Math 248B. Modular curves

Contents

1. Introduction

We seek an "arithmetic" construction of the theory of modular curves and modular forms (including Hecke operators, q-expansions, and so forth).

Example 1.1. For $k \geq 1$ and $N \geq 1$, we seek an integer $n(k,N)$ such that each modular form $f = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_1(N))$ with $a_n \in K$ for $1 \leq n \leq n(k,N)$ and some number field K has the following properties:

- the coefficients a_n are all in K, and
- the modular form f has bounded denominators: there exists $D \geq 1$ such that $a_n \in \frac{1}{D}\mathscr{O}_K$ for all n, *with control on* D.

We have in mind the following sort of application: sometimes a_0 will be the special value of an L-function (e.g. for f a suitable Eisenstein series), so we can study algebraicity and *integrality* properties of a_0 from those of the a_n's with $n > 0$.

Example 1.2. For $f = \sum a_n q^n$ and $g = \sum b_n q^n$ in $M_k(\Gamma_1(N))$ lying in $\frac{1}{D}\mathscr{O}_K[[q]]$, if $\mathfrak{p}$ is a prime of $\mathscr{O}_K$ away from D then we want $a_n \equiv b_n \pmod{\mathfrak{p}}$ for n up to some bound (depending only on k and N) to imply the same congruence for all n. (This should be proved using Riemann–Roch in characteristic $\mathfrak{p}$!)

There are four stages, and each step from stage to stage is a *huge* leap.

(I) The analytic theory over the upper half-plane $\mathfrak{h}$ with explicit definitions of modular forms, explicit $\mathrm{SL}_2(\mathbf{R})$-actions, etc.
(II) $\mathbf{C}$-analytic geometry: construct moduli spaces for $\mathbf{C}$-analytic "families" of elliptic curves, define modular forms as sections of line bundles built from "universal families", use *universal properties* in lieu of explicit formulas.
(III) The analogue of (II) in algebraic geometry over $\mathbf{C}$, establishing consistency with (II) via GAGA.

(IV) The theory over $\mathbf{Q}$ or $\mathbf{Z}[\frac{1}{N}]$ or $\mathbf{Z}$, recovering the $\mathbf{C}$-algebraic theory via base change.

Especially, don't try jumping from (I) to (III) or (IV) without first doing (II)! The principle here is that cheap definitions tend to be useless or misleading.

Example 1.3. If we try to define a "$\mathbf{C}$-analytic family" of elliptic curves with our bare hands using Weierstrass models, e.g.

$$\begin{array}{c}\{y^2z = x^3 + a(t)xz^2 + b(t)z^3\} \subset \mathbf{CP}^2 \times M \\ \Big\downarrow \\ M\end{array}$$

for a $\mathbf{C}$-analytic manifold M and holomorphic functions $a, b : M \to \mathbf{C}$, then we cannot detect intrinsic features and cannot do interesting constructions such as "quotient by a finte subgroup" — which is needed in families in order to define Hecke operators in stages (II)–(IV).

Remark 1.4. "Universal elliptic curves" over affine modular curves will *not* admit Weierstrass models globally.

Example 1.5. In the book of Diamond–Shurman, they "construct" the connected modular curve $X(N)_{\mathbf{C}}$ by ad hoc definition of its function field: over $\mathbf{C}(j)$, let $\mathbf{E}$ be the curve defined by

$$y^2 = x^3 - \frac{27j}{4(j-1728)}x - \frac{27j}{4(j-1728)}$$

or alternately by

$$y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$$

so that $j(\mathbf{E}) = j$, and take $X(N)$ to be the curve with function field $\mathbf{C}(j)(\mathbf{E}[N])$. This has big problems.

(i) One cannot prove much based on this definition.

(ii) At least as seriously, for $N \geq 3$ it's the wrong field! For the "correct" notion of $\begin{smallmatrix}\mathscr{E}\\ \downarrow \\ Y(N)\end{smallmatrix}$, the generic fiber $\mathscr{E}_\eta$ over $\mathbf{C}(Y(N))$ will *not* descend to the subfield $\mathbf{C}(j(\mathscr{E}_\eta)) \subset \mathbf{C}(Y(N))$.

2. The analytic Weierstrass family

Now we begin preparations for step (II) above. Let's first recall the classical meaning of $\mathfrak{h}$, or more canonically $\mathbf{C} - \mathbf{R} = \mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(\mathbf{R})$.

Proposition 2.1. *We have a bijection*

$$\mathbf{C} - \mathbf{R} \xrightarrow{\sim} \{(E, \varphi : \mathrm{H}_1(E, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 2})\}/\simeq$$

where E is an elliptic curve over $\mathbf{C}$ and the equivalence on the right is in the evident sense that $(E, \varphi) \simeq (E', \varphi')$ if there exists an isomorphism $\alpha : E \to E'$ such that $\varphi = \varphi' \circ \mathrm{H}_1(\alpha)$.

The bijection sends $\tau \in \mathbf{C} - \mathbf{R}$ to the pair consisting of the elliptic curve $E_\tau = \mathbf{C}/(\mathbf{Z}\tau \oplus \mathbf{Z})$, and the map φ that sends the homology class $[t \mapsto t\tau]$ to $(1, 0)$, and the homology class $[t \mapsto t]$ to $(0, 1)$. Its inverse sends (E, φ) to the quantity

$$\left(\int_{\varphi(1,0)} \omega\right) \Big/ \left(\int_{\varphi(0,1)} \omega\right)$$

for any nonzero holomorphic 1-form ω on E.

Proof. The main point is the uniformization of all elliptic curves as $\mathbf{C}/\Lambda$, and then one computes. Conceptually, the inclusion $\Omega^1(E) \hookrightarrow \mathrm{H}^1(E, \mathbf{C}) = \mathrm{H}^1(E, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}$ gives

$$\Omega^1(E) \oplus \overline{\Omega^1(E)} \cong \mathrm{H}^1(E, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C},$$

so that $\Omega^1(E) \cap H^1(E, \mathbf{R}) = 0$ and hence by "pure thought" one has $(\int_\gamma \omega)/(\int_{\gamma'} \omega) \notin \mathbf{R}$ for any $\omega \neq 0$ and any basis $\{\gamma, \gamma'\}$ of $H_1(E, \mathbf{R})$. ∎

Notation 2.2. If $\mathscr{F} \to X$ is a map and $x \in X$ then we will write $\mathscr{F}_x$ for the fiber over x.

We want to upgrade this by making a family of elliptic curves $\begin{smallmatrix}\mathscr{E}\\ \downarrow\\ \mathbf{C}-\mathbf{R}\end{smallmatrix}$ equipped with a continuously varying $\varphi_\tau : \mathrm{H}_1(\mathscr{E}_\tau, \mathbf{Z}) \simeq \mathbf{Z}^{\oplus 2}$ that is universal for such structures over *any* $\mathbf{C}$-manifold.

The fiber over $\tau \in \mathbf{C} - \mathbf{R}$ should recover the classical construction above. Thus we first need to define a good notion of "family of elliptic curves" $\begin{smallmatrix}E\\ \downarrow\\ M\end{smallmatrix}$ over a $\mathbf{C}$-manifold. Note that for a one-dimensional M we should have $\dim E = 2$, so that higher-dimensional $\mathbf{C}$-manifolds are unavoidable.

To prepare for this we need a brief digression on holomorphic maps in higher dimension.

Proposition 2.3. *Let $U \subset \mathbf{C}^n$ and $V \subset \mathbf{C}^m$ be open sets, and $f = (f_1, \ldots, f_m) : U \to V$ a C^∞-map. The following are equivalent.*

1. *Each $f_i(z_1, \ldots, z_n)$ is holomorphic in z_j, holding the z_i for $i \neq j$ fixed.*
2. *The differential $df(u) : \mathbf{C}^n \to \mathbf{C}^m$ is $\mathbf{C}$-linear for all $u \in U$.*
3. *Each f_i satisfies the Cauchy–Riemann equations with respect to each z_j.*
4. *Each f_i is locally an absolutely convergent power series in $z_1, \ldots, z_n$.*

Based on these, one gets a good notion of holomorphicity for f, as well as inverse function theorems, submersion/immersion theorems, and so a notion of $\mathbf{C}$-manifolds.

Let's give two examples of what *should* be families of elliptic curves.

Example 2.4. For $a, b : M \to \mathbf{C}$ holomorphic such that $4a^3 - 27b^2$ is non-vanishing, the subset

$$E = \{y^2 z = 4x^3 - axz^2 - bz^3\} \subset \mathbf{CP}^2 \times M$$

is a submanifold. The projection map $\begin{smallmatrix}E\\ \downarrow\\ M\end{smallmatrix}$ is equipped with a section $m \mapsto ([0, 1, 0], m)$.

Example 2.5. Consider the diagram

$$\begin{array}{ccc} \Lambda = \mathbf{Z}^2 \times (\mathbf{C} - \mathbf{R}) & \longrightarrow & \mathbf{C} \times (\mathbf{C} - \mathbf{R}) = V \\ & \searrow & \downarrow \\ & & \mathbf{C} - \mathbf{R} = M \end{array}$$

where the horizontal map takes $((m, n), \tau) \mapsto (m\tau + n, \tau)$. We want to form the "quotient over M":

$$\begin{array}{c} E = V/\Lambda\ . \\ \downarrow \\ M \end{array}$$

Rigorously, write $\Lambda_\tau = \mathbf{Z}\tau \oplus \mathbf{Z} \subset V_\tau = \mathbf{C}$ for the fibers of Λ and V above $\tau \in M$. Classically, the compact space $\mathbf{C}/\Lambda_\tau$ has a $\mathbf{C}$-structure making the covering map $\mathbf{C} \to \mathbf{C}/\Lambda_\tau$ a locally analytic isomorphism. We seek something similar in a *family* over $\mathbf{C} - \mathbf{R}$. Define $(z, \tau) \sim (z, \tau')$ when $\tau = \tau'$ and $z' - z \in \Lambda_\tau$ (i.e., "fiberwise lattice congruence"), and set $E = V/\Lambda := V/\sim$ with the quotient topology.

Proposition 2.6. *In Example 2.5, the map $\pi : V \to E$ is a covering map and E admits a unique $\mathbf{C}$-analytic structure making π a local analytic isomorphism. Moreover $E \to M$ is a proper submersion.*

Reading 2.7. In the handout "Vector bundles modulo relative lattices" we explain a vast generalization of this.

Proof. First we'll do the easier C^∞-analogue, and then improve the result to handle $\mathbf{C}$-analyticity. We remark that the improvement from C^∞ to $\mathbf{C}$-analytic is far from automatic. For instance, there is a 2-to-1 C^∞ covering map $\mathbf{CP}^1 \cong S^2 \to \mathbf{RP}^2$, but $\mathbf{RP}^2$ is non-orientable and therefore has no $\mathbf{C}$-structure. Another example: choose a genus 3 Riemann surface M with trivial automorphism group. Then there is a 2-to-1 cover $M \to S^2$ with no compatible $\mathbf{C}$-structure on S^2.

In the C^∞ case we can "untwist" via the following map over $\mathbf{C} - \mathbf{R} = M$:

$$\begin{array}{ccc} \mathbf{C} \times (\mathbf{C} - \mathbf{R}) & \xrightarrow[(x+iy,\tau)\mapsto(x+y\tau,\tau)]{\sim} & \mathbf{C} \times (\mathbf{C} - \mathbf{R}) = V \; . \\ {\scriptstyle (mi+n,\tau)} \nwarrow & & \nearrow {\scriptstyle j} \\ & \Lambda = \mathbf{Z}^2 \times (\mathbf{C} - \mathbf{R}) & \end{array}$$

(Exercise: write down an explicit C^∞-inverse to the horizontal map.) This untwisting converts j into the "constant map" $(\mathbf{Z}[i] \hookrightarrow \mathbf{C}) \times (\mathbf{C} - \mathbf{R})$, decoupling τ from the lattices. This makes the C^∞-version of everything easy to see.

For the $\mathbf{C}$-analytic case, the key is to make a $\mathbf{C}$-analytic structure on E making $\pi : V \to E$ holomorphic. Then this is necessarily a local analytic isomorphism because by the inverse function theorem one can check this on tangent spaces, and this relates things to the underlying C^∞ tangent maps. Likewise $E \to M$ is holomorphic because composition with π is, and is a submersion because the same is true in the C^∞-sense.

To make the construction, consider some connected open $U \subset E$ such over which the covering map $\pi : V \to E$ is trivial, i.e. $\pi^{-1}(U) = \coprod_{\alpha \in \mathbf{Z}^2} U_\alpha$ with the U_α's the connected components of $\pi^{-1}(U)$ and with π inducing C^∞-isomorphisms $U_\alpha \simeq U$. We want all U_α's to induce the *same* $\mathbf{C}$-structure on U, i.e. we want the composite $t : U_\alpha \xrightarrow{\pi} U \xleftarrow{\pi} U_{\alpha'}$ to be a holomorphic isomorphism, and then we can glue to globalize over E.

Choose $\xi \in U$, and take $v \in U_\alpha$, $v' \in U_{\alpha'}$ in the fibre of $\pi : V \to E$ over ξ. Say ξ lies over $\tau_0 \in \mathbf{C} - \mathbf{R} = M$, so that $v' - v = m\tau_0 + n$ for some $(m, n) \in \mathbf{Z}^2$. Now consider the $\mathbf{C}$-analytic automorphism $\mathbf{C} \times (\mathbf{C} - \mathbf{R}) \simeq \mathbf{C} \times (\mathbf{C} - \mathbf{R})$ sending $(z, \tau) \mapsto (z + m\tau + n, \tau)$. This leaves the map π invariant (as a C^∞-map), so induces a $\mathbf{C}$-analytic automorphism of $\pi^{-1}(U) = \coprod U_\alpha$ over U. This must shuffle the connected components, and takes v to v' by construction, so gives $U_\alpha \simeq U_{\alpha'}$. This isomorphism is rigged to be exactly the composite t above, completing the proof. ∎

Remark 2.8. We also expect $\pi : V \to E$ to be a true analytic quotient with respect to the equivalence relation $\sim$ in the sense that for any holomorphic map $V \to X$ constant on $\sim$-classes should yield a diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle \pi} \searrow & & \nearrow {\scriptstyle \exists!\, \mathbf{C}\text{-an}} \\ & E & \end{array} \; .$$

There is certainly a unique continuous such map and then $\mathbf{C}$-analyticity follows because π is a local analytic isomorphism.

We can describe our construction $\begin{smallmatrix} E \\ \downarrow \\ \mathbf{C}-\mathbf{R} \end{smallmatrix}$ in another way.

Proposition 2.9. *The map* $\mathbf{C} \times (\mathbf{C} - \mathbf{R}) \to \mathbf{C}$ *given by* $(w, \tau) \mapsto \wp_{\Lambda_\tau}(w)$ *is holomorphic, and the map*

$$E = V/\Lambda \to \{y^2 z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3\} \subset \mathbf{CP}^2 \times (\mathbf{C} - \mathbf{R})$$

sending $[(w, \tau)] \mapsto ([\wp_{\Lambda_\tau}(w) : \wp'_{\Lambda_\tau}(w) : 1], \tau)$ *(and* $[(0, \tau)] \mapsto ([0 : 1 : 0], \tau)$*) is an analytic isomorphism.*

Proof. On fibers, this recovers the classical theory. For the proof, see Homework 1. ∎

3. M-curves and M-groups

To discuss elliptic curves in a good way, we need to make sense of "holomorphically varying group stuctures" in fibers (a la group schemes), so we need (analytic) fiber products, at least of a special sort (cf. transversal intersection of manifolds).

Lemma 3.1. *Given a diagram of holomorphic maps of* $\mathbf{C}$*-manifolds*

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

with f *a submersion (so that locally on* X *the map* f *looks like* (*base*) $\times$ (*ball*) $\to$ (*base*)*), the topological fiber product* $X \times_S T \xrightarrow{f_T} T$ *("the base change with respect to* g*") is also a holomorphic submersion (which we will denote* X_T *when we want to emphasize the* T*-structure via* f_T*).*

Proof. Just as for schemes, we work locally on X, S, T. Then by the submersion theorem without loss of generality we may suppose that $X = S \times \mathbf{B}$ with $\mathbf{B} \subset \mathbf{C}^n$ an open ball/blob, and f the projection on the first factor. In that case we check that

$$\begin{array}{ccc} T \times \mathbf{B} & \xrightarrow{g \times \mathrm{id}} & S \times \mathbf{B} \\ \mathrm{pr}_1 \downarrow & & \downarrow \mathrm{pr}_1 \\ T & \xrightarrow{g} & S \end{array}$$

■

works. Note that f_T *is* a submersion by construction. (Use that manifold products are categorical.)

Example 3.2. If $T = \{s\} \hookrightarrow S$ is the inclusion of a single point, we get "analytic fibers" X_s over $s \in S$.

Example 3.3. If $X = \{\sum_I a_I(s)z^I = 0\} \subset \mathbf{C}^n \times S$ then $X_T = \{\sum_I (a_I \circ g)(t)z^I = 0\} \subset \mathbf{C}^n \times T$.

Remark 3.4. It's crucial in the above that we were able to work *locally on* X to get the product structure. In the $\mathbf{C}$-analytic case we cannot do better, but in the C^∞-case we can.

Theorem 3.5 (Ehresmann's fibration theorem)**.** *Any* C^∞ *proper submersion* $\pi : X \to M$ *between* C^∞*-manifolds is a* C^∞*-fiber bundle, i.e.,* M *is covered by open* $\{U_\alpha\}$ *such that*

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\sim} & U_\alpha \times (\mathrm{blob})_\alpha \\ & \searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

We saw this explicitly for our $V/\Lambda \to \mathbf{C} - \mathbf{R}$, even globally, with (blob) $= \mathbf{C}/\mathbf{Z}[i]$. But the corresponding statement in the $\mathbf{C}$-analytic case is *false*: consider the j-invariant on $\mathbf{C} - \mathbf{R}$, for instance!

