \times M),\psi)$. Every $\underline{\mathrm{H}}_1$-trivialized pair (E,ψ) over M admits such a form due to the relative uniformization of elliptic curves over complex manifolds (which is covariantly functorial in E and compatible with base change on M). ■

The bijection $\mathrm{Hom}(M, \mathbf{C} - \mathbf{R}) = F(M) \simeq F'(M)$ in the case of a one-point M identifies the set $\mathbf{C} - \mathbf{R}$ with the set of $\mathbf{C}$-vector space structures on $\mathbf{R}^2$ taken up to $\mathbf{C}$-linear isomorphism. Here is a more illuminating explanation of this fact. Observe that a complex structure on $\mathbf{R}^{2d}$ is an element of the set $\mathrm{Isom}_{\mathbf{R}}(\mathbf{R}^{2d}, \mathbf{C}^d)/\mathrm{GL}_d(\mathbf{C})$. Setting $d = 1$, if we use $(0, 1)$ as a $\mathbf{C}$-basis for the $\mathbf{C}$-line structure on $\mathbf{R}^2$ then forming the ratio $(1, 0)/(0, 1)$ relative this $\mathbf{C}$-linear structure identifies $\mathrm{Isom}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{C})/\mathbf{C}^\times$ with $\mathbf{C} - \mathbf{R}$. This recovers the above identification of the underlying set of $\mathbf{C} - \mathbf{R}$ with the set of variations of complex structure on $\mathbf{R}^2$ parameterized by a one-point M (i.e., the set of complex structures on the $\mathbf{R}$-vector space $\mathbf{R}^2$, taken up to $\mathbf{C}$-linear isomorphism).

Remark 4.8. In terms of the identification of $\mathbf{C} - \mathbf{R}$ with the moduli space $\mathfrak{M}$ for complex structures on $\mathbf{R}^2$, such a complex structure corresponds to a point in the connected component of $\mathbf{C} - \mathbf{R}$ containing the unique $i = \sqrt{-1}$ for which the common orientation class of $v \wedge iv$ for any nonzero $v \in \mathbf{R}^2$ is the *opposite* of the standard orientation of $\mathbf{R}^2$. More concretely, as we saw via the calculations in Example 4.5, a complex structure lies in the connected component of $\mathbf{C} - \mathbf{R}$ containing the unique i such that the matrix in $\mathrm{GL}_2(\mathbf{R})$ for the i-action has *positive* upper-right entry. (This entry is nonzero, since the i-action is a zero of the irreducible $X^2 + 1 \in \mathbf{R}[X]$ and hence cannot have any eigenvector on $\mathbf{R}^2$.)

In more intrinsic terms, two complex structures on $\mathbf{R}^2$ lie in the same connected component of $\mathfrak{M}$ precisely when the common orientation class of $v \wedge iv$ for any nonzero $v \in \mathbf{R}^2$ is the same for both complex structures. (This condition is independent of the choice of i.)

Upon identifying $\mathrm{Isom}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{C})$ with the set $\mathrm{Latt}(\mathbf{Z}^2, \mathbf{C})$ of lattice inclusions of $\mathbf{Z}^2$ into $\mathbf{C}$, the set $\mathrm{Latt}(\mathbf{Z}^2, \mathbf{C})/\mathbf{C}^\times$ of homothety classes of such inclusions is likewise identified with $\mathbf{C} - \mathbf{R}$ by forming the ratio $(1, 0)/(0, 1)$ relative to the lattice inclusion into $\mathbf{C}$. Hence, if we define an action by $g \in G := \mathrm{GL}_2(\mathbf{R})$ on $F'(M)$ by carrying a complex structure on $\mathbf{R}^2 \times M$ back to another one via precomposition with the g^t-action on $\mathbf{R}^2$ then this is a left action of G on the functor F' and and via $F \simeq F'$ it recovers the action of $\Gamma := \mathrm{GL}_2(\mathbf{Z})$ on F defined earlier. This G-action does not have a direct description in terms of elliptic curves, which is why we need to introduce the viewpoint of the functor F' to define it.

