Math 248B. Relative isogenies

Let $E \to M$ be an elliptic curve over a complex manifold M, and consider its canonically associated (in the covariant sense) relative uniformization V/L. Recall from class that if $C \subset E$ is a closed M-subgroup and $L' \subset V$ is the corresponding M-lattice containing L, then we considered the M-homomorphism

$$\phi : E = V/L \to V/L' =: E'$$

where the map in the middle is the natural one arising from the functoriality of the "V/L" construction (inducing the natural quotient map $V_m/L'_m \to V_m/L_m$ on fibers over each $m \in M$). We proved that ϕ is a finite analytic covering with functorial kernel represented by C. To justify considering E' (equipped with ϕ) as a "quotient" E/C over M, we require:

Proposition 0.1. *For any complex manifold X over M and any M-map $f : E \to X$ that is invariant under the action map $C \times_M E \to E$ (in the sense that $E(M') \to X(M')$ is $C(M')$-invariant for any $M' \to M$), the map f uniquely factors through $\phi : E \to E'$ over M.*

Proof. By the universal property of V/L' over M (as in the "V/Λ" handout), an M-morphism $E' \to X$ is "the same" as a holomorphic M-morphism $V \to X$ invariant by the L'-action on V over M. But L'-invariance implies L-invariance, so any such map from V uniquely factors through an M-map $V/L \to X$, and by Exercise 1 in HW4 the L'-invariance of $V \to X$ is equivalent to the L'/L-invariance of $V/L \to X$ (where the latter action is defined via the identification of L'/L as an M-subgroup of V/L; see Exercise 1 in HW4). But this is precisely a C-invariant map $E \to X$ over M. ■

In view of this proposition, it is legitimate to rename $\phi : E \to E'$ as $E \to E/C$. This notion of quotient has the usual functorial properties inherited from its universal property (e.g., good behavior with respect to inclusions $C \subset C'$ within E), and it also commutes with base change on M. That is, for any $M' \to M$, the natural map $E_{M'} \to (E/C)_{M'}$ induced by base change is not only $C_{M'}$-invariant, but we have:

Proposition 0.2. *The induced map $E_{M'}/C_{M'} \to (E/C)_{M'}$ is an isomorphism.*

Proof. We may work locally over M', hence locally over M, so we can assume that $C \to M$ has all fibers of a common size $d \geq 1$. The two natural maps $E_{M'} \to E_{M'}/C_{M'}$ and $E_{M'} \to (E/C)_{M'}$ are both degree-d analytic coverings, and the diagram of M'-maps

$$\begin{array}{ccc} E_{M'} & & \\ \downarrow & \searrow & \\ E_{M'}/C_{M'} & \longrightarrow & (E/C)_{M'} \end{array}$$

commutes, so the bottom map is a finite analytic covering (why?) with degree necessarily equal to 1. This is an M'-morphism which induces a degree-1 map between elliptic curves on fibers, so it is an isomorphism on fibers and hence (by the fibral isomorphism criterion from the handout on vector bundles and lattices) it is an isomorphism. ■

In particular, using base change to a point, the relative quotient E/C recovers the classical quotients E_m/C_m on fibers. We now run the procedure in reverse by relating general homomorphisms of elliptic curves over a base to those obtained by the preceding E/C construction:

Proposition 0.3. *Let $\phi : E \to E'$ be an M-homomorphism between elliptic curves over a complex manifold M.*

1. *The function $m \mapsto \deg(\phi_m) \geq 0$ is locally constant on M.*
2. *If $\deg(\phi_m) = d \geq 1$ for all $m \in M$ then ϕ is a degree-d analytic covering that is the quotient morphism by the closed M-subgroup $C = \ker \phi = E \times_{\phi, E', e'} M$ (with $C \to M$ also a degree-d analytic covering).*

Proof. To prove (1), consider the fibral homology map $\mathrm{H}_1(\phi_m) : \mathrm{H}_1(E_m, \mathbf{Z}) \to \mathrm{H}_1(E'_m, \mathbf{Z})$. This is naturally identified with the m-stalk of the map of local systems

$$\underline{\mathrm{H}}_1(\phi) : \underline{\mathrm{H}}_1(E/M) \to \underline{\mathrm{H}}_1(E'/M)$$

(why? Be rigorous!) of rank-2 finite free $\mathbf{Z}$-modules, so if it vanishes (resp. is an index-d lattice inclusion) on the stalk at some m then the same holds on all nearby stalks. This purely topological assertion can also be visualized via Ehresmann's fibration theorem. It remains to recall from the classical theory that $\phi_m = 0$ if and only if $\mathrm{H}_1(\phi_m) = 0$ and that ϕ_m is a degree-d isogeny if and only if $\mathrm{H}_1(\phi_m)$ is an index-d lattice inclusion. This completes the proof of (1).

Now assume that $\deg \phi_m = d \geq 1$ for all $m \in M$. Consider the covariantly associated relative uniformizations $\mathscr{V}/\Lambda$ for E and $\mathscr{V}'/\Lambda'$ for E'. Thus, by the very meaning of the functoriality of these uniformizations, ϕ gives rise to a map of line bundles $\widetilde{\phi} : \mathscr{V} \to \mathscr{V}'$ over M carrying Λ into Λ' such that the induced map $\mathscr{V}/\Lambda \to \mathscr{V}'/\Lambda'$ between relative quotients is ϕ. In particular, via the natural (covariantly functorial!) isomorphisms

$$\Lambda \simeq \underline{\mathrm{H}}_1(E/M), \ \ \Lambda' \simeq \underline{\mathrm{H}}_1(E'/M)$$

from Exercise 1 in HW3, the induced map $\Lambda \to \Lambda'$ is a degree-d lattice inclusion on stalks at each $m \in M$. Meanwhile, the map of line bundles $\mathscr{V} \to \mathscr{V}'$ over M is surjective on fibers (as the linear fibral map between $\mathbf{C}$-lines carries a lattice onto another lattice) and hence is an isomorphism. If we write V to denote this common line bundle, then we have identified E with V/L and E' with V/L' for a pair of M-lattices $L \subset L' \subset V$ such that the induced quotient map $V/L \to V/L'$ is exactly the original ϕ! But these were exactly the kinds of maps which were considered in the original "E/C" construction, so the assertions in (2) are now clear. ∎