Definition 3.6. An M*-group* is a holomorphic submersion $\begin{smallmatrix} G \\ \downarrow \\ M \end{smallmatrix}$ equipped with a identity section $e : M \hookrightarrow G$, a multiplication $G \times_M G \to G$ over M, and so forth.

Definition 3.7. An M*-curve* is a proper holomorphic submersion $\begin{smallmatrix} X \\ \downarrow \\ M \end{smallmatrix}$ such that the fibres X_m are connected of dimension 1.

Note that by Ehresmann's theorem the genus of X_m is a locally constant function of m, so "genus g" M-curve is a reasonable concept.

Definition 3.8. A *elliptic curve* over M is a genus 1 M-curve $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ equipped with section $e \in E(M) = \mathrm{Hom}_M(M, E)$.

Example 3.9. We've seen two nontrivial examples over $\mathbf{C} - \mathbf{R}$ that in fact are isomorphic (see HW 1).

Now, for an elliptic curve E over M, we want the following.

(1) Locally over M, the elliptic curve should have a Weierstrass model in $\mathbf{CP}^2 \times (\text{base})$. (We got this *globally* for example over $\mathbf{C} - \mathbf{R}$.)
(2) There should exist a unique M-group structure on (E, e) with identity e. Moreover, it should be communtative, functorial in the pair (E, e), and should commute with base change on M.
(3) There should exist a global canonical $E \simeq V_E/\Lambda_E$ with V_E a line bundle over M and Λ_E a relative lattice over M.

We'll prove (2), then (3), and finally (1) (and get a universal property for our family over $\mathbf{C} - \mathbf{R}$).

4. The M-elliptic curve group law

The classical approach to the elliptic curve group law via Riemann–Roch is to construct the isomorphism

$$E \simeq \operatorname{Pic}^0(E)$$

sending

$$x \mapsto \mathscr{I}_x^{-1} \otimes \mathscr{I}_e = \mathscr{O}(x - e) \tag{4.1}$$

with the right-hand side a group under $\otimes$ (and duals as inverses). Here $\mathscr{I}_x = \mathscr{O}(-x) \subset \mathscr{O}_E$ is the ideal sheaf of x (so that our $\mathscr{O}(D)$ is what is denoted $\mathscr{L}(D)$ in Hartshorne, i.e., $\mathscr{O}(D)$ has global sections $L(D)$).

In our setting there is a new wrinkle: $\operatorname{Pic}(M)$ may not be trivial. So a relative version of (4.1) will need a new idea (once we sort out a replacement for $\mathscr{I}_x$).

We want to make an M-group law on $(E, e) \to M$, via line bundles.

Remark 4.1. For higher-dimensional versions, such as complex tori, one needs a completely different approach with a "relative exponential map" via ODEs with parameters. In the algebraic theory we'll also use line bundles, and again this won't work for higher relative dimension — one needs totally different methods. See Chapter 6, §3 of [GIT].)

Our first task is to relativize the construction of the invertible ideal sheaf $\mathscr{I}_x \subset \mathscr{O}_X$ for a point x on a curve X.

We'll consider the following setup. Let $\pi : X \to M$ be a separated submersion with fibers of pure dimension 1. For instance X might be some suitable $\{\sum_I a_I(m) z^I = 0\} \subset \mathbf{CP}^2 \times M$. Consider a section $x : M \to X$ of π. Just as for schemes, topologically x is a closed embedding. By the immersion theorem, $x : M \to X$ is a closed submanifold, so $\mathscr{O}_X \twoheadrightarrow x_* \mathscr{O}_M$.

Lemma 4.2. *$\mathscr{I}_x := \ker(\mathscr{O}_X \twoheadrightarrow x_*\mathscr{O}_M)$ is invertible, and commutes with base change in the sense that for*

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle x'}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle x} \\ M' & \longrightarrow & M \end{array}$$

the map $\widetilde{f}^*(\mathscr{I}_x) \to \mathscr{O}_{X'}$ induced from $\mathscr{I}_x \to \mathscr{O}_X$ is an isomorphism onto $\mathscr{I}_{x'}$.

Proof. We can work locally on M, and then on X around $x(M)$. Then by the submersion theorem we are reduced to the case of

$$\begin{array}{c} X = M \times \Delta \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle x(m)=(m,0)} \\ M \end{array} .$$

Then $\mathscr{I}_x = z \cdot \mathscr{O}_X$, and everything is clear. ■

Remark 4.3. See HW2, Exercise 1 for generalizations of this lemma.

If $f : (E, e) \to M$ is an elliptic curve, we get a map

$$E(M) \to \mathrm{Pic}^0(E \to M) = \{\mathscr{L} \in \mathrm{Pic}(E) \text{ of degree 0 on fibers}\}$$

sending $x \mapsto \mathscr{I}_x^{-1} \otimes \mathscr{I}_e$. Note that this is injective because we can pass to (analytic) fibers and use the classical case! But is this surjective? No! This is false if $\mathrm{Pic}(M) \neq 1$. Indeed, the map

$$\mathrm{Pic}(M) \xrightarrow{f^*} \mathrm{Pic}^0(E \to M)$$

is injective because $e^* \circ f^* = \mathrm{id}$. For $\mathscr{N} \in \mathrm{Pic}(M)$ the line bundle $f^*\mathscr{N}$ is *trivial* on fibers E_m, so for $\mathscr{N} \not\simeq \mathscr{O}_M$ we see that $f^*\mathscr{N}$ cannot have the form $\mathscr{I}_x^{-1} \otimes \mathscr{I}_e$. In fact the construction "$\mathscr{I}_x^{-1} \otimes \mathscr{I}_e$" is wrong in the relative case when $\mathrm{Pic}(M) \neq 1$. To fix it, we need a new concept, which rests on the following lemma.

Lemma 4.4. *Let $f : X \to M$ be a proper holomorphic submersion with connected and nonempty fibers. Then the maps $\mathscr{O}_M \to f_*\mathscr{O}_X$ and $\mathscr{O}_M^\times \to f_*(\mathscr{O}_X^\times)$ are isomorphisms (i.e. $\mathscr{O}_M(U) \simeq \mathscr{O}_X(f^{-1}U)$ for all open $U \subset M$ and so forth).*

Remark 4.5. When M is a point, this says that the only global sections of $\mathscr{O}_X$ are the constant functions. This is a consequence of the maximum principle (note that the properness of f in this case is equivalent to X being compact).

Remark 4.6. For a line bundle $\mathscr{L}$ on X, it follows that

$$\mathrm{Aut}(\mathscr{L}) \simeq \mathscr{O}(X)^\times \simeq \mathscr{O}(M)^\times,$$

i.e., every $\mathscr{O}_X$-linear automorphism of $\mathscr{L}$ is scaling by a non-vanishing holomorphic function *on M*.

Proof. Since we can replace M with U it suffices to consider global sections. By Remark 4.5 we are done on fibers. In general, some $h \in \mathscr{O}(X)$ is therefore constant on each X_m, so $h = \overline{h} \circ f$ for some function $M \to \mathbf{C}$. We need $\overline{h}$ to be holomorphic. But locally on M by the submersion theorem the map f has a holomorphic section e, and then we can see that $\overline{h}$ is holomorphic by rewriting

$$\overline{h} = \overline{h} \circ \mathrm{id}_M = \overline{h} \circ f \circ e = h \circ e.$$

■

The upshot is that if line bundles $\mathscr{L}, \mathscr{L}'$ on X are isomorphic, a choice of isomorphism $\mathscr{L}' \simeq \mathscr{L}'$ is unique up to $\mathrm{Aut}(\mathscr{L}) = \mathscr{O}(M)^\times$. So, to remove this ambiguity, we'll impose more structure.

Definition 4.7. In the setup of the Lemma, and with a holomorphic section $e : M \to X$ of f, an *e-rigidified* line bundle on X is a pair $(\mathscr{L}, i)$ where $i : e^*\mathscr{L} \simeq \mathscr{O}_M$ (i.e., i carries the data of a nowhere vanishing section s of $e^*\mathscr{L}$).

Example 4.8. If M is a point, so that e is simply a point of X, an e-rigidification is simply a basis of the fiber $\mathscr{L}(e)$ at $e \in X$.

Example 4.9. Given any $\mathscr{L}$ on X, consider the twist $\delta_e(\mathscr{L}) := \mathscr{L} \otimes f^*e^*\mathscr{L}^{-1}$. (Note that we're twisting $\mathscr{L}$ by something that comes from $\mathrm{Pic}(M)$.) Then we have

$$e^*(\delta_e(\mathscr{L})) = e^*\mathscr{L} \otimes e^*f^*e^*\mathscr{L}^{-1} = e^*\mathscr{L} \otimes e * \mathscr{L}^{-1} \simeq \mathscr{O}_M$$

so that $\delta_e(\mathscr{L})$ has a canonical e-rigidification. (Think about that this means when M is a point.)

An isomorphism $(\mathscr{L}, i) \simeq (\mathscr{L}', i')$ is $\alpha : \mathscr{L} \simeq \mathscr{L}'$ such that $e^*(\alpha)$ pulls back i' to i

Lemma 4.10. *For $(\mathscr{L}, i)$, $(\mathscr{L}', i')$, any $\alpha : \mathscr{L} \simeq \mathscr{L}'$ as line bundles has a unique $\mathscr{O}(M)^\times$-multiple $(u \circ f) \cdot \alpha$ that respects i, i'. In particular $\mathrm{Aut}(\mathscr{L}, i) = \{1\}$, so e-rigidified pairs are* uniquely *isomorphic if at all isomorphic.*

Proof. Consider

$$\mathscr{O}_M \overset{i^{-1}}{\simeq} e^*\mathscr{L} \overset{e^*\alpha}{\simeq} e^*\mathscr{L}' \overset{i'}{\simeq} \mathscr{O}_M.$$

This is multiplication by some unit on M. Its reciprocal is u. ■

Definition 4.11. We set $\mathrm{Pic}_e(X \to M) = \{(\mathscr{L}, i)\}/\simeq$. This is a commutative group via $(\mathscr{L}, i) \otimes (\mathscr{L}', i') := (\mathscr{L} \otimes \mathscr{L}', i \otimes i')$ and $(\mathscr{L}, i)^{-1} := (\mathscr{L}^\vee, (i^\vee)^{-1})$.

The basic idea for the M-group law on an elliptic curve $(E, e) \to M$ is to show that the map

$$E(M) \to \mathrm{Pic}^0_e(E \to M)$$

sending $x \mapsto \delta_e(\mathscr{I}_x^{-1} \otimes \mathscr{I}_e)$ is bijective. But this is too weak — insufficiently functorial, like showing that $E(\mathbf{Q}) \simeq \mathrm{Pic}^0(E)$ for some elliptic curve $E/\mathbf{Q}$. To succeed, we'll need to adapt this to $E(M')$ for any $M' \to M$ (so that we can take $M' = E \times_M E$, for instance).

For an elliptic curve $(E, e) \to M$, we seek to understand $E(X)$, the collection of diagrams

$$\begin{array}{ccc} X & \longrightarrow & E \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & M & \end{array},$$

for varying $(g : X \to M)$, in terms of line bundles. We'll give this a group structure functorially in $(X \to M)$ with identity $e \circ g$, which by Yoneda will make (E, e) into an M-group. By the classical theory on fibers, this group structure will necessarily be unique, commutative, and compatible with base change. The crux is the following "base-change trick." In the following fiber product diagram

$$\begin{array}{ccc} Y_{M'} = Y \times_M M' & \longrightarrow & Y \\ {\scriptstyle p_2}\downarrow \uparrow & \nearrow & \downarrow{\scriptstyle \mathrm{sub}} \\ M' & \xrightarrow{f} & M \end{array}$$

the universal property of the fiber product means that giving one of the dotted arrows is equivalent to giving the other (where the vertical dotted arrow is understood to be a section of p_2). So to understand M-maps $M' \to Y$ we see that the key case is to understand sections of $\pi : Y \to M$ (at the cost of renaming M' as M and $Y_{M'}$ as Y, so we must be allowed to work over general bases M and with a fair bit of generality in Y). Compatibility with base change in M will encode functoriality of the above diagram in M'.

We will see on Homework 2 that a section

$$\begin{array}{c} Y \\ {\scriptstyle \pi}\downarrow \uparrow{\scriptstyle s} \\ M \end{array}$$

is an embedded submanifold, even closed when π is separated. So to construct sections we can try to think geometrically and make embeddings $j : Z \hookrightarrow Y$ such that $\pi \circ j : Z \simeq M$ is an isomorphism.

Example 4.12. For an M-curve $\pi : Y \to M$, consider a section

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & Y \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

By Lemma 4.2 the sheaf $\mathscr{I}_s := \ker(\mathscr{O}_Y \twoheadrightarrow j_*\mathscr{O}_Z)$ is invertible and $\mathscr{I}_s|_{Y_m} = \mathscr{I}_{s(m)} \subset \mathscr{O}_{Y_m}$ (the equality coming from base change by $\{m\} \to M$) is of degree -1 for all $m \in M$.

Thus for an M-map $y : M' \to Y$ the associated section $\widetilde{y}$ of $Y_{M'} \to M'$ we get an invertible $\mathscr{I}_{\widetilde{y}}$ on $Y_{M'}$ (not Y!) with fibral degree -1. Consider the map $Y(M') \to \mathrm{Pic}(Y_{M'})$ sending $y \mapsto \mathscr{I}_{\widetilde{y}}$.

Lemma 4.13. *This is natural in $M' \to M$, i.e., for $h : M'' \to M'$ the diagram*

$$\begin{array}{ccc} Y(M') & \longrightarrow & \mathrm{Pic}(Y_{M'}) \\ \big\downarrow h & & \big\downarrow (1_Y \times h)^* \\ Y(M'') & \longrightarrow & \mathrm{Pic}(Y_{M''}) \end{array}$$

commutes.

Proof. The point is that $(1_Y \times h)^*(\mathscr{I}_{\widetilde{y}'}) \simeq \mathscr{I}_{\widetilde{y}''}$, as seen in Lemma 4.2 (also see HW2). ■

To clarify that $\mathrm{Pic}(Y_{M'})$ should be viewed functorially in $M' \to M$, we make the following definition.

Definition 4.14. For a submersion $f : Y \to M$, we define

$$\mathrm{Pic}_{Y/M} : (\mathbf{C} - \mathrm{man}/M) \to \mathrm{Ab}$$

sending $M' \rightsquigarrow \mathrm{Pic}(Y_{M'})$, contravariant via pullback. We make three variants of this.

First, when f is also proper with connected fibres and we have $e \in Y(M)$,

$$\mathrm{Pic}_{Y/M,e} : M' \rightsquigarrow \mathrm{Pic}_{e_{M'}}(Y_{M'})$$

where the right-hand side is the collection of $e_{M'}$-rigidified line bundles on $Y_{M'}$ up to isomorphism.

Second, for an M-curve $Y \to M$ we define

$$\mathrm{Pic}^d_{Y/M} : M' \rightsquigarrow \{\mathscr{L} \in \mathrm{Pic}(Y_{M'}) \colon \deg_{Y_{M',m'}}(\mathscr{L}_{m'}) = d \text{ for all } m' \in M'\}.$$

Third, when both sets of additional conditions are satisfied we can make $\mathrm{Pic}^d_{Y/M,e}$.

Example 4.15. Let Y be an M-curve. Then there is a map of functors $h_Y := \mathrm{Hom}_M(-, Y) \to \mathrm{Pic}^{-1}_{Y/M}$ sending $y \mapsto \mathscr{I}_{\widetilde{y}}$.

Example 4.16. Again let Y be an M-curve. Given $e \in Y(M)$, we get $\mathrm{Pic}^d_{Y/M} \to \mathrm{Pic}^d_{Y/M,e}$ sending $\mathscr{L} \mapsto (\delta_e(\mathscr{L}), \mathrm{can.})$ where $\delta_e(\mathscr{L}) = \mathscr{L} \otimes f^* e^*(\mathscr{L}^{-1})$.

Example 4.17. Putting these together, for an M-curve Y with $e \in Y(M)$, we get $h_Y \to \mathrm{Pic}^{-1}_{Y/M,e}$ sending $y \mapsto \delta_e(\mathscr{I}_{\widetilde{y}})$.

The main theorem, then, is the following.

Theorem 4.18 (Abel)**.** *For an elliptic curve $(E, e) \to M$, the maps of functors $h_E \to \mathrm{Pic}^{\mp 1}_{E/M,e}$ which map $E(M') \to \mathrm{Pic}^{\mp 1}_{e_{M'}}(E_{M'})$ by $y \mapsto \delta_e(\mathscr{I}^{\pm 1}_{\widetilde{y}})$ are isomorphisms of functors on $\mathbf{C}$-manifolds over M.*

Corollary 4.19. *The map $h_E \to \mathrm{Pic}^0_{E/M,e}$ sending $y \mapsto \delta_e(\mathscr{I}_{\widetilde{y}} \otimes \mathscr{I}^{-1}_{e_Y})$ is an isomorphism, so gives an M-group structure to (E, e).*

Proof of Theorem 4.18. By the above preparations, we can rename M' as M and $E_{M'}$ as E to reduce ourselves to the case $M' = M$. That is, it is necessary and sufficient that the map

$$E(M) \to \mathrm{Pic}^1_e(E)$$

sending $y \mapsto \delta_e(\mathscr{I}^{-1}_y)$ is bijective. We know injectivity already via fibers.

(This use of the base-change trick hides a number of complexities. If we unravel what's going on, the group law is constructed via Yoneda with $M' = E \times_M E$.)

For surjectivity, we choose $(\mathscr{L}, i)$, and we seek a (necessarily unique) closed submanifold $j : Z \hookrightarrow E$ such that $\pi \circ j : Z \simeq M$ and such that there is an isomorphism $\delta_e(\mathscr{I}^{-1}_Z) \simeq \mathscr{L}$ respecting e-rigidifications.

Reading 4.20. This is explained in detail in the second handout "Line bundles and sections". Here we give a brief sketch.

The first observation is that since Z (if it exists) is unique and since $(\mathscr{L}, i)$ has no nontrivial automorphisms, this problem is local over M.