In this way we get a left action of G on $\mathbf{C} - \mathbf{R}$ that restricts to the Γ-action as defined earlier; denote this as $[g]$ for $g \in G$. Arguments similar to those used for the Γ-action on $\mathbf{C} - \mathbf{R}$ show that the G-action on $\mathbf{C} - \mathbf{R}$ is the classical one: $[g](\tau) = (a\tau + b)/(c\tau + d)$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. Thus, by inspection the action map $G \times (\mathbf{C} - \mathbf{R}) \to (\mathbf{C} - \mathbf{R})$ is real-analytic. This real-analyticity (and in particular, continuity) of the action can be seen by pure thought as well, but this requires an additional concept, as we now explain.

Definition 4.9. A (real-analytic) *variation of complex structure* on $\mathbf{R}^{2d}$ parameterized by a real-analytic manifold M exactly as in the holomorphic case: it is a pair consisting of a rank-d real-analytic complex vector bundle $V \to M$ and an isomorphism $\varphi : \mathbf{R}^{2d} \times M \simeq V$ of real-analytic vector bundles. The notion of *isomorphism* between such pairs (V, φ) is defined in the evident manner (an isomorphism $V \simeq V'$ that carries φ to φ').

The difficulty as in Example 4.2 doesn't arise in the real-analytic case: two such pairs are isomorphic if and only if for each $m \in M$ the resulting $\mathbf{C}$-linear structures on the m-fiber $\mathbf{R}^{2d}$ coincide. Thus, an isomorphism class of real-analytic variations of complex structure on $\mathbf{R}^{2d}$ parameterized by M is nothing more or less than a specified $\mathbf{C}$-linear structure on the fibers of $\mathbf{R}^{2d} \times M \to M$ making it into a real-analytic complex vector bundle. (More concretely, if $\{v_1, \dots, v_d\}$ in $\mathbf{R}^{2d}$ is a basis relative to the $\mathbf{C}$-linear structure on the fibral $\mathbf{R}^{2d}$ at some point of M then they are also a basis for the $\mathbf{C}$-linear structure on the fibral $\mathbf{R}^{2d}$ at all nearby points in M.) The same goes through if we replace "real-analyic" with C^∞ everywhere.

As in the holomorphic case, we can define "pullback" for such variations over real-analytic manifolds, and so can make sense of the functor $F'_{\mathbf{R}}$ of isomorphism classes of real-analytic variations of complex structure on $\mathbf{R}^{2d}$. The link with the earlier notion of "holomorphic variation of complex structure" is given by the next result.

Proposition 4.10. *Consider the universal holomorphic variation of complex structure* $(\mathscr{L} \to \mathfrak{M}, \varphi)$ *on* $\mathbf{R}^2$. *The underlying real-analytic variation of complex structure on the underlying real-analytic objects is universal in the real-analytic category. The same holds in the* C^∞ *sense.*

Proof. The construction of the universal object over $\mathbf{C}-\mathbf{R}$ in the holomorphic case works verbatim in the real-analytic case, upon replacing "holomorphic" with "real-analytic" (or C^∞) everywhere. ∎

Returning to the issue of verifying the real-analyticity of the action of $G = \mathrm{GL}_2(\mathbf{R})$ on $\mathfrak{M}$ by pure thought, the preceding proposition shows that this action can be described entirely within the real-analytic category (which is where G lives!) by the same functorial recipe $\varphi \mapsto \varphi \circ g^t$ as in the holomorphic case. But the real-analytic Lie group G represents the group-valued functor $\mathrm{Aut}_{\mathbf{R}^2}$ assigning to any real-analytic manifold M the group of automorphisms of the trivial rank-2 vector bundle $\mathbf{R}^2 \times M$, so the action map $G \times \mathfrak{M} \to \mathfrak{M}$ represents the natural transformation $\mathrm{Aut}_{\mathbf{R}^2} \times F'_{\mathbf{R}} \to F'_{\mathbf{R}}$ and so it is real-analytic.