Now, we are looking for an analogue of the classical result that for any divisor D of degree 1 on a curve of genus 1 we have $D \sim [z]$ for a unique $z \in E$. This uses cohomology (Riemann–Roch), so we need relative cohomology methods. This is used to prove that the sheaf $\pi_*\mathscr{L}$ is *invertible* and that there is an isomorphism

$$(\pi_*\mathscr{L})_m/\mathfrak{m}_m(\pi_*\mathscr{L})_m \simeq H^0(E_m, \mathscr{L}_m) \tag{4.2}$$

for all $m \in M$. (This can also be done via elliptic PDE methods of Kodaira–Spencer.) The handout discusses this stuff (also cf. Chapter III §12 of Hartshorne). One then localizes M to make $\pi_*\mathscr{L} \simeq \mathscr{O}_M$. By (11.1) the corresponding section $s \in H^0(E, \mathscr{L})$ has $s(m) \in H^0(E_m, \mathscr{L}_m)$ *nonzero* for all $m \in M$. Thus multiplication by s gives a map $s : \mathscr{O}_M \to \mathscr{L}$ that is fibrally nonzero, which we dualize to get $\mathscr{L}^\vee \hookrightarrow \mathscr{O}_M$. This invertible sheaf cuts out Z, and we use $\deg(\mathscr{L}_m) \equiv 1$ to show that $\delta_e(Z) \simeq \mathscr{L}$. ■

5. Relative uniformization of M-elliptic curves

Notation 5.1. For X a $\mathbf{C}$-manifold over M, rather than write h_X for $\mathrm{Hom}_M(-, X)$ we'll just denote it as X when thinking functorially.

For instance, for an elliptic curve $(E, e) \to M$ we'll write $E \xrightarrow{\sim} \mathrm{Pic}^0_{E/M,e}$ as functors on $(\mathbf{C}-\mathrm{man}\,/M)$. This *defined* the M-group structure on E (via tensor product of line bundles) and now we'll give another application of the line bundle viewpoint: the existence of a relative uniformization (i.e. a canonical $E \xrightarrow{\sim} V_E/\Lambda_E$).

Remark 5.2. In some sense this is the "wrong" viewpoint because Pic is contravariant and V, Λ are covariant in $A := V/\Lambda$ (via $V = T_e(A)$ and $\Lambda = H_1(A, \mathbf{Z})$). The "right" method uses an exponential map $\exp_{G/M} : T_eG = e^*(\Omega^1_{G/M})^\vee \to G$ for M-groups G. First we'll make V_E, Λ_E that are contravariant in E and "fix" this later.

We will use the cohomological description of $\mathrm{Pic}^0(E)$ in the classical case to construct a uniformization without the use of an exponential map. Then we'll need more work to fix up the variance. To clarify ideas, let's think about $\mathrm{Pic}^0(X)$ for any compact connected Riemann surface X. We have

$$\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X) \simeq \check{H}^1(X, \mathscr{O}_X^\times) = H^1(X, \mathscr{O}_X^\times).$$

We seek to describe $\mathrm{Pic}^0(X)$ via "cohomology operations", revealing it as a complex torus V/Λ. This rests on an exponential sequence, as follows.

Let $\mathbf{Z}(1) = \ker(\exp : \mathbf{C} \to \mathbf{C}^\times) = 2\pi i\mathbf{Z}$. For any $\mathbf{C}$-manifold Z, the exponential sequence on Z is an exact sequence of abelian sheaves

$$0 \to \mathbf{Z}(1) \to \mathscr{O}_Z \xrightarrow{\exp} \mathscr{O}_Z^\times \to 1.$$

This gives an exact sequence

$$0 \to \mathrm{coker}(H^1(\mathbf{Z}(1)) \to H^1(\mathscr{O}_Z)) \to \mathrm{H}^1(Z, \mathscr{O}_Z^\times) = \mathrm{Pic}(Z) \xrightarrow{c_1 = \delta} H^2(Z, \mathbf{Z}(1)).$$

(We remark that the map $H^1(\mathbf{Z}(1)) \to H^1(\mathscr{O}_Z)$ is injective if Z is compact and connected.) Take $Z = X$, so that we have a diagram

$$\begin{array}{ccc} H^2(X, \mathbf{Z}(1)) & \hookrightarrow & H^2(X, \mathbf{C}) \\ \simeq\downarrow & & \downarrow \frac{1}{2\pi i}\int_{X,i} \\ \mathbf{Z} & \hookrightarrow & \mathbf{C} \end{array}$$

(with the right-hand vertical map independent of the choice of i) as well as an inclusion

$$H^1(X, \mathbf{Z}(1)) \hookrightarrow H^1(X, \mathscr{O}_X)$$

where the left-hand side is free of rank $2g$ and the right-hand side is a $\mathbf{C}$-vector space of dimension $g = \mathrm{genus}(X)$ by Serre duality.

Lemma 5.3. *The image of $H^1(X, \mathbf{Z}(1)) \hookrightarrow H^1(X, \mathscr{O}_X)$ is a lattice.*

Proof. This is a consequence of Hodge theory; see HW3. ■

Thus we get

$$0 \to H^1(X, \mathscr{O}_X)/H^1(X, \mathbf{Z}(1)) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbf{Z}(1)) \simeq \mathbf{Z}$$

as groups with $H^1(X, \mathscr{O}_X)/H^1(X, \mathbf{Z}(1))$ a $\mathbf{C}$-torus, hence infinitely divisible. But we also have

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z}$$

with $\mathrm{Pic}^0(X)$ infinitely divisible by the theory of the Jacobian, and so necessarily

$$H^1(X, \mathscr{O}_X)/H^1(X, \mathbf{Z}(1)) \xrightarrow{\sim} \mathrm{Pic}^0(X)$$

as groups.[1]

We seek to adapt the preceding argument for M-curves (and then focus on genus 1). Let $f : X \to M$ be an M-curve of genus $g \geq 1$. We seek to glue $\{H^1(X_m, \mathscr{O}_{X_m})\}_{m \in M}$ into the fibers of a rank g vector bundle $V_X \to M$, and similarly glue $\{H^1(X_m, \mathbf{Z}(1))\}_{m \in M}$ into a local system $L_x \to M$ of free abelian groups of rank $2g$, so that

(1) we can make sense of the M-group V_x/L_x, and
(2) prove that this represents $\mathrm{Pic}^0_{X/M,e}$ for $e \in X(M)$.

This "fibral gluing" requires a digression into the handout on *cohomology and base change*, where we review methods for discussing variation in cohomology of fibres of a map. The key concept we will need is that of the higher direct images of $f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(M)$. The $\mathrm{R}^i f_*$'s are the derived functors of f_*. Concretely $\mathrm{R}^i f_*(\mathscr{F})$ is the sheafification of $U \mapsto H^i(f^{-1}U, \mathscr{F})$, from which we deduce the existence of a map

$$\mathrm{R}^i f_*(\mathscr{F})_m = \varinjlim_{U \ni m} H^i(f^{-1}U, \mathscr{F}) \to H^i(X_m, \mathscr{F}|_{X_m}), \tag{5.1}$$

the sheaf $\mathscr{F}|_{X_m}$ being the topological pullback along $X_m \hookrightarrow X$.

Example 5.4. If M is a point, then $\mathrm{R}^i f_* = H^i(X, -)$.

Example 5.5. If $\mathscr{F} = \underline{A}$ on X then $\mathscr{F}|_{X_m} = \underline{A}$ on X_m, so get a map $\mathrm{R}^i f_*(\underline{A})_m \to H^i(X_m, \underline{A})$.

Example 5.6. If M is a manifold and $f : X \to M$ is a topological fiber bundle (e.g. f is a proper C^∞-submersion, so Ehresmann's theorem applies) then there is an open cover $\{U_i\}$ of M on which $f^{-1}U_i \simeq U_i \times Y_i$ and $X_m \simeq Y_i$ for all $m \in U_i$. For open balls $m \in V \subset U \subset U_i$ we have $f^{-1}U \simeq U \times X_m$ and $f^{-1}V \simeq V \times X_n$, and so we have a diagram

$$\begin{array}{ccc} H^i(f^{-1}U, A) & \longrightarrow & H^i(f^{-1}V, A) \\ & {\scriptstyle\simeq}\searrow \quad \swarrow{\scriptstyle\simeq} & \\ & H^i(X_m, A). & \end{array}$$

Therefore $\mathrm{R}^i f_*(A)|_{U_i}$ is the constant sheaf associated to $H^i(X_m, A)$ and

$$\mathrm{R}^i f_*(A)_m \xrightarrow{\sim} H^i(X_m, A)$$

for all $m \in M$. This is the *base change* morphism associated to the inclusion $\{m\} \hookrightarrow M$.

Example 5.7. For an M-curve $f : X \to M$ of genus g, the sheaf $\mathrm{R}^1 f_* \mathbf{Z}$ is a local system of $\mathbf{Z}^{2g}$'s, and the sheaf $\mathrm{R}^2 f_* \mathbf{Z}$ is a local system of $\mathbf{Z}$'s.

Example 5.8. For the family of elliptic curves $V/\Lambda \to \mathbf{C} - \mathbf{R}$ constructed in previous lectures, one can show (see HW3) that $(\mathrm{R}^1 f_* \mathbf{Z})^\vee \cong \Lambda$ as local systems over $\mathbf{C} - \mathbf{R}$ (necessarily split, topologically, since $\mathbf{C} - \mathbf{R}$ is simply connected). On the other hand (again see HW3) the analytic Tate curve $f : E \to \Delta^\times = \{0 < |q| < 1\}$ has $\mathrm{R}^1 f_* \mathbf{Z}$ not globally free.

---

[1] This also shows that $c_1 = n \cdot \deg$ for some $n \in \mathbf{Z}$. We expect $n = \pm 1$, but which one is it?

Reading 5.9. We need variants of the above for "holomorphic coefficients". Now see the handout "Base change morphisms".

If $g : X' \to X$ is a map of ringed spaces and $\mathscr{F}$ is a sheaf on X, recall that $g^*\mathscr{F}$ and the topological pullback $\mathscr{F}|_{X'}$ are related by $g^*\mathscr{F} = \mathscr{F}|_{X'} \otimes_{\mathscr{O}_X|_{X'}} \mathscr{O}_{X'}$. Thus for example if M is a $\mathbf{C}$-manifold and $h : \{m\} \to M$ is the inclusion of a point, then the pullback $\mathscr{O}_M|_m$ is the stalk $\mathscr{O}_{M,m}$ whereas $h^*\mathscr{O}_M \simeq \mathscr{O}_{M,m}/\mathfrak{m}_{M,m} \simeq \mathbf{C}$ via evaluation at m.

Remark 5.10. Topological pullback is a special case (in which the structure sheaves are simply the constant sheaves $\underline{\mathbf{Z}}$, and $g * \mathscr{F}$ and $\mathscr{F}|_{X'}$ are the same thing.

In the "Base change morphisms" handout it is explained that to a commutative diagram of ringed spaces

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow[h]{} & M \end{array}$$

and to an $\mathscr{O}_X$-module $\mathscr{F}$ there is a canonical $\mathscr{O}_{M'}$-linear morphism

$$h^*\mathrm{R}^i f_*(\mathscr{F}) \to \mathrm{R}^i f'_*(g^*\mathscr{F}),$$

the associate *base change morphism*. The functorial properties of this morphism are explained in the handout.

Example 5.11. If $f : X \to M$ is a topological fiber bundle, $h : \{m\} \to M$ is the inclusion of a point, $X' = X_m$, and $\mathscr{F} = A$ is a constant sheaf, this recovers the map of Example 5.6.

We now consider the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{j_m} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \{m\} & \xrightarrow[h]{} & M \end{array}$$

with $f : X \to M$ a holomorphic submersion of $\mathbf{C}$-manifolds. We will write $\mathscr{F}_m := j_m^*(\mathscr{F})$ for the ringed space pullback $\mathscr{F}|_{X_m}/\mathfrak{m}_m\mathscr{F}|_{X_m}$.

Then the source of the corresponding base change morphism is $(\mathrm{R}^i f_*\mathscr{F})(m)$ where $\mathscr{G}(m) := \mathscr{G}_m/\mathfrak{m}_{M,m}\mathscr{G}_m$, while the target is $H^i(X_m, \mathscr{F}_m)$. Thus the base change morphism in this case is a $\mathbf{C}$-linear map

$$(\mathrm{R}^i f_*\mathscr{F})(m) \to H^i(X_m, \mathscr{F}_m) \tag{5.2}$$

The same map is obtained by composing the the map $\mathrm{R}^i f_*(\mathscr{F})_m \to H^i(X_m, \mathscr{F}|_{X_m})$ of (5.1) with the map $H^i(X_m, \mathscr{F}|_{X_m}) \to H^i(X_m, \mathscr{F}_{X_m})$ obtained from evaluation $\mathscr{O}_{M,m} \to \mathscr{O}_{M,m}/\mathfrak{m}_{M,m} \cong \mathbf{C}$ at m, and observing that the composition factors through $(\mathrm{R}^i f_*\mathscr{F})(m)$.

Example 5.12. When $i = 0$, we get the map $(f_*\mathscr{F})(m) \to H^0(X_m, \mathscr{F}_m)$ which already arose in the handout on line bundles and sections to elliptic curves (and which was seen there to be an isomorphism if f was proper and $\mathscr{F}$ was a vector bundle).

Theorem 5.13 (Kodaira–Spencer, Grauert)**.** *In the preceding discussion, if f is* proper *and $\mathscr{F}$ is locally free of finite rank such that $\dim H^i(X_m, \mathscr{F}_m) = d$ for all $m \in M$, then:*

- $\mathrm{R}^i f_*(\mathscr{F})$ *is locally free of rank d, and*
- *the map* (5.2) *is an isomorphism for all $m \in M$.*

Proof. See the references in the "line bundles and sections" handout for $i = 0$: the same references apply for all i. ■

Example 5.14. Let $f : X \to M$ be an M-curve of genus g (recall that M-curves are proper by hypothesis). By Ehresmann's theorem (Example 5.6), the sheaf $L_{X/M} := \mathrm{R}^1 f_*\mathbf{Z}(1)$ is a local system of free abelian groups of rank $2g$ on M. By Theorem 5.13, the sheaf $V_{X/M} := \mathrm{R}^1 f_*\mathscr{O}_X$ is a rank g vector bundle over M.

There is a natural map $L_{X/M} \to V_{X/M}$, and unraveling the discussion of base change we see that the map $(L_{X/M})_m \to (V_{X/M})(m)$ *is* the map $H^1(X_m, \mathbf{Z}(1)) \hookrightarrow H^1(X_m, \mathscr{O}_{X_m})$, which is a lattice for all $m \in M$. Thus by the first handout $P_{X/M} := V_{X/M}/L_{X/M}$ makes sense as an M-group with g-dimensional connected fibers.

Remark 5.15. In HW3 you'll establish precise "base change compatibilities" for $V_{X/M}$, $L_{X/M}$, and $P_{X/M}$.

Now suppose that we have $e \in X(M)$. For and $M' \to M$ and the resulting $f' : X_{M'} \to M'$, we apply the δ-functor $\mathrm{R}^\bullet f'_*$ to the exponential sequence

$$0 \to \mathbf{Z}(1) \to \mathscr{O}_{X'} \to \mathscr{O}_{X'}^\times$$

on X' to get the following exact sequence of abelian sheaves *on* M':

$$f'_*\mathscr{O}_{X'} \overset{\exp}{\to} f'_*\mathscr{O}_{X'}^\times \to L_{X'} \to V_{X'} \to \mathrm{R}^1 f'_*\mathscr{O}_{X'}^\times \overset{\delta}{\to} \mathrm{R}^2 f'_*\mathbf{Z}(1).$$

The leftmost map is surjective since (by Theorem 5.13) locally it is just $\exp : \mathscr{O}_M \to \mathscr{O}_M^\times$. What is $\mathrm{R}^1 f'_*\mathscr{O}_{X'}^\times$? It's the sheafification of $U' \mapsto H^1(X_{U'}, \mathscr{O}_{X'}^\times) = \mathrm{Pic}(X_{U'})$, i.e., it's $\underline{\mathrm{Pic}}_{X'/M'}$, the sheafification of $\mathrm{Pic}_{X'/M'}$ regarded as a presheaf on the topological space M'. There is a (forgetful) map from the sheaf $\mathrm{Pic}_{X'/M',e_{M'}}$ to $\underline{\mathrm{Pic}}_{X'/M'}$ which is an isomorphism (as can be checked locally). One sees (by checking on fibres) that the kernel of δ is precisely the line bundles of fibral degree zero, and putting all this information together we have a short exact sequence

$$0 \to L_{X'} \to V_{X'} \to \mathrm{Pic}^0_{X'/M',e_{M'}} \to 0$$

of sheaves on M', and we get an isomorphism $P_{X'} \xrightarrow{\sim} \mathrm{Pic}^0_{X'/M',e_{M'}}$, again of sheaves on M'. Finally we deduce that

$$P_{X/M}(M') = P_{X'/M'}(M') = \mathrm{Pic}_{X'/M',e_{M'}}(M') = \mathrm{Pic}^0_{X/M,e}(M')$$

functorially in M', and we have an isomorphism of functors $P_{X/M} \xrightarrow{\sim} \mathrm{Pic}^0_{X/M,e}$ on $\mathbf{C} - \mathrm{man}/M$.

Now set $g = 1$. Then for a relative elliptic curve $(E, e) \to M$, we get an isomorphism

$$\alpha_{E/M} : E \xrightarrow{\sim} \widehat{E} := \mathrm{Pic}^0_{E/M,e} \simeq P_{E/M}$$

of functors on $\mathbf{C} - \mathrm{man}/M$, respecting base change over M (see HW3, exercise 4 for this). Reviewing the construction gives functoriality, as follows.

Proposition 5.16. *For a map $\varphi : (E', e') \to (E, e)$ of elliptic curves over M, there is a commutative square*

$$\begin{array}{ccc} \mathrm{R}^1 f_*\mathscr{O}_E/\mathrm{R}^1 f_*\mathbf{Z}(1) & \xrightarrow{\sim} & \widehat{E} \\ \downarrow & & \downarrow \widehat{\varphi} \\ \mathrm{R}^1 f'_*\mathscr{O}_{E'}/\mathrm{R}^1 f'_*\mathbf{Z}(1) & \xrightarrow[\sim]{} & \widehat{E}' \end{array}$$

where the left vertical arrow is cohomology pullback and the right vertical arrow $\widehat{\varphi}$ is Pic_e*-functoriality.*

To fix the contravariance, we iterate twice.

Proposition 5.17. *The composite $E \xrightarrow{\alpha_E} \widehat{E} \xrightarrow{\alpha_{\widehat{E}}} \widehat{E}^\wedge$ is covariant. Thus $(E, e) \to M$ admits relative uniformization that is* covariant *in (E, e) and respects base change in M.*

Reading 5.18. See the handout "Double duality" for details.

Proof. The issue is with functoriality. If we start with $\varphi : E \to E'$, note that the map $\widehat{\varphi} : \widehat{E}' \to \widehat{E}$ goes the other direction from φ as well as its double-duals, and moreover the small squares in the resulting diagram do not commute (i.e. $\alpha_E \neq \widehat{\varphi} \circ \alpha_{E'} \circ \varphi$). However they do each commute up to (the same) factor in $\mathbf{Z}$ — to see this one reduces by base change to the classical case (fibers)! — which can be cancelled. ∎

6. The universal property of the analytic Weierstrass family

Next we turn to the universal property of the Weierstrass family over $\mathbf{C}-\mathbf{R}$.

Definition 6.1. For $f:(E,e)\to M$, an H_1-trivialization is an M-group isomorphism $\mathbf{Z}^2\times M \overset{\psi}{\xrightarrow{\sim}} \underline{H}_1(E/M) := (\mathrm{R}^1 f_*\mathbf{Z})^\vee$ (where the target is covariant in E/M and respects base change on M).

Example 6.2. HW3, Exercise 1 shows that if $E=V/\Lambda$ over M then $\underline{H}_1(E/M)\cong\Lambda$ *canonically* in such a way that the isomorphism induces the canonical $\Lambda_m\cong H_1(V_m/L_m,\mathbf{Z})$ on fibers. Also any $(E,e)\to M$ admits an H_1-trivialization *locally* on M (why?).

Example 6.3. By construction, the Weierstrass family $\mathscr{E}\to\mathbf{C}-\mathbf{R}$ is equipped with a *specific* H_1-trivialization $\Psi: H_1(\mathscr{E}_\tau,\mathbf{Z})\simeq\mathbf{Z}\tau\oplus\mathbf{Z}$. Explicitly, Ψ is the map that sends $((m,n),\tau)\in\mathbf{Z}^2\times\mathbf{C}-\mathbf{R}$ to $m\tau+n\in\Lambda_\tau\cong H_1(\mathscr{E}_\tau,\mathbf{Z})$.

But HW3 Exercise 2(iii) shows that the Tate family $\mathbb{E}\to\Delta^\times$ does *not* have an H_1-trivialization over $\Delta^\times$ (though OK with a slitted unit disk).

Definition 6.4. For H_1-trivialized pairs (E,ψ) and (E',ψ') over M, an isomorphism between them is an M-isomorphism $\varphi:(E,e)\xrightarrow{\sim}(E',e')$ such that the dual base change morphism $H_1(\varphi):\underline{H}_1(E/M)\to\underline{H}_1(E'/M')$ pulls back ψ to ψ'.

Lemma 6.5. *If such φ exists, it's unique (so its existence is local on M).*

Proof. We can pass to fibers, and classically the association $E\rightsquigarrow H_1(E,\mathbf{Z})$ is faithful. ■

Definition 6.6. Let $F:\mathbf{C}-\mathrm{man}\to\mathrm{Set}$ be the functor sending M to $\{(E,\psi)\text{ over }M\}/\simeq$. This is made into a contravariant functor via base change: for $M'\xrightarrow{h}M$, the map $F(M)\to F(M')$ sends (E,ψ) to the pair

$$(E_{M'},\mathbf{Z}^2\times M'\xrightarrow{h^*\psi}h^*\underline{H}_1(E/M)\simeq\underline{H}_1(E_{M'}/M')).$$

(The right-hand map is the dual of the base change morphism, which is an isomorphism because $\underline{H}_1(E/M)$ and $\underline{H}_1(E'/M')$ are both a locally a constant sheaf, and constant sheaves topologically pull back to constant sheaves.) Loosely speaking, for continuously varying bases $(e_1(m),e_2(m))$ of $H_1(E_m,\mathbf{Z})$, on $E'=E_{M'}$ we use the family $(e_1(h(m')),e_2(h(m')))$ on the fiber $E'_{m'}=E_{h(m')}$.

Theorem 6.7. *The functor F is represented by the Weierstrass family $(\mathscr{E},\Psi)$ over $\mathbf{C}-\mathbf{R}$. That is, for any $(E,\psi)\to M$, there exists a Cartesian square*

$$\begin{array}{ccc}(E,\psi) & \xrightarrow{\tilde h} & (\mathscr{E},\Psi)\\ \downarrow & & \downarrow\\ M & \xrightarrow{h} & \mathbf{C}-R\end{array}$$

Example 6.8. The case where M is a point says that all classical (E,ψ) are isomorphic to $(\mathbf{C}/\Lambda_\tau,\{\tau,1\})$ for a unique $\tau\in\mathbf{C}-\mathbf{R}$. This is a consequence of our original description of $\mathbf{C}-\mathbf{R}$ in Proposition 2.1. Let's remind ourselves how this goes. We have $E=V/\Lambda$ with $\Lambda=\mathbf{Z}e_1\oplus\mathbf{Z}e_2$ via ψ. Identify $V\simeq\mathbf{C}$ via the basis e_2, and then $e_1=\tau e_2$ via the $\mathbf{C}$-structure on $\Lambda_{\mathbf{R}}\xrightarrow{\sim}V$. So the map h sends the point m to the isomorphism class of the pair (E,ψ) which is a point of $\mathbf{C}-\mathbf{R}$ as in Proposition 2.1.

Proof of Theorem 6.7. We will relativize the case where M is a point. Use the relative uniformization $V/\Lambda\simeq E$ of Proposition 5.17 to get maps

$$\mathbf{Z}^2\times M\xrightarrow{\sim}\underline{H}_1(E/M)\simeq\Lambda\hookrightarrow V$$

where the first map is ψ, the second comes from Example 6.2, and the third is the inclusion of an M-lattice into the line bundle V. So the maps $\psi(1,0),\psi(0,1)\in V(M)$ are nowhere vanishing holomorphic sections. It follows that $\psi(1,0)=h\cdot\psi(0,1)$ for a holomorphic function $h:M\to\mathbf{C}^\times$, valued in $\mathbf{C}-\mathbf{R}$ since the image of Λ_m in V_m is always a lattice.

Use the map $\psi(0,1)$ to trivialize $V \to M$, yielding a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\sim} & \mathbf{C} \times M \\ \uparrow & & \uparrow \\ \Lambda & \xleftarrow[\psi]{\sim} & \mathbf{Z}^2 \times M \end{array} \qquad \begin{array}{c} (ah(m)+b, m) \\ \uparrow \\ ((a,b), m) \end{array}$$

In other words the inclusion $(\Lambda \hookrightarrow V)$ over M is the h-pullback of the Weierstrass construction over $\mathbf{C} - \mathbf{R}$. This yields $E \simeq \mathscr{E}_M$ on quotients as desired. ■

Corollary 6.9. *Every $(E, e) \to M$ has a Weierstrass model locally over M.*

"Bad" proof. Use H_1-trivializations locally on M. ■

Corollary 6.10. *(Exercise.) Fix a choice of $i \in \mathbf{C}$, thus fixing a choice of upper half plane $\mathfrak{h}_i$. Restrict the family $\mathscr{E} \to \mathbf{C} - \mathbf{R}$ to the upper half plane $\mathscr{E}^{(i)} \to \mathfrak{h}_i$. Then $\mathscr{E}^{(i)} \to \mathfrak{h}_i$ is universal for H_1-trivializations that are i-oriented via $H_1(E_m, \mathbf{Z}) \to T_0(E_m) = \mathbf{C} -$ line.*

7. Analytic level structures

Let us now consider the following application. There is a left $\mathrm{GL}_2(\mathbf{Z})$-action on the functor F via $\gamma \cdot (E \to M, \psi) = (E \to M, \psi \circ \gamma^t)$.

Reading 7.1. See the handout on "$\mathrm{GL}_2(\mathbf{Z})$-action and modular forms".

By Yoneda's lemma, the left action of $\mathrm{GL}_2(\mathbf{Z})$ on the functor $F \simeq \mathrm{Hom}(-, \mathbf{C} - \mathbf{R})$ induces a left action of $\mathrm{GL}_2(\mathbf{Z})$ on $\mathbf{C} - \mathbf{R}$ as well as a lift of this to the universal object $\mathscr{E} \to \mathbf{C} - \mathbf{R}$. This is made explicit in the handout: we have a diagram

$$\begin{array}{ccc} (\mathscr{E}, \Psi \circ \gamma^t) & \xrightarrow[\sim]{[\gamma]_{\mathscr{E}}} & (\mathscr{E}, \Psi) \\ \downarrow f & & \downarrow f \\ \mathbf{C} - \mathbf{R} & \xrightarrow[{[\gamma]}]{\sim} & \mathbf{C} - \mathbf{R} \end{array}$$

where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $[\gamma](\tau) = \frac{a\tau+b}{c\tau+d}$ and $[\gamma]_{\mathscr{E}} : \mathbf{C}/\Lambda_\tau \to \mathbf{C}/\Lambda_{[\gamma](\tau)}$ induced by $z \mapsto z/(c\tau + d)$. That is, the action of γ on $\mathbf{C} - \mathbf{R}$ is the classical action, while the on $\mathscr{E}$ encodes scaling by the familiar "factor of automorphy".

Remark 7.2. There is also a a functorial $\mathrm{GL}_2(\mathbf{R})$-action, due to Deligne, via "variation of $\mathbf{C}$-structure" where we fix the lattice and very the $\mathbf{C}$-structure on $\Lambda_{\mathbf{R}}$. Again see the $\mathrm{GL}_2(\mathbf{Z})$-action handout.

To use $\mathscr{E} \to \mathbf{C} - \mathbf{R}$ to represent moduli problems with level structure, we must discuss isogenies and the Weil pairing in the relative setting.

Definition 7.3. An *analytic covering* of a $\mathbf{C}$-manifold M is a surjective local analytic isomorphism $h : M' \to M$ that is also a covering space. Call it *finite* if all the fibers are constant (and then the function $m \mapsto \#h^{-1}(m)$ is locally constant).

Remark 7.4. Any covering space of M admits a *unique* holomorphic structure making it an analytic covering: work locally on M to reduce to the case of split covers.

Example 7.5. For an elliptic curve $E \to M$, its relative uniformization $V/\Lambda \simeq E$ makes $V \to E$ an analytic covering. On HW4 you'll show bijectivity of the map from intermediate M-lattices $\Lambda \subset \Lambda' \subset V$, to closed M-subgroups $C \subset E$ such that $C \to M$ is a *finite* analytic cover, sending $\Lambda' \mapsto (\Lambda'/\Lambda \hookrightarrow V/\Lambda = E)$. Here Λ'/Λ is the quotient local system.

Consider a closed M-subgroup $C \subset E$. We seek to make a good quotient "E/C". The first step is to pass to the associated lattice $\Lambda \subset \Lambda' \subset V$, and let $E' := V/\Lambda'$. Consider the map of elliptic curves over M

$$E = V/\Lambda \xrightarrow{\varphi} V/\Lambda' = E'$$

coming from the functoriality of quotients. In the diagram

$$\begin{array}{ccc} & V & \\ \swarrow & & \searrow \\ E & \xrightarrow{\varphi} & E' \end{array}$$

the two maps out of V are analytic coverings, from which it follows that φ is also an analytic covering, with finite fibers (each $E_m \to E'_m$ is an isogeny). It follows that we can form the kernel of φ via the cartesian square

$$\begin{array}{ccc} \ker(\varphi) & \hookrightarrow & E \\ \downarrow & & \downarrow \varphi \\ M & \hookrightarrow & E' \end{array}$$

and then $C \subset \ker(\varphi)$ inside E (a map of finite analytic coverings of M). But then $C_m = \Lambda'_m/\Lambda_m = \ker(\varphi)_m$ inside E_m, and we conclude that $C = \ker(\varphi)$.

Proposition 7.6. *The map $\varphi : E \to E'$ of M-groups has the universal property for quotients by the M-group $C = \ker(\varphi)$: any holomorphic M-map $E \to X$ that is invariant by the action $C \times_M E \to E$ factors uniquely through φ.*

Reading 7.7. See the "Isogenies" handout for the proof.

Definition 7.8. We will denote the map $\varphi : E \to E'$ as $E \to E/C$.

Remark 7.9. For $M' \to M$, the map $E_{M'} \to (E/C)_{M'}$ is the quotient by $C_{M'}$. (Why? Again, see the "Isogenies" handout.)

Proposition 7.10. *Let $\varphi : E \to E'$ be a homomoprhism between elliptic curves over M. Then:*

(1) *The function $m \mapsto \deg(\varphi_m) \geq 0$ is a locally constant function on M.*
(2) *If $\deg(\varphi_m) = d \geq 1$ for all $m \in M$, then φ is a degree d analytic cover that* is *the quotient by the closed M-subgroup*

$$\begin{array}{ccc} C := \ker(\varphi) & \hookrightarrow & E \\ \downarrow & & \downarrow \varphi \\ M & \underset{e'}{\hookrightarrow} & E' \end{array}$$

and the map $C \to M$ is a degree d analytic cover.

Proof. Once more, see the "Isogenies" handout. ■

Example 7.11. For $E = V/\Lambda$ and $\varphi = [N]_E$ for some $N \neq 0$, we get $E[N] = \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda$ a degree N^2 analytic cover of M, a local system with fibers $(\mathbf{Z}/N\mathbf{Z})^2$.

Definition 7.12. An *isogeny* $\varphi : E \to E'$ is an M-homomorphism with finite fibres (equivalently, all φ_m are isogenies), which we say has degree d if $\deg \varphi_m = d$ for all $m \in M$.

Example 7.13. $[N] : E \to E$ is a degree N^2 isogeny. A degree 1 isogeny is an isomorphism (recall the fibral isomorphism criterion of an earlier handout).

Definition 7.14. For $N \geq 1$ a *full level-N structure* on $E \to M$ is an M-group isomorphism $\phi : (\mathbf{Z}/N\mathbf{Z})^2 \times M \xrightarrow{\sim} E[N]$. On HW4 it is checked that this is equivalent to giving (P, Q) in $E[N](M)$ such that $P(m), Q(m)$ are a bases of $E_m[N]$ for all m.

Example 7.15. An elliptic curve E admitting an H_1-trivialization has a full level-N structure for all N! For $\mathscr{E} \to \mathbf{C}-\mathbf{R}$, this is P_N, Q_N given by $\tau \to (\tau/N, \tau)$ and $\tau \to (1/N, \tau)$ respectively. Or in $\mathbf{CP}^2 \times (\mathbf{C}-\mathbf{R})$ these are $([\wp_{\Lambda_\tau}(\tau/N) : \wp'_{\Lambda_\tau}(\tau/N) : 1], \tau)$ and $([\wp_{\Lambda_\tau}(1/N) : \wp'_{\Lambda_\tau}(1/N) : 1], \tau)$. And then for general E admitting an H_1-trivialization, use the universal property.

Proposition 7.16. *For $E \to M$, there exists a unique $\mathbf{Z}/N\mathbf{Z}$-bilinear M-map*

$$\langle\, ,\, \rangle_{E,N} : E[N] \times_M E[N] \to \mu_N \times M$$

that agrees with the Weil pairing on fibers. That is, the Weil pairing varies holomorphically in m.

Proof. Uniqueness is clear, so we can work locally. See homework 4 for the "right" proof relating the Weil pairing to the intersection pairing on $H_1(E_m, \mathbf{Z})$ for each fiber. Here we give a cheap proof. Localize on M so that we get an H_1-trivialiazation. Then E is a pullback of $\mathscr{E} \to \mathbf{C} - \mathbf{R}$, and it's enough to do the construction for $\mathscr{E}$. We set $\langle P_N, Q_N\rangle_N(\tau) = e^{2\pi i_\tau/N}$ where i_τ is the square root of -1 in the half-plane of $\mathbf{C} - \mathbf{R}$ containing τ. (Caution: this is the opposite from the convention used in Exercise 1.15 of Silverman [AEC2].) ∎

Remark 7.17. For a full level-N structure $P, Q \in E[N](M)$, the pairing $\langle P, Q\rangle_{E,N} : M \to \mu_N$ is locally constant and valued in the primitive roots $\mu_N^\times \subset \mu_N$.

Definition 7.18. Let $[\Gamma(N)\text{–str}] : \mathbf{C}\text{–man} \to \text{Set}$ be the functor sending M to $\{(E \to M, \text{full level-}N \text{ str.})\}/\cong$, contravariant via base change.

For $\zeta \in \mu_N^\times(\mathbf{C})$ we set $[\Gamma(N) - \text{str}]^\zeta$ be the functor sending M to $\{(E, (P, Q)) \colon \langle P, Q\rangle = \zeta\}/\cong$.

Example 7.19. Choose $i = \sqrt{-1} \in \mathbf{C}$, $\zeta = e^{2\pi i r/N}$ for some $r \in (\mathbf{Z}/N\mathbf{Z})^\times$. Then over $\mathfrak{h}_i$ the pair $(\mathscr{E}^{(i)}, (rP_N, Q_N))$ lies in $[\Gamma(N) - \text{str}]^\zeta(\mathfrak{h}_i)$. But over $\mathfrak{h}_{-i}$ this lies in $[\Gamma(N) - \text{str}]^{\zeta^{-1}}$!

Next time we'll represent these using quotients of the Weierstrass construction.

8. Analytic modular curves

We've defined functors $[\Gamma(N) - \text{str}]$ as wel as $[\Gamma(N) - \text{str}]^\zeta$ for $\zeta \in \mu_N^\times(\mathbf{C})$, and now we will prove the representability of the latter.