Example 4.11. Here is an interpretation of the weight-k operation $f \mapsto f|_k g$ for $g \in G$ (and holomorphic functions f on $\mathbf{C}-\mathbf{R}$) in terms of moduli of variations of complex structure; here, by definition, $f|_k g : \tau \mapsto (c\tau + d)^{-k} f([g](\tau))$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Our argument will be an adaptation of what we have already seen for Γ using moduli of elliptic curves. Consider the universal variation over $\mathbf{C}-\mathbf{R}$: a pair $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is a holomorphic line bundle over $\mathbf{C}-\mathbf{R}$ and $\varphi : \mathbf{R}^2 \times (\mathbf{C}-\mathbf{R}) \simeq \mathscr{L}$ is a real-analytic isomorphism of real vector bundles such that $\varphi(1,0), \varphi(0,1) \in \mathscr{L}(\mathbf{C}-\mathbf{R})$. For any $g \in G$, $(\mathscr{L}, \varphi \circ g^t)$ is another variation of complex structures on $\mathbf{R}^2$ parameterized by $\mathbf{C}-\mathbf{R}$, so by universality there is a unique cartesian square of *holomorphic* maps

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{[g]_{\mathscr{L}}} & \mathscr{L} \\ \downarrow & & \downarrow \\ \mathbf{C}-\mathbf{R} & \xrightarrow[{[g]}]{} & \mathbf{C}-\mathbf{R} \end{array}$$

in which the top side is compatible with $\varphi \circ g^t$ on the source and φ on the target. This expresses the G-action on $(\mathscr{L}, \varphi)$ obtained via its universality for the functor F'.

We now have holomorphic line bundle isomorphisms $[g]_{\mathscr{L}} : \mathscr{L} \simeq [g]^*(\mathscr{L})$ over $\mathbf{C}-\mathbf{R}$ carrying $\varphi \circ g^t$ to $[g]^*(\varphi)$. Dualizing yields isomorphisms $[g]^*(\mathscr{L}^\vee) \simeq \mathscr{L}^\vee$, or equivalently $\mathscr{L}^\vee \simeq [g]_*(\mathscr{L}^\vee)$. Relative to the holomorphic trivialization of $\mathscr{L}^\vee$ by the dual basis to $\varphi(0,1)$, this becomes an isomorphism $\mathscr{O} \simeq [g]_*(\mathscr{O})$. Arguments as in the proof of Proposition 3.2 show this to be $f \mapsto f|_1 g$. Hence, passing to kth tensor powers for $k \geq 1$ and using the dual to $\varphi(0,1)^{\otimes k}$ as the trivializing frame for $(\mathscr{L}^\vee)^{\otimes k}$ yields the action $f \mapsto f|_k g$.

The preceding example is consistent with the use of the Weierstrass family to represent F' and our earlier calculations with $\omega_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}^{\otimes k}$ in the sense that there is a similarity in the explicit formulas, now using G instead of the discrete group Γ. This consistency can be explained without direct inspection of formulas, by identifying $\omega_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$ with $\mathscr{L}^\vee$ via the following result applied to $\mathscr{E} \to \mathbf{C}-\mathbf{R}$:

Proposition 4.12. *For uniformized elliptic curves $f : E = V/\Lambda \to M$, the dual $V^\vee$ of the line bundle V is naturally identified with $\omega_{E/M}$ in a manner compatible with base change on M. More specifically, there is a unique way to define such an isomorphism so that in the classical case it is the isomorphism $\mathrm{Cot}_0(E) \simeq \Gamma(E, \Omega^1_E)$ defined by translation-invariance.*

Proof. The requirements on the isomorphism specify it on fibers, so the uniqueness is immediate (given existence) and our task is to prove that the classical isomorphism on fibers induces an isomorphism at the level of holomorphic line bundles in the relative setting (i.e., the classical isomorphism "varies holomorphically" in analytic families of elliptic curves). The key point is that the natural map of line bundles

$$\phi : \omega_{E/M} = f_* \Omega^1_{E/M} \to e^* \Omega^1_{E/M}$$

defined over an open set $U \subset M$ by carrying $\omega \in \Omega^1_{E/M}(f^{-1}(U))$ to $e^*(\omega) \in (e^*\Omega^1_{E/M})(U)$ is an isomorphism. To verify this isomorphism result, first note that the formation of ϕ naturally commutes with base change on M (upon reviewing the sense in which the formation of $\omega_{E/M}$ commutes with such base change). In the classical case ϕ is the map $\Gamma(E, \Omega^1_E) \simeq \mathrm{Cot}_0(E)$ defined by $\omega \mapsto \omega(0)$, which is an isomorphism due to translation-invariance (and is inverse to the classical isomorphism in the statement of the proposition). Thus, in the relative case ϕ is an isomorphism on fibers over M and so it is an isomorphism.