Reading 8.1. The handout "Classical analytic modular curves" handles all others, as well as the "coarse space" situation.

Lemma 8.2. *For $N \geq 3$, the objects in $[\Gamma(N) - \text{str}](M)$ have* no *nontrivial automorphisms. In particular, given a cartesian diagram of $\Gamma(N)$-structures*

$$\begin{array}{ccc} E' & \xrightarrow{\widetilde{h}} & E \\ {\scriptstyle P',Q'} \big\uparrow\downarrow & & \downarrow\big\uparrow {\scriptstyle P,Q} \\ M' & \xrightarrow[h]{} & M \end{array}$$

the map $\widetilde{h}$ is uniquely determined *by h.*

Proof. Suppose that

$$\begin{array}{ccc} E & \xrightarrow[\sim]{\alpha} & E \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

with $\alpha_m : E_m \xrightarrow{\sim} E_m$ inducing the identity on $E_m[N]$. By the classical theory α_m is the identity for all m. ∎

We will follow the following construction principle: we will represent functors with "more structure" first, then take quotients to remove it. Recall that we have a $\mathrm{GL}_2(\mathbf{Z})$-action on $\mathscr{E} \to \mathbf{C} - \mathbf{R}$ with a universal $\Psi : \mathbf{Z}^2 \times \mathbf{C} - \mathbf{R} \xrightarrow{\sim} \underline{H}_1(\mathscr{E}/\mathbf{C} - \mathbf{R})$:

$$\begin{array}{ccc} (\mathscr{E}, \Psi \circ \gamma^t) & \xrightarrow[\sim]{[\gamma]_{\mathscr{E}}} & (\mathscr{E}, \Psi) \\ \downarrow f & & \downarrow f \\ \mathbf{C} - \mathbf{R} & \xrightarrow[{[\gamma]}]{\sim} & \mathbf{C} - \mathbf{R} \end{array}$$

where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $[\gamma](\tau) = \frac{a\tau+b}{c\tau+d}$ and $[\gamma]_{\mathscr{E}} : \mathbf{C}/\Lambda_\tau \to \mathbf{C}/\Lambda_{[\gamma](\tau)}$ sends $(z, \tau) \mapsto (z/(c\tau+d), [\gamma](\tau))$. Over τ we explicitly have $\Psi = \{\tau, 1\}$ and $\Psi \circ \gamma^t = \{a\tau + b, c\tau + d\}$.

Fix $i = \sqrt{-1} \in \mathbf{C}$. For $\tau \in \mathfrak{h}_i$, the map Ψ_τ carries determinant on $\mathbf{Z}^2$ to the negative of the i-oriented cap product on $H_1(\mathscr{E}_\tau, \mathbf{Z})$, so the $\mathrm{SL}_2(\mathbf{Z})$-action *preserves* $\mathfrak{h}_i$ (also clear by the formula for $[\gamma]$). Recall that for $\zeta = e^{2\pi i r/N} \in \mu_N^\times(\mathbf{C})$ with $r \in (\mathbf{Z}/N\mathbf{Z})^\times$ we get $(\mathscr{E}^{(i)}, (rP_N, Q_N)) \in [\Gamma(N) - \mathrm{str}]^\zeta(\mathfrak{h}_i)$.

For $\gamma \in \Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$ we have $\Psi \circ \gamma^t \equiv \Psi$ as maps $(\mathbf{Z}/N\mathbf{Z})^2 \xrightarrow{\sim} \underline{H}_1(\mathscr{E})/N \simeq \mathscr{E}[N]$. Thus we have

$$\begin{array}{ccc} \mathscr{E}^{(i)} & \xrightarrow{[\gamma]_{\mathscr{E}}} & \mathscr{E}^{(i)} \\ \downarrow & & \downarrow \\ \mathfrak{h}_i & \xrightarrow[{[\gamma]}]{} & \mathfrak{h}_i \end{array}$$

respecting the $\Gamma(N)$-structure (rP_N, Q_N) with Weil pairing ζ.

Lemma 8.3. *For $N \geq 3$, the $\Gamma(N)$-action on $\mathfrak{h}_i$ is* free*, hence also on $\mathscr{E}^{(i)}$. ("Free" means that each x has trivial stabilizer.)*

Proof without linear fractional mess. If γ fixes $\tau \in \mathfrak{h}_i$ then we get an automorphism $[\gamma]_{\mathscr{E},\tau}$ of $\mathscr{E}_\tau$ inducing the identity on $\mathscr{E}_\tau[N]$. By rigidity (using $N \geq 3$) $[\gamma]_{\mathscr{E},\tau}$ is the identity on the H_1-rigidified elliptic curve $\mathscr{E}_\tau$. But the action of γ on $\Psi_\tau = \Psi_{\gamma\tau}$ is via γ^t, so γ is the identity. ■

It follows from Lemma 8.3 that we can form quotient $\mathbf{C}$-manifolds $\Gamma(N)\backslash\mathscr{E}^{(i)}$ and $\Gamma(N)\backslash\mathfrak{h}_i$. For details see Section 4 of the "$\mathrm{GL}_2(\mathbf{Z})$-action and modular forms handout". Here is the gist of it. If X is a (Hausdorff) $\mathbf{C}$-manifold with an analytic group $H \times X \to X$, the formation of a quotient works as well as possible (e.g. the map $X \to H\backslash X$ is a local analytic isomorphism) when the action is properly discontinuous, i.e., free and discontinuous. Those properties for the action on $\mathscr{E}^{(i)}$ follow formally from those for the action on $\mathfrak{h}_i$ (and the compatibilities of the two actions), and we have just proved freeness. For discontinuity, one uses the fact that $\mathfrak{h}_i$ with its left action of $\Gamma(N)$ can be identified with $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R})$ with its left action of $\Gamma(N)$ (cf. Proposition 4.15 of the handout, which says that if G is a locally compact and separable Hausdorff group, Γ is a discrete subgroup, and K is a compact subgroup, then the left action by Γ on G/K is discontinuous).

As a result we get an $\Gamma(N)\backslash\mathfrak{h}_i =: Y_\zeta(N)$-elliptic curve

$$\begin{array}{c} \Gamma(N)\backslash\mathscr{E}^{(i)} =: (E_\zeta, (P_\zeta, Q_\zeta)) \\ \downarrow \uparrow (r\overline{P}_N, \overline{Q}_N) \\ Y_\zeta(N), \end{array}$$

an element of $[\Gamma(N) - \mathrm{str}]^\zeta(\Gamma(N)\backslash\mathfrak{h}_i)$ with full level-N structure taking the class of τ to the class of $(\mathbf{C}/\Lambda_\tau, (\frac{r\tau}{N}, \frac{1}{N}))$.

Remark 8.4. If $N = 1, 2$ we can still form these quotients. The quotient $\Gamma(N)\backslash\mathfrak{h}_i$ makes sense as a smooth curve because $\dim \mathfrak{h}_i = 1$. But $\dim \mathscr{E}^{(i)} = 2$ and the quotient (properly defined) is *not* smooth. This gives a "coarse moduli space" (just a normal "$\mathbf{C}$-analytic space"). See the "Classical analytic modular curves" handout for all this.

Theorem 8.5. *For $N \geq 3$, the object $(E_\zeta \to Y_\zeta(N), (P_\zeta, Q_\zeta))$ represents $[\Gamma(N) - \mathrm{str}]^\zeta$.*

Remark 8.6. As previously mentioned, see the handout for $Y(N) = \coprod_\zeta Y_\zeta(N)$, $Y_1(N)$, $Y_0(N)$, and coarse spaces in general.

Remark 8.7.
(1) Taking **C**-valued points recovers the classical fact that

$$[\Gamma(N) - \mathrm{str}]^\zeta(\mathbf{C}) = \{(\mathbf{C}/\Lambda_\tau, (\frac{r\tau}{N}, \frac{1}{N}))\}/\{\tau \sim \Gamma(N)\tau\}$$

but Theorem 8.5 is *much* stronger.
(2) The "explicit" $\Gamma(N)\backslash\mathfrak{h}_i$ representing $[\Gamma(N) - \mathrm{str}]^\zeta$ involves i but not ζ. Ah, but its identification *as* the representing object involves r (hence involves $\zeta = e^{2\pi i r/N}$) via $r\overline{P}_N$. A moduli space is *always* viewed with a universal object over it.
(3) Best to think "abstractly" not "explicitly" whenever possible, to avoid a mess. Some examples follow.

Example 8.8. For $a \in (\mathbf{Z}/N\mathbf{Z})^\times$ there is an isomorphism $[\Gamma(N)-\mathrm{str}]^\zeta \xrightarrow{\sim} [\Gamma(N)-\mathrm{str}]^{\zeta^a}$ sending $(E, (P, Q)) \mapsto (E, (aP, Q))$, so to prove the theorem it suffices to treat *one* ζ (e.g. $e^{2\pi i/N}$).

Example 8.9. There is an isomorphism $\langle a \rangle : [\Gamma(N)-\mathrm{str}]^\zeta \xrightarrow{\sim} [\Gamma(N)-\mathrm{str}]^\zeta$ sending $(E, (P, Q)) \mapsto (E, (aP, a^{-1}Q))$ which induces an automorphism

$$\begin{array}{ccc} E_\zeta & \xrightarrow{\sim} & E_\zeta \\ {\scriptstyle (aP, a^{-1}Q)} \uparrow\downarrow & & \downarrow\uparrow {\scriptstyle (P,Q)} \\ Y_\zeta(N) & \xrightarrow[\sim]{} & Y_\zeta(N) \end{array}$$

For $a = r^{-1}$ this encodes that $(\Gamma(N)\backslash\mathscr{E}^{(i)}, (\overline{P}_N, r\overline{Q}_N))$ is also universal and identifies explicit $\Gamma(N)\backslash\mathfrak{h}_i$ as a moduli space in another way. For our explicit triple in Theorem 8.5, what are the isomorphisms in the above diagram? See HW4.

Proof of Theorem 8.5. Without loss of generality fix $\zeta = e^{2\pi i/N}$. By the classical theory, the map $H_1(E, \mathbf{Z}) \twoheadrightarrow H_1(E, \mathbf{Z}/N\mathbf{Z})$ carries the negative of the i-oriented cap product onto our Weil pairing, using $\mathbf{Z}/N\mathbf{Z} \to \mu_N$ via the basis ζ. (Look back at that Silverman AECII exercise again.)

Consider an elliptic curve $f : E \to M$ with full level-N structure (P, Q) with Weil pairing ζ. To build a unique cartesian diagram

$$\begin{array}{ccc} E & \dashrightarrow & E_\zeta \\ \downarrow & & \downarrow \\ M & \dashrightarrow & Y_\zeta(N) \end{array}$$

compatible with level structures it is enough to work locally on M, so without loss of generality there exists an H_1-trivialization $\varphi : \mathbf{Z}^2 \times M \to \underline{H}_1(E/M)$. Changing the trivialization by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ if necessary on connected components of M, we can assume that it carries det over to "$-$fibral cap" with respect to the fibration $E \to M$, and we have a diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathscr{E}^{(i)} \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathfrak{h}_i \end{array}$$

Then $(E \to M, \varphi)$ yields

$$\begin{array}{ccccc} E & \longrightarrow & \mathscr{E}^{(i)} & \longrightarrow & \Gamma(N)\backslash\mathscr{E}^{(i)} \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & \mathfrak{h}_i & \longrightarrow & \Gamma(N)\backslash\mathfrak{h}_i \end{array}$$

which however might not yet respect the level-N structure. But φ mod N carries det mod N to the Weil pairing, and $(P,Q) : (\mathbf{Z}/N\mathbf{Z})^2 \times M \xrightarrow{\sim} \underline{H}_1(E)/N$ does too, so these "differ" by an element of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$. But $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ is *surjective* (strong approximation) so we can change φ by an element of $\mathrm{SL}_2(\mathbf{Z})$ so that φ mod N is (P,Q). This φ is *unique* up to $\Gamma(N)$, so with the new H_1-trivialization the above diagram does the job and is unique. ■

9. Γ-structures

Consider a pair $(N \geq 1, \Gamma \hookrightarrow \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$. For an elliptic curve $E \to M$, consider the sheaf of sets on M defined by

$$[\Gamma(N) - \mathrm{str}]_{E/M} : U \mapsto \{\text{full level-}N \text{ structures on } U\}.$$

This has a left $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-action that's simply transitive on stalks, so we have

$$\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \times [\Gamma(N) - \mathrm{str}]_{E/M} \xrightarrow{\sim} [\Gamma(N) - \mathrm{str}]_{E/M} \times_M [\Gamma(N) - \mathrm{str}]_{E/M}$$

as sheaves on M. This is a $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-torsor over M (its H^0 could be empty). On HW4 Exercise 4 you study the following definition.

Definition 9.1. A Γ-structure on E/M is an element $\alpha \in H^0(M, \Gamma\backslash[\Gamma(N) - \mathrm{str}]_{E/M})$. Roughly it's "locally a full level-N structure up to Γ-action". Notions of isomorphism $(E \to M, \alpha) \simeq (E' \to M, \alpha')$ and base change are evident.

Caution: this concept depends on (and includes the data of) N, not just Γ: consider $\Gamma = 1$!

Example 9.2. (See HW4.) If $\Gamma = 1$, a Γ-structure is just a full level-N structure.

For $\Gamma = \mathrm{G}\,\Gamma_0(N) = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$ a Γ-structure is equivalent to the data of $C \hookrightarrow E[N]$ a closed M-subgroup such that $C \to M$ is a degree-N analytic cover with cyclic fibers.

For $\Gamma = \mathrm{G}\,\Gamma_1(N) = \left\{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\right\}$ a Γ-structure is equivalent to the data of $P \in E[N](M)$ such that $P(m) \in E_m$ has exact order N for all $m \in M$.

Observe that if $N \mid N_1$ and Γ_1 is the preimage of Γ under $\mathrm{GL}_2(\mathbf{Z}/N_1\mathbf{Z}) \twoheadrightarrow \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$, then

$$\Gamma_1\backslash[\Gamma(N_1) - \mathrm{str}]_{E/M} \xrightarrow{\sim} \Gamma\backslash[\Gamma(N) - \mathrm{str}]_{E/M}$$

as sheaves on M. What is the intrinsic concept encoding pairs (N, Γ) up to this sort of equivalence? A compact open subgroup $K \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$! (Equivalently, a compact open subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$, up to conjugacy. See HW5.) We'll return to this later.

In the "Classical modular curves" handout, the solution to the $[\Gamma(N) - \mathrm{str}]$ moduli problem for $N \geq 3$ is used to prove the following.

Theorem 9.3.

(1) *All pairs $(E \to M, \alpha = \Gamma - \mathrm{str})$ are rigid if and only if the preimage of $\Gamma \cap \mathrm{SL}_2(\mathbf{Z}/N/Z)$ under $\mathrm{SL}_2(\mathbf{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ is torsion-free.*

(2) *The contravariant functor $F_\Gamma : M \rightsquigarrow \{(E \to M, \alpha)\}/\simeq$ with α a $frm[o]--Gamma$-structure has a coarse moduli space Y_Γ of pure dimension 1, with $\pi_0(Y_\Gamma)$ a torsor for $(\mathbf{Z}/N\mathbf{Z})^\times/\det\Gamma$ (so Y_Γ is connected if and only if $\det\Gamma = (\mathbf{Z}/N\mathbf{Z})^\times$).*

(3) *Suppose* Γ*-structures are rigid, and arrange that* $N \geq 3$*. Then there exists a "fine moduli space"* Y_Γ*, i.e., a universal* $(E_\Gamma \to Y_\Gamma, \alpha_\Gamma)$*. Explicitly*

$$(E_\Gamma \to Y_\Gamma, \alpha_\Gamma) = (\Gamma\backslash E_N \to \Gamma\backslash Y(N), \Gamma \circ \phi)$$

where $(E_N \to Y(N), \phi) = \coprod_{\zeta \in \mu_N^\times}(E_\zeta \to Y_\zeta(N), \phi_\zeta)$ *is universal for* $[\Gamma(N) - \mathrm{str}]$.

Recall that a coarse moduli space $\mathscr{M}$ for a functor F is a natural transformation $F \to \mathrm{Hom}(-, \mathscr{M})$ that is initial for all natural transformations out of F, and such that $F(*) \to \mathscr{M} \simeq \mathrm{Hom}(*, \mathscr{M})$ is bijective (where $*$ is the one-point complex manifold). Equivalently, it is a complex manifold $\mathscr{M}$ together with an identification $\iota : F(*) \simeq \mathscr{M}$ such that for any $\xi \in F(M)$ the the map $M \to \mathscr{M}$ sending $m \mapsto \iota((F(M) \xrightarrow{m} F(*))(\xi))$ is holomorphic. For instance if $N = 1, 2$ then the space $Y(N) = \Gamma(N)\backslash\mathfrak{h}^{(i)}$ is a coarse moduli space for $[\Gamma(N) - \mathrm{str}]$.

Example 9.4. Here are some examples of rigid Γ:

- $\{1\} \subset \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ for $N \geq 3$
- $\mathrm{G}\,\Gamma_1(N)$ for $N \geq 4$,
- $\mathrm{G}\,\Gamma_1(N) \times \mathrm{G}\,\Gamma_0(N')$ for $(N, N') = 1$ and $N \geq 4$.

The latter classifies triples (E, P, C) with P of order N and C cyclic of order N'. On the other hand $\Gamma = \mathrm{G}\,\Gamma_0(N)$ always contains $-I$ and so is never rigid.

Example 9.5. Assume $\det\Gamma = (\mathbf{Z}/N\mathbf{Z})^\times$, and let $\widetilde{\Gamma}$ be the preimage of Γ in $\mathrm{GL}_2(\mathbf{Z})$. By construction $Y_\Gamma = \widetilde{\Gamma}\backslash\mathfrak{M}$ where $\mathscr{E} \to \mathfrak{M} = \mathbf{C} - \mathbf{R} = \mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(\mathbf{R})$ is the Weierstrass family.

Let $G = \mathrm{GL}_2$, $G' = \mathrm{SL}_2$, $Z = \left\{\begin{pmatrix} t & 0 \\ 0 & t\end{pmatrix}\right\}$ the center of G (in the sense of algebraic groups). The classical action of $G(\mathbf{Z})$ on $\mathfrak{M}$ was recovered via moduli of elliptic curves, and in the handout "$\mathrm{GL}_2(\mathbf{Z})$ and modular forms" the classical $G(\mathbf{R})$-action extending the $G(\mathbf{Z})$ action is explained via "variation of $\mathbf{C}$-structure". We also prove conceptually that this is *transitive* with stabilizers $Z(\mathbf{R}) \cdot K_\infty^\circ$ where K_∞ is a maximal compact of $G(\mathbf{R})$ (i.e. conjugate to $\mathrm{O}_2(\mathbf{R})$). The same goes for the $G'(\mathbf{R})$-action on connected components $\mathfrak{M}_{\pm i}$ of $\mathfrak{M}$, with stabilizers $K'_\infty = K_\infty^\circ$ a maximal compact. (The point in the first case is that $\mathfrak{M}$ is the space of complex structures on $\mathbf{R}^2$, equivalently the space of embeddings $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$, and by the Skolem–Noether theorem all such embeddings are conjugate. Then $\mathfrak{M}$ has exactly two connected components because $\mathrm{SL}_2(\mathbf{R})$ is connected.)

The choice of a base point in $\mathfrak{M}$ corresponds to a choice of $K_\infty \subset G(\mathbf{R})$, and similarly a base point in $\mathfrak{M}_{\pm i}$ corresponds to a choice of $K'_\infty \subset G'(\mathbf{R})$. Choosing a base point in $\mathfrak{M}$ *defines*

$$\begin{array}{ccc} G(\mathbf{R})/Z(\mathbf{R})K_\infty^\circ & \xrightarrow{\sim} & \mathfrak{M} \\ \uparrow & & \uparrow \\ G'(\mathbf{R})/K'_\infty & \xrightarrow[\sim]{} & \mathfrak{M}_{\pm i} \end{array}$$

if the base point is in $\mathfrak{M}_{\pm i}$. Therefore

$$Y_\Gamma = \widetilde{\Gamma}\backslash G(\mathbf{R})/Z(\mathbf{R})K_\infty^\circ \xleftarrow{\sim} \widetilde{\Gamma}'\backslash G'_\infty/K'_\infty.$$

The classical base point $\pm i$ is irrelevant.

Remark 9.6. The $\mathbf{C}$-structure on Y_Γ comes from the $G(\mathbf{R})$-equivariant $\mathbf{C}$-structure on $\mathfrak{M}$ (which a choice of base point identifies with $G(\mathbf{R})/Z(\mathbf{R})K_\infty^\circ$). We'll later characterize this in terms of $G(\mathbf{R})$ with "extra group-theoretic structure" (a "Shimura datum").

Now let's return to the adelic interpretation, which is needed to elegantly handle the case $\det\Gamma \neq (\mathbf{Z}/N\mathbf{Z})^\times$. Let $K \subset G(\widehat{\mathbf{Z}})$ be the compact open subgroup associated to the pair (N, Γ) for $N \geq 1$, i.e. the preimage of Γ under $G(\widehat{\mathbf{Z}}) \to G(\mathbf{Z}/N\mathbf{Z})$. Via a choice of i we have

$$Y(N) = \coprod_{\mu_N^\times} \Gamma(N)\backslash\mathfrak{M}_i \tag{9.1}$$

using $(\mathbf{C}/\Lambda_\tau, \{\frac{r\tau}{N}, \frac{1}{N}\})$ as the level-N structure for $(\tau, e^{2\pi i r/N})$, i.e., identifying $(\mathbf{Z}/N\mathbf{Z})^\times \xrightarrow{\sim} \mu_N^\times$ via $r \to e^{2\pi r i/N}$.

Let's describe how the left $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-action on the left-hand side of (9.1), under which γ sends $(E,\phi) \mapsto (E, \phi \circ \gamma^t)$, goes over to the right-hand side of (9.1). The details are on HW5.

Consider the section $r \mapsto \langle r \rangle = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ to $\det : \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \to (\mathbf{Z}/N\mathbf{Z})^\times$. Then $g \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ carries (m, r) to

$$((\langle r \cdot \det(g)\rangle^{-1} g \langle r \rangle)^{\sim} \cdot m, (\det g) r\rangle$$

where the tilde denotes a lift under $\mathrm{SL}_2(\mathbf{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$, and $g \cdot m$ for $g \in \mathrm{SL}_2(\mathbf{Z})$ and $m \in \mathfrak{M}$ is such that $g(m,1) = (g \cdot m, 1)$. Then we get a $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-equivariant map

$$Y(N) = \coprod_{\mu_N^\times} \Gamma(N)\backslash \mathfrak{M}_i \xrightarrow{\sim} \mathrm{GL}_2(\mathbf{Z})\backslash(\mathfrak{M} \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}))$$

sending $(m, e^{2\pi i r/N}) \mapsto (m, \langle r \rangle^{-1})$ if on the right-hand side we use the action $g \cdot (m_0, g_0) = (m_0, g_0 g^{-1})$. (The action of $\mathrm{GL}_2(\mathbf{Z})$ on $\mathfrak{M} \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ is the diagonal action.) Thus

$$Y(N) \cong G(\mathbf{Z})\backslash(G(\mathbf{R}) \times G(\widehat{\mathbf{Z}}))/(Z(\mathbf{R})K_\infty^\circ \times K(N))$$

where $K(N) = \{g \in G(\widehat{\mathbf{Z}}) \colon g \equiv 1 \pmod N\}$. This is moreover equivariant for actions of $\Gamma \cong K/K(N)$, and so we have

$$Y_\Gamma \cong G(\mathbf{Z})\backslash(G(\mathbf{R}) \times G(\widehat{\mathbf{Z}}))/(\mathbf{Z}(R)K_\infty^\circ \times K).$$

To "adelize" we need:

Theorem 9.7 (Strong approximation for $\mathrm{SL}_2 = G'$)**.** *We have* $G'(\mathbf{A}_f) = G'(\mathbf{Q})G'(\widehat{\mathbf{Z}})$.

Reading 9.8. See the "Strong approximation" handout for a simple proof over any global field using strong approximation for $\mathbf{A}$.

Since $\mathbf{A}_f^\times = \mathbf{Q}^\times \cdot \widehat{\mathbf{Z}}^\times$, and we have a section $\langle \cdot \rangle : \mathbf{G}_m \to \mathrm{GL}_2$ to det, we deduce that $G(\mathbf{A}_f) = G(\mathbf{Q})G(\widehat{\mathbf{Z}})$, so $G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})G(\widehat{\mathbf{Z}})$ (and the last two subgroups commute with one another).

There are *two* natural ways to topologize $\mathrm{GL}_n(\mathbf{A})$: the restricted product topology (using the $\mathbf{Z}$-structure on GL_n), or the "functor of points" topology using a *closed* embedding $\mathrm{GL}_n \hookrightarrow$ (affine space). In fact these topologies coincide: see Prop 2.1 and equation (3.5.1) through Thm. 3.6 of the notes "Weil and Grothendieck approaches to adelic points" on Brian Conrad's website. Note that this is *not* the topology of $n \times n$ matrix entries (consider $n = 1$!).

This makes an open inclusion $G(\mathbf{R}) \times G(\widehat{\mathbf{Z}}) \hookrightarrow G(\mathbf{A})$, so the set-theoretic bijections

$$\begin{array}{ccc} Y_\Gamma \xrightarrow{\sim} G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^\circ \times K) & & (g_\alpha, c_\alpha) \\ \uparrow \simeq & & \uparrow \\ \coprod_{c_\alpha} \Gamma_\alpha \backslash G'(\mathbf{R})/K'_\infty & & g_\alpha \end{array}$$

are homeomorphisms, where the c_α are representatives of $G(\mathbf{Q})\backslash G(\mathbf{A}_f)/K \underset{\det}{\simeq} \widehat{\mathbf{Z}}^\times/\det(K)$ and $\Gamma_\alpha := c_\alpha K c_\alpha^{-1} \cap G'(\mathbf{Q})$. The vertical map is canonical and independent of the choice of c_α's. This is all a GL_2-version of translating between adelic and classical class field theory.

Remark 9.9.

1. Relating $\pi_0(Y_\Gamma)$ with a "class group" is a ubiquitous phenomenon for Shimura varieties. For instance full level N corresponds to $(\mathbf{Z}/N\mathbf{Z})^\times = \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$.
2. The composite inverse maps $f_\alpha : \mathfrak{M}_{\pm i} \cong G'(\mathbf{R})/K_\infty^\circ \to Y_\Gamma$ *are* the classical analytic maps onto connected components.

(3) When $N \mid N_1$ and $\mathrm{GL}_2(\mathbf{Z}/N_1\mathbf{Z}) \to \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ carries Γ_1 into Γ_0 (equivalently $K_1 \subset K$ inside $G(\widehat{\mathbf{Z}})$ the diagram

$$\begin{array}{ccc} Y_{\Gamma_1} & \xrightarrow{\sim} & G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^\circ \times K_1) \\ \downarrow & & \downarrow \\ Y_\Gamma & \xrightarrow{\sim} & G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^\circ \times K) \end{array}$$

commutes.

Note that the "adelic side" is more group theoretic and makes sense for *any* compact open subgroup $K \subset G(\mathbf{A}_f)$ (i.e. any open subgroup that conjugates into $G(\widehat{\mathbf{Z}})$) so the $\mathbf{Z}$-structure on G is irrelevant, only the $\mathbf{Q}$-structure matters. But on the left-hand side the $\mathbf{Z}$-structure on G is essential. (The analogue of adelic vs classical class field theory for $\mathbf{G}_m$ as a $\mathbf{Q}$-group vs $\mathbf{Z}$-group. Consider $\mathbf{G}_m(\mathbf{A})$ vs $\mathbf{G}_m(\mathbf{Z}/N\mathbf{Z})$.)

So then, for a general compact open subgroup $K \subset G(\mathbf{A}_f)$ does the Riemann surface

$$G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^\circ \times K) \simeq \coprod_{\text{finite}} \Gamma_\alpha \backslash G'(R)/K'_\infty$$

have a moduli theoretic meaning? The answer is yes, via the following idea due to Deligne.

We consider the *isogeny category of elliptic curves*: for $E = V/\Lambda$ and $E' = V'/\Lambda'$ over M, note that $\mathrm{Hom}_M(E', E)$ is the set of maps $f \in \mathrm{Hom}_M(\Lambda', \Lambda)$ such that there is a diagram

$$\begin{array}{ccc} \Lambda' \otimes \mathbf{C} & \xrightarrow{f_{\mathbf{C}}} & \Lambda \otimes \mathbf{C} \\ \downarrow & & \downarrow \\ V' & \overset{\exists}{\dashrightarrow} & V \end{array}$$

Note that $f_{\mathbf{C}}$ depends on f only through $f_{\mathbf{Q}}$. We define the *isogeny category* to have as objects $(E \to M)$, with morphisms $\mathrm{Hom}^\circ_M(E', E)$ the set of maps $h \in \mathrm{Hom}_M(\Lambda'_{\mathbf{Q}}, \Lambda_{\mathbf{Q}})$ such that there is a diagram

$$\begin{array}{ccc} \Lambda'_{\mathbf{C}} & \xrightarrow{h_{\mathbf{C}}} & \Lambda_{\mathbf{C}} \\ \downarrow & & \downarrow \\ V' & \overset{\exists}{\dashrightarrow} & V. \end{array}$$

Let $E_{\mathbf{Q}}$ denote "E viewed in the isogeny category".

Definition 9.10. Let $T_f(E)$ be the local system $\varprojlim_N E[N] = \prod_\ell \varprojlim_n E[\ell^n]$ of rank two $\widehat{\mathbf{Z}}$-modules, and $V_f(E)$ the local system $\mathbf{Q} \otimes_{\mathbf{Z}} T_f(E) = \mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} T_f(E)$. Observe that $V_f(E)$ can be viewed as a functor of $E_{\mathbf{Q}}$.

Lemma 9.11. *Consider the following categories:*

- $\mathscr{E}ll_M$, *elliptic curves over* M,
- $\mathscr{E}ll^\circ_M$, *the isogeny category over* M.
- $\mathscr{E}ll'_M$, *the category of triples* (E', L, ϕ) *with* $E' \in \mathscr{E}ll^\circ_M$, L *a local system of rank two* $\widehat{\mathbf{Z}}$*=modules, and* $\phi : \mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} L \xrightarrow{\sim} V_f(E')$ *an* $\mathbf{A}_f$*-linear map (an "adelic* H_1*-trivialization for* L*").*

Then the functor $\mathscr{E}ll_M \to \mathscr{E}ll'_M$ *sending*

$$E \mapsto (E_{\mathbf{Q}}, T_f(E), \mathrm{can} : \mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} T_f(E) \xrightarrow{\sim} V_f(E_{\mathbf{Q}}))$$

is an equivalence of categories, respecting base change.

Reading 9.12. Start by thinking about what this says for M a point.... See the handout on K-structures for the details.

For any $E' \in \mathscr{E}ll^\circ_M$ we have a sheaf $I_{E'/M} : U \mapsto \mathscr{I}som_I(\mathbf{A}^2_{f,M} \xrightarrow{\sim} V_f(E'))$, carrying an action of $\mathrm{GL}_2(\mathbf{A}_f)$.

Definition 9.13. A K*-structure* on E'/M is $\alpha \in H^0(M, K\backslash I_{E'/M})$, with the evident notions of isomorphism and base change.

Concretely there exists $g \in \mathrm{GL}_2(\mathbf{A}_f)$ such that $gKg^{-1} \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ (see HW5). Since $\mathrm{GL}_2(\mathbf{A}_f) = \mathrm{GL}_2(\widehat{\mathbf{Z}})G(\mathbf{Q})$, without loss of generality we can take $g \in G(\mathbf{Q})$, even in $\mathrm{Mat}_2(\mathbf{Z})$. Then by Lemma 9.11 the diagram

$$\begin{array}{ccc} \mathbf{A}^2_{f,M} & \xrightarrow[\phi\circ g^t]{\sim} & V_f(E') \\ \uparrow & & \uparrow \\ \widehat{\mathbf{Z}}^2_M & \xrightarrow{\sim} & T_f(E) \end{array}$$

pins down a *specific* elliptic curve E/M, and $\phi \circ K^t$ determines $(\phi \circ g^t) \circ (gKg^{-1})^t$ a "classical" gKg^{-1}-structure on E. See the handout on K-structures, especially Proposition 3.4.

Example 9.14. The special case $g = 1$ says that if $K \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ then a K-structure α on $E' \in \mathscr{E}ll^\circ_M$ specifies a *particular* local system $L \subset V_f(E')$ of finite free rank-2 $\widehat{\mathbf{Z}}$-modules such that $\mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} L \xrightarrow{\sim} V_f(E')$, and the associated E/M is equipped (via α) with a "classical" K-structure.

This proves the following (see also Cor. 3.5 in the handout).

Proposition 9.15.

(1) *K-structures are rigid if and only if $K \cap G'(\mathbf{Q})$ is torsion-free. (If $K \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$, note that $K \cap G'(\mathbf{Q}) = K \cap \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$.)*

(2) *The functor F_K of isomorphism classes of K-structures on $\mathscr{E}ll^\circ_M$ admits a coarse moduli space Y_K of pure dimension 1, with $\pi_0(Y_K)$ a torsor for $\mathbf{Q}^\times \backslash \mathbf{A}_f^\times / \det(K) = \widehat{\mathbf{Z}}^\times / \det(K)$, and in the rigid case this is a fine moduli space.*

Indeed, upon choosing $i = \sqrt{-1} \in \mathbf{C}$ and a base point $\tau_0 \in \mathfrak{h}_i$ (equivalently a maximal compact $K^\circ_\infty \subset \mathrm{SL}_2(\mathbf{R})$) we get an explicit presentation

$$Y_K = G(\mathbf{Q}) \backslash G(\mathbf{A}) / (\mathbf{Z}(\mathbf{R}) K^\circ_\infty \times K)$$

in which the pair $g = (g_\infty, g_f) \in G(\mathbf{A})$ is sent to the elliptic curve $\mathbf{C}/\Lambda_\tau$ with $\tau = [g_\infty](\tau_0)$ together with the map $\alpha_g : \mathbf{A}^2_f \xrightarrow{\sim} \mathbf{A}_f \otimes_{\mathbf{Z}} \Lambda_\tau$ given by $(g_f^{-1})^t$ with respect to the basis $\{\tau, 1\}$ on the right.

(For $K \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$, what's the associated elliptic curve with "classical" K-structure? You have to replace the lattice Λ_τ with the lattice $(\Lambda_\tau)_{\mathbf{Q}} \cap (g_f^{-1})^t(\widehat{\mathbf{Z}})^2$... ugh!)

Example 9.16. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{Q})$, why does γg give the same pair up to isomorphism in this $\mathscr{E}ll^\circ_M$ picture? The key is that even though perhaps $\gamma \notin \mathrm{GL}_2(\mathbf{Z})$, we nevertheless have an isomorphism *in the category* $\mathscr{E}ll^\circ_M$:

$$\begin{array}{ccc} \mathbf{C}/\Lambda_\tau & \xrightarrow{\sim} & \mathbf{C}/\Lambda_{[\gamma](\tau)} \\ z & \longmapsto & \frac{z}{c\tau+d} \\ (\Lambda_\tau)_{\mathbf{Q}} & \xrightarrow[(\gamma^t)^{-1}]{\sim} & (\Lambda_{[\gamma](\tau)})_{\mathbf{Q}} \end{array}$$

carrying α_g to $\alpha_{\gamma g}$. This is "fake" in both directions when $\gamma, \gamma^{-1} \notin \mathrm{Mat}_2(\mathbf{Z})$ so that neither Λ_τ nor $\Lambda_{[\gamma](\tau)}$ is carried into the other. See Remark 4.3 in the K-structures handout for details.