In view of the description of ϕ on fibers, it suffices to prove that V is naturally identified with $e^*((\Omega^1_{E/M})^\vee)$ in a manner that recovers the classical isomorphism $V_m \simeq \mathrm{Tan}_0(V_m/\Lambda_m)$ on fibers (and then we dualize to get an isomorphism $V^\vee \simeq e^*(\Omega^1_{E/M}) \simeq \omega_{E/M}$ which satisfies the properties we seek). Note that the classical isomorphism $V_m \simeq \mathrm{Tan}_0(V_m/\Lambda_m)$ is built in two steps: we first identify the vector space V_m with its own tangent space $\mathrm{Tan}_0(V_m)$ at the origin, and then we use that the analytic covering map $V_m \to V_m/\Lambda_m$ respects the origins and hence induces an isomorphism between the tangent spaces at the respective origins. Adapting this to the relative setting, since the analytic covering map $\pi : V \to E$ is an analytic isomorphism between open neighborhoods of the zero section 0 of V and the identity section of E (this applies equally well with suitable open neighborhoods around any holomorphic sections $s \in V(M)$ and $\pi \circ s \in E(M)$), via π we get an isomorphism between $0^*(\Omega^1_{V/M})$ and $e^*(\Omega^1_{V/M})$.

Now we can now ignore E and focus entirely on V: we seek to show that for any vector bundle $V \to M$ (such as a line bundle above), there is an isomorphism $V^\vee \simeq 0^*(\Omega^1_{V/M})$ as vector bundles over M such that on fibers its dual recovers the classical isomorphism $V_m \simeq \mathrm{Tan}_0(V_m)$ (defined via directional derivative operators); such an isomorphism is certainly unique if it exists since we are specifying it on fibers, and it is compatible with base change on M and functorial in V over M due to the fibral specification (and the functoriality of the classical isomorphism on fibers). The advantage of this formulation of the problem is that the intervention of Λ has been removed, so we can make use of the linear structure on the fibers. In view of the uniqueness (granting existence), we may work locally on M for existence and so may assume that M is open in some $\mathbf{C}^n$ and V admits a trivialization.

Now our setup is a restriction of the special case $M = \mathbf{C}^n$ and $V = \mathbf{C}^r \times \mathbf{C}^n = \mathbf{C}^r \times M$ (as a rank-r vector bundle $p_2 : V \to M$), so we may focus on this special case and use global coordinates. Letting $z_1, \dots, z_{r+n}$ denote the standard coordinates on V viewed as a complex manifold, $\Omega^1_{V/M} := \Omega^1_V / p_2^*(\Omega^1_M)$ admits the global frame given by $\mathrm{d}z_1, \dots, \mathrm{d}z_r$ and V admits the global frame $e_1, \dots, e_r$ corresponding to $e_j(m) = (0, \dots, 1, \dots, 0, m)$ (with a 1 in the jth slot and 0's elsewhere among the first r components). Using the dual frame $\{e_j^\vee\}$ to trivialize $V^\vee$, the map $V^\vee \simeq 0^*(\Omega^1_{V/M})$ defined by $e_j^\vee \mapsto 0^*(\mathrm{d}z_j)$ is a vector bundle isomorphism whose effect on fibers as readily checked to be as desired (dual to the classical description of the tangent space at any point of a vector space). ■

We conclude our discussion of the moduli-theoretic interpretation of the $\mathrm{GL}_2(\mathbf{R})$-action by using it to give a conceptual proof of the transitivity of its action on the moduli space $\mathfrak{M}$ of complex structures on $\mathbf{R}^2$ without reference to the identification of $\mathfrak{M}$ with $\mathbf{C} - \mathbf{R}$ or the resulting explicit linear fractional transformation formulas for the $\mathrm{GL}_2(\mathbf{R})$-action on $\mathbf{C} - \mathbf{R}$. In particular, we deduce by pure thought (without linear fractional transformation calculations) that $\mathfrak{M}$ has exactly two connected components and recover the classical description of these components as coset spaces for $\mathrm{SL}_2(\mathbf{R})$ modulo a maximal compact subgroup.

The effect of $g \in \mathrm{GL}_2(\mathbf{R})$ on a complex structure on $\mathbf{R}^2$ is precomposition of the $\mathbf{C}$-action with the g^t-action on $\mathbf{R}^2$. These complex structures correspond to $\mathbf{R}$-algebra embeddings $\mathbf{C} \hookrightarrow \mathrm{Mat}_2(\mathbf{R})$, in terms of which the g-action becomes composition with the conjugation $x \mapsto (g^t)^{-1} x g^t$ on $\mathrm{Mat}_2(\mathbf{R})$. By the Skolem-Noether theorem, or direct arguments in our case (e.g., using rational canonical form), any two embeddings $\mathbf{C} \hookrightarrow \mathrm{Mat}_2(\mathbf{R})$ are related through conjugation by some $g \in \mathrm{GL}_2(\mathbf{R})$. This proves conceptually that the $\mathrm{GL}_2(\mathbf{R})$-action on $\mathfrak{M}$ is *transitive*. Also, under this action the subgroup $\mathbf{R}^\times$ acts trivially since it is central in $\mathrm{GL}_2(\mathbf{R})$.

The $\mathrm{GL}_2(\mathbf{R})$-stabilizer of a point in $\mathfrak{M}$ is the centralizer of $\mathbf{C}^\times$ in $\mathrm{GL}_2(\mathbf{R})$ under the embedding. The centralizer is the embedded $\mathbf{C}^\times = \mathbf{C} - \{0\}$ since a maximal commutative subfield of a central simple algebra is its own centralizer. The determinant on $\mathrm{Mat}_2(\mathbf{R})$ goes over to the norm $\mathrm{N}_{\mathbf{C}/\mathbf{R}}$ on $\mathbf{C}$ (by the definition of the field-theoretic norm), so the embedded $\mathbf{C}^\times$ is $\mathbf{R}^\times \times K$ where the embedded circle group $K \subset \mathrm{SL}_2(\mathbf{R})$ is a maximal compact subgroup. (To see this maximality, use considerations with inner products to first prove the more classical fact that $\mathrm{O}_n(\mathbf{R})$ is a maximal compact subgroup of $\mathrm{GL}_n(\mathbf{R})$ with every compact subgroup contained in one of its conjugates, and then set $n = 2$.) We now recall the following general fact (see §1.6–§1.7 in Chapter III of Bourbaki's "Lie groups and Lie algebras" for a proof, or Theorem 1.2.1 in Miyake's "Modular Forms" for an analogous result in the topological case):

Proposition 4.13. *Let X be a (paracompact) Hausdorff real-analytic manifold equipped with an analytic action $H \times X \to X$ by an analytic real Lie group H. For any $x \in X$, let H_x denote the stabilizer at x. This is a real-analytic closed Lie subgroup of H and the natural map $H/H_x \to X$ defined by $h \mapsto h.x$ is a real-analytic isomorphism. The same holds in the complex-analytic and C^∞ categories.*

Proof. See §1.6–§1.7 in Chapter III of Bourbaki's "Lie groups and Lie algebras" for the proof in the real-analytic and complex-analytic cases. The same argument works in the C^∞ case. (See Theorem 1.2.1 in Miyake's "Modular Forms" for a proof of an analogous result with locally compact separable Hausdorff spaces.) ■

By choosing a base point in $\mathfrak{M}$, we get a real-analytic isomorphism

$$\mathfrak{M} \simeq \mathrm{GL}_2(\mathbf{R})/(\mathbf{R}^\times K) = (\{\langle \pm 1 \rangle\} \ltimes \mathrm{SL}_2(\mathbf{R}))/K$$

where $\langle a \rangle = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since $\mathrm{SL}_2(\mathbf{R})$ is connected (why?) and its maximal compact subgroup K is also connected, we conclude from this description of $\mathfrak{M}$ as a Lie group coset space that $\mathfrak{M}$ has exactly two connected components (related to the fact that $\mathrm{GL}_2(\mathbf{R})$ has two connected components) and that $\mathrm{SL}_2(\mathbf{R})$ acts transitively on each connected component with stabilizer at any point equal to a maximal compact subgroup. (This is seen very explicitly in the classical theory in terms of the action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathfrak{h}_i$ via linear fractional transformations. The conceptual method just used has the virtue of adapting to higher-dimensional settings where explicit formulas can be unwieldy.)