10. Hecke operators

The entire collection $\{F_K\}$ or $\{Y_K\}$ admits a very rich structure of $\mathrm{GL}_2(\mathbf{A}_f)$-actions! Consider K, K' and $g \in \mathrm{GL}_2(\mathbf{A}_f)$ such that $gK'g^{-1} \subset K$. For instance:

(i) $g = 1$, $K' \subset K$,

(ii) any $g \in \mathrm{GL}_2(\mathbf{A}_f)$ and K, with $K' := K \cap g^{-1}Kg$ so that $K', gK'g^{-1}$ are both in K. (Take $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $K = K_0(N)$ or $K_1(N)$ for classical Hecke operators.)

Then we have $J_{K',K}(g) : F_{K'} \twoheadrightarrow F_K$ sending

$$(E', \phi' : \mathbf{A}_f^2 \xrightarrow{\sim} V_f(E')) \bmod K' \mapsto (E', \phi \circ g^t) \bmod K.$$

Then Corollary 4.5 of the K-structures handout shows that the associated map

$$\begin{array}{ccc} Y_{K'} & \longrightarrow & Y_K \\ G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^\circ \times K') & \longrightarrow & G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^\circ \times K) \\ x & \longmapsto & x(1, g^{-1}) \end{array}$$

is a *finite* branched covering.

For $K, K' \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ and $g \in \mathrm{GL}_2(\widehat{\mathbf{Z}})$ this is easily expressed via classical level structures (using $K(N) \subset K \cap K'$ for some N) and the classical action of $g \bmod N \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ on full level-N structures. But if $g \in \mathrm{Mat}_2(\widehat{\mathbf{Z}})$ with $\det(g) \in \mathbf{A}_f^\times, \notin \widehat{\mathbf{Z}}^\times$, we have to "change" the elliptic curve to make the K-structure become classical.

For instance for $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ we have $(g^t)^{-1}(\widehat{\mathbf{Z}}^2) \cap \mathbf{Q}^2 = \{(a, \frac{b}{p}) : a, b \in \mathbf{Z}\}$ so $\mathbf{C}/\Lambda_\tau$ is replaced with $\mathbf{C}/\langle \tau, \frac{1}{p} \rangle \simeq \mathbf{C}/\Lambda_{p\tau}$. Then we get a pair of maps

$$\begin{array}{ccccc} & & Y_{K\cap g^{-1}Kg} & & \\ & \swarrow^{J_{K',K}(1)} & & \searrow^{J_{K',K}(g)} & \\ Y_K & & & & Y_K. \end{array}$$

Elementary general properties:

1. (1) $J_{K',K}(g)J_{K'',K'}(h) = J_{K'',K}(gh)$.
2. (2) $J_{K,K}(g) = 1$ for $g \in K$.
3. (3) If $K' \lhd K$ then $J_{K',K'}(g)$ for $g \in K/K'$ yields a K/K'-action on $Y_{K'}$ with $\mathrm{can} : Y_{K'} \to Y_K$ as quotient.
4. (4) Each $J_{K',K}(g)$ factors into building blocks

$$J_{K',K}(g) : Y_{K'} \xrightarrow[J_{K',gK'g^{-1}}(g)]{\sim} Y_{gK'g^{-1}} \xrightarrow[J_{gKg^{-1},K}(1)]{} Y_K$$

where the first part is the "mystery" part and the second part is forgetful.

Why care? Consider

$$\widetilde{H}^1(Y_K, \mathbf{C}) = \mathrm{im}(H_c^1(Y_K, \mathbf{C}) \to H^1(Y_K, \mathbf{C})) \xleftarrow{\sim} H^1(X_K, \mathbf{C}).$$

(This is $S_2(\Gamma_K, \mathbf{C})$ when $\det(K) = \widehat{\mathbf{Z}}^\times$.) Consider $V := \varinjlim_K \widetilde{H}^1(Y_K, \mathbf{C})$, the limit taken with respect to inclusion, so that the $K \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ are cofinal and the Y_K's are highly disconnected. Then V carries an action of $\mathrm{GL}_2(\mathbf{A}_f)$ via the $J_{K',K}(g)$'s (or just $J_{K,gKg^{-1}}(g)$'s and inclusions to the limit) with $\widetilde{H}^1(Y_K, \mathbf{C}) \xrightarrow{\sim} V^K$. When you build the *algebraic* theory one can do the "same" thing: take $\varinjlim \widetilde{H}^1_{\text{ét}}((Y_K)_{\overline{\mathbf{Q}}}, \overline{\mathbf{Q}}_\ell)$ with an action of $\mathrm{GL}_2(\mathbf{A}_f)$, recovering the $\overline{\mathbf{Q}}_\ell$-version of the above $\mathbf{C}$-construction. But in the algebraic case we have a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-action commuting with the $\mathrm{GL}_2(\mathbf{A}_f)$-action, and this underlies the dictionary relating modular forms, Galois representations, and automorphic representations.

11. Classical modular forms

Let us now explain the correspondence between:

- classical modular forms,
- sections of holomorphic line bundles built from universal elliptic curves over modular curves, and
- via GAGA, the algebro-geometric theory of modular forms,

each with their Hecke actions. We'll apply the following fact to universal elliptic curves.

Recall that for an elliptic curve $f : E \to M$ the sheaf $\Omega^1_{E/M}$ is defined to be the quotient $\Omega^1_E/f^*\Omega^1_M$. On the stalk at $x \in E$ the natural map $f^*(\Omega^1_M) \to \Omega^1_E$ is precisely the dual of the map $T_x(E) \twoheadrightarrow T_{f(x)}(M)$ (surjective because f is a submersion); see HW2. Thus the stalk $\Omega^1_{E/M,x}$ is dual to $\ker(T_x(E) \to T_{f(x)}(M))$.

Fact 11.1 (Prop 4.12 in the handout "$\mathrm{GL}_2(\mathbf{Z})$-action..."). For $f : E = V/\Lambda \to M$, we have an isomorphism of line bundles $\omega_{E/M} := f_*\Omega^1_{E/M} \simeq V^\vee$ that

1. commutes with base change on M, and
2. on fibers at $m \in M$ recovers the classical $H^0(E_m, \Omega^1_{E_m}) \xrightarrow{\sim} \mathrm{Cot}_0(E_m)$.

The main point is that there is a natural map $f_*\Omega^1_{E/M} \to e^*\Omega^1_{E/M}$ which on fibers is the classical isomorphism of (2), and so is an isomorphism. It remains to identify $e^*\Omega^1_{E/M}$ with $V^\vee$. This follows from the fact that $e^*\Omega^1_{E/M}$ and $0^*\Omega^1_{V/M}$ can be identified, together with the general fact that $V^\vee \simeq 0^*(\Omega^1_{V/M})$ for any vector bundle $V \to M$.

Example 11.2. The line bundle $\omega_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$ has global basis "dz" where z is a coordinate on $\mathscr{E}_\tau = \mathbf{C}/\Lambda_\tau$. Indeed, $\mathscr{E} = V/\Lambda$ with $\Lambda = \underline{\mathbf{Z}}e_1 \oplus \underline{\mathbf{Z}}e_2$, and then the identification $\omega_{\mathscr{E}/(\mathbf{C}-\mathbf{R})} \simeq V^\vee$ sends dz to e_2^*, the dual frame to e_2. (Try visualizing this above $\tau \in \mathbf{C} - \mathbf{R}$.)

To exploit this, we combine it with the functoriality of $\omega_{E/M}$. From a commutative diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ M' & \xrightarrow[h]{} & M \end{array}$$

we get a map $\omega_{E/M} \to h_*\omega_{E'/M'}$ via $\Gamma(E_U, \Omega^1_{E/M}) \to \Gamma(E'_{h^{-1}(U)}, \Omega^1_{E'/M'})$.

Example 11.3. We have the diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{[\gamma]_{\mathscr{E}}} & \mathscr{E} \\ \downarrow & & \downarrow \\ \mathbf{C}-\mathbf{R} & \xrightarrow[{[\gamma]}]{} & \mathbf{C}-R \end{array}$$

in which $[\gamma]_{\mathscr{E}}$ is the map $\mathbf{C}/\Lambda_\tau \to \mathbf{C}/\Lambda_{[\gamma]\tau}$ given by multiplication by $1/(c\tau+d)$, and corresponding map on ω's sends dz to $\frac{dz}{c\tau+d}$.

The upshot is that for $\omega = \omega_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$ on $\mathbf{C}-\mathbf{R}$ and $k \geq 1$ we get a map $\omega^{\otimes k} \to [\gamma]_*(\omega^{\otimes k})$ sending

$$f \cdot (dz)^{\otimes k} \text{ on } U \mapsto (f \circ [\gamma])(cz+d)^{-k}(dz)^{\otimes k} = (f|_k\gamma)(\tau)(dz)^{\otimes k} \text{ on } [\gamma]^{-1}(U).$$

Example 11.4. Suppose $\det(\Gamma) = (\mathbf{Z}/N\mathbf{Z})^\times$ and that Γ-structures are rigid. Then we get a quotient

$$\begin{array}{ccc} \mathscr{E}^{(i)} & \longrightarrow & E_\Gamma \\ \downarrow & & \downarrow \\ \mathfrak{h}_i & \longrightarrow & Y_\Gamma \end{array}$$

for the properly discontinous action of $\widetilde{\Gamma}' = \widetilde{\Gamma} \cap \mathrm{SL}_2(\mathbf{Z})$. The sheaf $\omega_{\Gamma'} = \omega_{E_\Gamma/Y_\Gamma}$ is the $\widetilde{\Gamma}'$-descent of $\omega_{\mathscr{E}^{(i)}/\mathfrak{h}_i}$, and is usually *not* globally free. This is related to E_Γ not having a Weierstrass model in $\mathbf{CP}^2 \times Y_\Gamma$. Thus

$$H^0(Y_\Gamma, \omega_\Gamma^{\otimes k}) \cong \{f : \mathfrak{h}_i \xrightarrow{\text{hol}} \mathbf{C} \colon f|_h\gamma = f \text{ for all } \gamma \in \widetilde{\Gamma}'\}$$

is HUGE. For instance if $\Gamma = \mathrm{G}\,\Gamma_1(N)$, $N \geq 4$ then $\widetilde{\Gamma}' = \Gamma_1(N) \subset \mathrm{SL}_2(\mathbf{Z})$.

Fact 11.5. For $\Gamma = \Gamma_1(N)$, $N \geq 5$, the algebraic theory provides

$$\begin{array}{ccc} E_\Gamma & \hookrightarrow & \overline{E}_\Gamma \\ \downarrow & & \downarrow \\ Y_\Gamma & \hookrightarrow & X_\Gamma \end{array}$$

with X_Γ compact and $\overline{E}_\Gamma$ the universal "generalized elliptic curve". This yields a canonical "relative dualizing sheaf" $\overline{\omega}_\Gamma$ on X_Γ extending ω_Γ such that $H^0(X_\Gamma, \overline{\omega}_\Gamma^{\otimes k}) \hookrightarrow H^0(Y_\Gamma, \omega_\Gamma^{\otimes k})$ goes onto $M_k(\Gamma, \mathbf{C})$. The key input will be an algebraic theory of q-expansions that agrees with the analytic one, and detects membership in $\overline{\omega}_\Gamma^{\otimes k}$ near cusps.

Hecke aspects. Take $\Gamma = \Gamma_1(N)$, $N \geq 5$. If $a \in (\mathbf{Z}/N\mathbf{Z})^\times$, then the assignment $(E, P) \mapsto (E, aP)$ gives a diagram

$$\begin{array}{ccc} E_\Gamma & \xrightarrow{\langle a \rangle_E} & E_\Gamma \\ {\scriptstyle aP}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle P} \\ Y_\Gamma & \xrightarrow[\langle a \rangle_Y]{} & Y_\Gamma. \end{array}$$

The map $\langle a \rangle_Y$ is made explicit on HW6, which shows that it is exactly the usual diamond operator $\langle r \rangle$ on M_k.

Example 11.6. If $a = -1$ then $\langle a \rangle_Y = \mathrm{id}$ (why?) yet $\langle a \rangle_E = -1$, so $\langle a \rangle = (-1)^k$ on M_k! (Look on $\omega^{\otimes k}$.)

Now fix a prime p and write

$$Y := Y_1(N, p) = \{(E, P, C) \colon \langle P \rangle \cap C = 0\}$$

where P has order N, C is cyclic of order p, and the last condition is automatic unless $p \mid N$. We then have a diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ Y_\Gamma & & Y_\Gamma \end{array}$$

coming from the functors $(E, P, C) \mapsto (E, P)$ and $(E, P, C) \mapsto (E/C, P \bmod C)$ respectively. This extends to a diagram

$$\begin{array}{ccccccc} & & \mathscr{E} & \xrightarrow[\text{isog}]{\varphi} & \mathscr{E}/\mathcal{C} & & \\ & \swarrow & & \searrow\ \swarrow & & \searrow & \\ E_\Gamma & & & Y & & & E_\Gamma \\ & \searrow & {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} & \swarrow & \\ & & Y_\Gamma & & Y_\Gamma & & \end{array}$$

which yields

$$\omega_\Gamma^{\otimes k} \to \pi_{2*}(\omega_{(\mathscr{E}/\mathcal{C})/Y}^{\otimes k}) \xrightarrow{\varphi^*} \pi_{2*}(\omega_{\mathscr{E}/Y}^{\otimes k}) = \pi_{2*}(\pi_1^* \omega_\Gamma^{\otimes k}).$$

Passing to global sections we get

$$H^0(Y_\Gamma, \omega_\Gamma^{\otimes k}) \to H^0(Y, \pi_1^* \omega_\Gamma^{\otimes k}) = H^0(Y_\Gamma, \pi_{1*}\pi_1^* \omega_\Gamma^{\otimes k}).$$

But $\pi_{1*}\pi_1^*\omega_\Gamma^{\otimes k} \simeq \omega_\Gamma^{\otimes k} \otimes_{\mathscr{O}_{Y_\Gamma}} \pi_{1*}\mathscr{O}_Y$. Since $\pi_{1*}\mathscr{O}_Y$ is finite locally free over $\mathscr{O}_{Y_\Gamma}$ there is a trace map to $\mathscr{O}_{Y_\Gamma}$, and so composing with $1 \otimes \mathrm{Tr}$ we finally get a map $H^0(Y_\Gamma, \omega_\Gamma^{\otimes k}) \to H^0(Y_\Gamma, \omega_\Gamma^{\otimes k})$. Using $\mathbf{C}/\Lambda_\tau$-models and writing $f(\tau)(dz)^{\otimes k}$, this is exactly the classical pT_p without any cusp condition, so it induces pT_p on $H^0(X_\Gamma, \overline{\omega}^{\otimes k}) = M_k$. (Recall that the classical T_p involves a factor of p^{k-1}!)

Cusp forms. Continue to assume that $\Gamma \subset \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ is rigid and that $\det \Gamma = (\mathbf{Z}/N\mathbf{Z})^\times$, and take $k \geq 2$. We want $S_k(\widetilde{\Gamma}', \mathbf{C})$ to correspond to $\widetilde{H}^1(Y_\Gamma, ?_k)$ with coefficients in some local system depending on k. Write $f_\Gamma : E_\Gamma \to Y_\Gamma = \widetilde{\Gamma}' \backslash \mathfrak{h}_i$, so that $\pi_1(Y_\Gamma) = \widetilde{\Gamma}'$. Define

$$\mathscr{F}_{k-2} = \mathrm{Sym}^{k-2}(\mathrm{R}^1 f_{\Gamma *} \underline{\mathbf{Z}}) = \mathrm{Sym}^{k-2}(\mathbf{Z}e_1^* \oplus \mathbf{Z}e_2^*)$$

which intuitively you can think of as "homogeneous polynomials in e_1^* and e_2^* of degree $k-2$". Extending coefficients to $\mathbf{C}$ we have $\mathscr{F}_{k-2,\mathbf{C}}$.

Then

$$\begin{array}{ccc} H^1(Y_\Gamma, \mathscr{F}_{k-2,\mathbf{C}}) & \xrightarrow[Y_\Gamma = \widetilde{\Gamma}' \backslash \mathfrak{h}_i]{\sim} & H^1(\widetilde{\Gamma}', \mathrm{Sym}^{k-2}(\mathbf{C}e_1^* \oplus \mathbf{C}e_2^*)) \\ \uparrow & & \uparrow \\ \widetilde{H}^1(Y_\Gamma, \mathscr{F}_{k-2,\mathbf{C}}) & \xrightarrow{\sim} & H^1_{\mathrm{par}}(\widetilde{\Gamma}', \mathrm{Sym}^{k-2}(\mathbf{C}e_1^* \oplus \mathbf{C}e_2^*)) \end{array}$$

in which the action of $\widetilde{\Gamma}'$ in the upper-right is the standard representation (the transposes cancel out — in fact there is self-duality coming from that on $\mathrm{R}^1 f_{\Gamma *}\mathbf{Z}$!) and the parabolic condition in the lower-right is a group-theoretic condition relative to unipotent subgroups stabilizing cusps.

We will make a map

$$H^0(Y_\Gamma, \omega_\Gamma^{\otimes k}) \to H^1(Y_\Gamma, \mathscr{F}_{k-2,\mathbf{C}}) \tag{11.1}$$

which concretely takes

$$f \mapsto \left(\gamma \mapsto \int_{\tau_0}^{[\gamma](\tau_0)} (2\pi i)^{k+1} (\tau e_1^* + e_2^*)^{k-2} f(\tau) d\tau \right).$$

Once this is done, a local analysis near the cusps shows that it carries

$$S_k(\widetilde{\Gamma}', \mathbf{C}) \to \widetilde{H}^1(Y_\Gamma, \mathscr{F}_{k-2,\mathbf{C}})$$

and comparing the Petersson inner product on the left with the topological cup product on the right shows that

$$\mathrm{ES}_\Gamma : \mathbf{C} \otimes_{\mathbf{R}} S_k(\widetilde{\Gamma}', \mathbf{C}) \to \widetilde{H}^1(Y_\Gamma, \mathscr{F}_{k-2,\mathbf{C}})$$

is injective. One can compute that the dimensions on both sides of this map are equal (use $k \geq 2$!) and so the map ES_Γ is an isomorphism. The map (11.1) comes from the following rather general fact.