We recall that a left action of a group Γ on a Hausdorff topological space X is called *discontinuous* if each $x \in X$ has finite stabilizer Γ_x and admits an open neighborhood U such that $\gamma(U) \cap U \neq \emptyset$ if and only if $\gamma \in \Gamma_x$. If the action is moreover free (i.e., $\Gamma_x = 1$ for all $x \in X$) then it is called *properly discontinuous.* Note that in such cases the topological quotient map $X \to \Gamma\backslash X$ (using the quotient topology) is a covering space. Moreover:

Example 4.14. Suppose X is a complex manifold and Γ acts properly discontinuously on X through holomorphic automorphisms. Then we claim that there is a unique complex manifold structure on $\Gamma\backslash X$ making $X \to \Gamma\backslash X$ a local analytic isomorphism, and this is initial among Γ-invariant holomorphic maps from X to complex manifolds. (The same holds in the real-analytic and C^∞ categories, by similar arguments.) In view of the uniqueness assertion, the problem is local on Γ-stable opens in X. By proper discontinuity we thereby reduce to the case when X has a disjoint covering by connected open sets $\{U_\gamma\}$ transitively permuted by Γ via holomorphic isomorphisms. In other words, by selecting one connected component we get a *holomorphic* Γ-equivariant identification $X = \Gamma \times U$, and then the topological quotient is identified with the projection $p_2 : X \to U$. The desired results are then clear.

In an important special case, discontinuity of the action is automatic:

Proposition 4.15. *Let G be a locally compact and separable Hausdorff topological group, Γ a discrete subgroup, and K a compact subgroup. The left action by Γ on G/K is discontinuous, and it is properly discontinuous if and only if Γ meets every conjugate of K trivially. In case G is a Lie group with finite component group and K is a maximal compact subgroup, this latter property occurs if and only if Γ is torsion-free.*

In view of the above $\mathrm{SL}_2(\mathbf{R})$-equivariant description of the connected components of $\mathfrak{M}$, this proposition recovers the classical fact that a discrete subgroup Γ in $\mathrm{SL}_2(\mathbf{R})$ acts discontinuously on $\mathfrak{M}$ and properly discontinuously if and only if Γ is torsion-free. We also remark that (by some real work) when G is a Lie group, it is separable if and only if it has countable component group, and it has finite component group whenever it is the group of $\mathbf{R}$-points of any linear algebraic $\mathbf{R}$-group (as can be proved "by hand" in special cases). The reader is referred to §3 in Chapter XV of Hochschild's book "Structure of Lie Groups" (see Theorems 3.1 and 3.7) for a proof that in any Lie group with finite component, the following hold: (i) all compact subgroups lie in a maximal one, (ii) all maximal compact subgroups are conjugate, (iii) all maximal component subgroups meet every connected component.

Proof. The discontinuity of the action is Theorem 1.5.2 in Miyake's book "Modular Forms". The freeness amounts to the requirement that if $\gamma \in \Gamma$ and $g \in G$ satisfy $\gamma g K = gK$ then $\gamma = 1$. The equality of cosets says exactly that $\gamma \in gKg^{-1}$, so freeness is equivalent to Γ having trivial intersection with every conjugate of K, as desired. Finally, in case G is a Lie group with finite component group and K is a maximal compact subgroup, by the general results from Hochschild's book mentioned above we see that Γ meets all conjugates of K trivially if and only if Γ meets every compact subgroup of G trivially. But the intersection of the discrete Γ with a compact subgroup of G is necessarily finite, so it is equivalent to say that Γ contains no nontrivial finite subgroups, which is to say that it is torsion-free. ■