Proposition 11.7. *For* any $f : E \to M$, *with* $\mathscr{F} := \mathrm{R}^1 f_* \mathbf{C}$ *and* $k \geq 2$, *there exists a natural map*

$$H^0(M, \omega_{E/M}^{\otimes k}) \to H^1(M, \mathrm{Sym}^{k-2} \mathscr{F}).$$

Proof. On HW5 we see that there is a Kodaira–Spencer map $\mathrm{KS}'_{E/M} : \omega_{E/M}^{\otimes 2} \to \Omega_M^1$ functorially in $E \to M$.

Concretely, for an M-curve $f : X \to M$ we dualize $0 \to f^*\Omega_M^1 \to \Omega_X^1 \to \Omega_{X/M}^1 \to 0$ to get $0 \to T_{X/M} \to T(X) \to f^*T(M) \to 0$ and then $\mathrm{KS}_{X/M}$ is the first connecting map for the long exact sequence associated to f_*. That is, $\mathrm{KS}_{X/M} : f_* f^* T_M \to \mathrm{R}^1 f_* T_{X/M}$. The source is just $T(M)$ again, and the target is naturally dual to $f_*((\Omega_{X/M}^1)^{\otimes 2})$, so dualizing gives $\mathrm{KS}'_{X/M} : \omega_{X/M}^{\otimes 2} \to \Omega_M^1$.

On the universal Weierstrass model the map $\mathrm{KS}'_{\mathscr{E}/(\mathbf{C}-\mathbf{R})}$ is an isomorphism taking $(2\pi i_\tau dz)^{\otimes 2} \mapsto 2\pi i_\tau d\tau$. For $E_\Gamma \to Y_\Gamma$ it is the $\widetilde{\Gamma}'$-equivariant descent of this, and is still an isomorphism. This gives

$$H^0(M, \omega_{E/M}^{\otimes k}) \to H^0(M, \omega_{E/M}^{\otimes (k-2)} \otimes \Omega_M^1).$$

On HW6 you build a map $\omega_{E/M} = f_* \Omega_{E.M}^1 \to \mathscr{O}_M \otimes_{\underline{\mathbf{C}}} \mathscr{F}$ inducing the canonical map $H^0(\Omega_{E_m}^1) \to H^1(E_m, \mathbf{C})$ on fibers. Applying Sym^{k-2} (which is nothing other than $\otimes^{k-2}$ on a line bundle!) gives $\omega_{E/M}^{\otimes(k-2)} \to \mathscr{O}_M \otimes_{\underline{\mathbf{C}}} \mathrm{Sym}^{k-2} \mathscr{F}$, and so we get

$$H^0(M, \omega_{E/M}^{\otimes k}) \to H^0(M, \omega_{E/M}^{\otimes (k-2)} \otimes \Omega_M^1) \xrightarrow{\mathrm{dR}} H^1(M, \mathrm{Sym}^{k-2} \mathscr{F}).$$

∎

Remark 11.8. Using pullback/trace on $\widetilde{H}^1(Y_\Gamma, \mathscr{F}_{k-2})$ (over $\mathbf{Z}$) makes ES_Γ Hecke-compatible without needing $\frac{1}{p}$ stuff (see HW6). Thus the Hoecke eigenvalues are algebraic integers lying in a single number field.

Remark 11.9. We can make $\widetilde{H}^1(Y_\Gamma, \mathscr{F}_{k-2,\overline{\mathbf{Q}}_\ell})$ using étale cohomology on modular curves over $\mathbf{Z}[\frac{1}{N}]$, and then the algebraic Hecke operators are $G_{\mathbf{Q}}$-compatible and cut out 2-dimensional $G_{\mathbf{Q}}$-representations that are unramified at primes $p \nmid N\ell$. The chain of changes-of-coefficients $\mathbf{C} \leftarrow \overline{\mathbf{Q}} \leftarrow \mathbf{Q} \leftarrow \mathbf{Z}[\frac{1}{N}] \rightarrow \mathbf{F}_p \rightarrow \overline{\mathbf{F}}_p$ and comparison between $\widetilde{H}^1_{\mathrm{an}}$ and $\widetilde{H}^1_{\text{ét}}$ (Artin) shows that the action of T_p can be understood using mod p geometry of the Hecke correspondence diagram in characteristic p (see Katz–Mazur, Kronecker). This shows that the T_p eigenvalue is the sum of two Frob_p-eigenvalues, and this is instrumental in showing that Weil II implies Ramanjuan.

12. Introduction to the algebraic theory

Now we begin the algebraic theory of elliptic curves. Idea: as in the analytic case we wish to consider proper "submersions" $f: E \to S$ with a section e and fibers E_s that are geometrically connected of dimension 1 and genus 1. But how to make sense of "submersion"?

The naive approach would be to require $f: E \to S$ Zariski-locally on S to be given (over an open U) by a Weierstrass equation in $\mathbf{P}^2_U$ such that the discriminant lies in $\mathscr{O}(U)^\times$. But this is bad for several reasons: besides being too extrinsic, it's also bad for quotients and for making constructions via descent. (Recall that we constructed $E_\Gamma \to Y_\Gamma$ by a quotient process, albeit a very non-algebraic one.

The dictionary between the analytic and algebraic approaches will be as follows.

$\mathbf{C}$-analyic geometry	Algebraic geometry
submersion	smooth morphism
local analytic isomorphism	étale morphism
finite analytic covering	finite étale surjection

For instance the algebraic analogue of the local analytic isomorphism $z \mapsto z^n$ on $\mathbf{C}\setminus\{0\}$ is the étale morphism $z \mapsto z^n$ on $\mathbf{A}^1 \setminus \{0\}$ over a $\mathbf{Z}[\frac{1}{n}]$-algebra, and the algebraic analogue of the finite analytic covering $E[N] \to M$ is $E[N] \to S$ for S a $\mathbf{Z}[\frac{1}{n}]$-scheme.

We will need to discuss the above notions, but before we do, let's take for granted that they are reasonable and discuss what to do with them.

Definition 12.1. A *genus-g S-curve* is a smooth and proper map $f: X \to S$ such that all geometric fibers $X_{\overline{s}}$ are connected curves of genus g.

Example 12.2. Suppose $F \in \Gamma(\mathbf{P}^2_S, \mathscr{O}(d)) = \mathscr{O}(S)[U_0, U_1, U_2]_d$, i.e. F is a homogeneous polynomial of degree d in the variables U_i, such that $F_s \in k(s)[U_0, U_1, U_2]$ is nonzero for all s. Then $Z(F) \subset \mathbf{P}^2_S$ is an S-curve of genus $g = (d-1)(d-2)/2$ if and only if $Z(F)$ is disjoint from $Z(\frac{\partial f}{\partial U_0}, \frac{\partial f}{\partial U_1}, \frac{\partial f}{\partial U_2})$. If d is a unit on S, this is the same as saying $\mathrm{disc}(F) \in \Gamma(S, \mathscr{O}_S^\times)$.

Example 12.3. Over a field k, a genus-1 curve X (which might have $X(k) = \varnothing$...) may not occur in $\mathbf{P}^2_k$!

Definition 12.4. An *elliptic curve over S* is a proper smooth map $f: E \to S$ along with a section $e \in E(S)$ such that all geometric fibers $E_{\overline{s}}$ are connected of dimension 1 and genus 1.

What basic results do we want?

(1) Zariski locally on S, there should exist a Weierstrass model.
(2) There should exist a unique S-group scheme structure with identity e, and any $(E, e) \to (E', e')$ should automatically be an S-group map. (This forces commutativity, since $g \mapsto g^{-1}$ is an S-group self-map of (E, e).) The construction will use Grothendieck's work on $\mathrm{Pic}^0_{X/S,e}$, akin to the $\mathbf{C}$-analytic case.
(3) For all $N \neq 0$, the map $[N]: E \to E$ should be finite locally free of rank N^2. When N is a unit on S, this is étale, and there should exist an étale surjection $S' \to S$ such that $E[N]_{S'} \simeq (\mathbf{Z}/N\mathbf{Z})^2 \times S'$ as S'-group schemes.

Then we can contemplate level structures, moduli, and so forth.

Remark 12.5.

(i) Working over étale covers and descending back down replaces working "locally for the analytic topology".
(ii) We'll largely focus on the case where N is a unit on S, so making $Y_1(N)$ over $\mathbf{Z}[\frac{1}{N}]$. We'll go a tiny bit beyond, to get $Y_1(N,p)$ over $\mathbf{Z}[\frac{1}{N}]$ rather than $\mathbf{Z}[\frac{1}{Np}]$, which is needed to relate T_p with Frob_p on the Galois representation. Really working over $\mathbf{Z}$ needs "Drinfeld structures" — see the Katz–Mazur book.
(iii) How to handle cusps algebraically? Via moduli, we need to allow certain degenerations of elliptic curves (akin to $\overline{\mathscr{M}}_g$ for $g > 2$). A "good" degeneration will be something like $y^2 = x(x+1)(x-\lambda)$ degenerating to $y^2 = x^2(x+1)$ as $\lambda \to 0$, and a "bad" degeneration will be something like $y^2 = x(x+\lambda)(x-\lambda)$ degenerating to $y^2 = x^3$ as $\lambda \to 0$.

13. Smooth and étale maps

We will assume that all schemes are locally noetherian (one needs "finite presentation" stuff in general). References: chapter 2 of "Néron models" by Bosch–Lutkebohmert–Raynaud , and Vakil's algebraic geometry notes.

Remark 13.1. The key to all nontrivial content in these foundations is Zariski's Main Theorem, one form of which is as follows: if Y is a quasi-compact separated scheme and $f : X \to Y$ is a separated, quasi-finite, finitely presented morphism then there is a factorization $f : X \to Z \to Y$ where the first map is an open immersion and the second one is finite.

Definition 13.2. A map $f : X \to S$ is *étale* if any of the following (non-trivially) equivalent conditions hold:

(1) (Structure theorem.) Locally f is of the form $U \to \operatorname{Spec} R$ where U is an open subscheme of $\operatorname{Spec}(R[T]/g)[\frac{1}{g'}]$.
(2) The map f is locally of finite type and flat (hence open!) and for all $s \in S$ we have $X_s = \coprod_i \operatorname{Spec} k_i(s)$ with $k_i(s)/k(s)$ finite separable.
(3) The map f is locally of finite type and flat with $\Omega^1_{X/S} = 0$.
(4) The map f is locally of finite type and functorially étale: for any closed immersion

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A \\ & \searrow & \downarrow \\ & & S \end{array}$$

along an ideal $I \subset A$ such that $I^2 = 0$, we have $X(A) \xrightarrow{\sim} X(A_0)$, i.e., in the following diagram the dotted arrow exists and is unique:

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A \\ \downarrow & \swarrow & \downarrow \\ X & \xrightarrow{f} & S. \end{array}$$

(This is something like an inverse function theorem, involving recursively solving n equations in n unknowns with invertible linear part.)
(4′) Same as (4) but A is artin local, even with an algebraically closed residue field.

The hard part of the equivalence among these definitions is that (4) implies (1), because no flatness is mentioned in (4).

Facts 13.3.

- In a diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

where both maps to S are étale, the map f is also étale. (Use (4)!) Recall that local analytic isomorphisms behaved similarly.
- If S is locally of finite type over an algebraically closed field k, then $X \to S$ is étale if and only if $\mathscr{O}^{\wedge}_{S,f(x)} \xrightarrow{\sim} \mathscr{O}^{\wedge}_{X,x}$ for all $x \in X(k)$.
- If X, S are locally of finite type over $\mathbf{C}$ and smooth, then $X \xrightarrow{f} S$ is étale if and only if the analytification f^{an} is a local analytic isomorphism.

Example 13.4. X open in $\operatorname{Spec}(R[t_1, \ldots, t_n]/(f_1, \ldots, f_n)) \left[\frac{1}{\det\left(\frac{\partial f_i}{\partial t_j}\right)}\right]$. This is good for (4), bad for (1)–(3)!

Example 13.5. A finite map $\operatorname{Spec} A' \to \operatorname{Spec} A$ of Dedekind domains is étale if and only if it is everywhere unramified, because $\delta_{A'/A} = \operatorname{ann}(\Omega^1_{A'/A})$.

Definition 13.6. A map $f : X \to S$ of (locally noetherian) schemes is *smooth* if it satisfies any of the following equivalent conditions.

(1) (Structure theorem) X can be covered by Zariski open subsets U on which f factors as $U \xrightarrow{(t_1,\ldots,t_n)} \mathbf{A}^n_S \to S$ in which the first map is étale and the second map is the structure map. We call the $\{t_i\}$ "local étale coordinates".
(2) It is locally of finite type and flat (hence open!) and all geometric fibers $X_{\overline{s}}$ are regular.
(3) It is locally of finite type, flat, and $\Omega^1_{X/S}$ is locally free near $x \in X$ of rank equal to $\dim_x X_{f(x)}$. (In criterion (1) the differentials dt_i are a basis near x.)
(4) It is locally of finite type and functorially smooth: for any closed immersion

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A \\ & \searrow & \downarrow \\ & & S \end{array}$$

along an ideal $I \subset A$ such that $I^2 = 0$, we have $X(A) \twoheadrightarrow X(A_0)$, i.e., in the following diagram the dotted arrow exists:

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A \\ \downarrow & \swarrow & \downarrow \\ X & \xrightarrow{f} & S. \end{array}$$

(This is something like an implicit function theorem, involving recursively solving n equations in $m \geq n$ unknowns with invertible linear part.)
(4′) Same as (4) using artin local A with algebraically closed residue field.

Example 13.7. A locally of finite type scheme $X \to \operatorname{Spec} k$ is smooth if and only if $X_{\overline{k}}$ is regular (cf. (2)). A non-example due to Zariski: if $a \in k \setminus k^p$, $p \neq 2$, then the curve $(y^2 = x^p - a)$ is regular but not geometrically regular (and therefore not smooth), because over $\overline{k}$ we have $y^2 = (x - \sqrt[p]{a})^p$.

Example 13.8. If $f : \operatorname{Spec} k' \to \operatorname{Spec} k$ is finite non-separable then $\Omega^1_{k'/k}$ is a nonzero k'-vector space and f is not smooth.

Facts 13.9.
- If S is locally of finite type over algebraically closed k, then $X \to S$ is smooth if and only if for all $x \in X(k)$ we have $\mathscr{O}^{\wedge}_{S,f(x)}[[t_1, \ldots, t_n]] \xrightarrow{\sim} \mathscr{O}^{\wedge}_{X,x}$.

- If S is locally of finite type over $\mathbf{C}$ and X, S regular then $f : X \to S$ is smooth if and only if f^{an} is a submersion.
- If $S = \mathrm{Spec}(k)$ with k a perfect field and $X \to \mathrm{Spec}(k)$ is locally of finite type, then it is smooth if and only if X is regular.

Theorem 13.10.

(1) *Smooth/étale are preserved by composition and base change.*
(2) *If $X \to S$ is a smooth surjection then many "homological" properties of X and S are equivalent: reduced, normal, regular....*
(3) *In a cartesian diagram*

$$\begin{array}{ccc} X_S & \longrightarrow & X \\ {\scriptstyle f_S}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\text{fpqc}]{} & S \end{array}$$

we have f smooth/étale if and only if f_S is smooth/étale. Examples: $S' \to S$ is extension of scalars from $\mathbf{Z}_{(p)}$ to $\mathbf{Z}_p$, or from k to K.

14. Weierstrass models

Proposition 14.1. *Suppose that $f : X \to S$ is flat and separated, and let $e \in X(S)$ be a section.*

(1) *The map $e : S \hookrightarrow X$ is a closed immersion.*
(2) *Define $\mathscr{I}_e = \ker(\mathscr{O}_X \twoheadrightarrow e_*\mathscr{O}_S)$. This is S-flat, meaning that $\mathscr{I}_{e,x}$ is $\mathscr{O}_{S,f(x)}$-flat for all $x \in X$.*
(3) *Let $\pi : S' \to S$ be an S-scheme. In the cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{\pi}} & X \\ {\scriptstyle e'}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle e} \\ S' & \xrightarrow[\pi]{} & S \end{array}$$

we have $\widetilde{\pi}^*(\mathscr{I}_e) \xrightarrow{\sim} \mathscr{I}_{e'} \subset \mathscr{O}_{X'}$, just as in the analytic case.

Proof. See HW8, #1. ■

When X is an S-curve, we want to define "series expansion along e".

Lemma 14.2. *Assume that f is smooth and pure of relative dimension 1.*

(1) *$\mathscr{I}_e$ is invertible (so $\mathscr{I}_e^{\otimes n} \xrightarrow{\sim} \mathscr{I}_e^n$ for all $n \geq 0$).*
(2) *For $x = e(s)$, after shrinking around s in S there exists an open U with $e(S) \subset U \subset X$ such that $\mathscr{I}_e|_U = \mathscr{O}_U \cdot t$ with $t : \mathscr{O}_U \hookrightarrow \mathscr{O}_U$ (a "local coordinate at e") and t retains this property after base change.*

Proof. Throughout the proof we shrink S to suppose that $S = \mathrm{Spec}\, R$ is affine. By smoothness, we can take an open $U \subset X$ containing x such that $f|_U$ factors as $U \xrightarrow{\varphi} \mathbf{A}^1_S \to S$ with φ étale. Replace S with $e^{-1}(U)$ so now $e(S) \subset U$. Translate the coordinate T on $\mathbf{A}^1_S$ by $\varphi \circ e \in R$ so that $\varphi \circ e = 0$. Since φ is flat, $t := \varphi^*(T)$ acts injectively on $\mathscr{O}_U$. (You can see from this definition that t will retain its defining property after base change.) Then $t \cdot \mathscr{O}_U$ cuts out $\varphi^{-1}(0) \to S$ (the base change of φ with respect to $S \xrightarrow{0} \mathbf{A}^1_S$). This is étale and separated with section e, so e is an étale closed immersion. Therefore $\varphi^{-1}(0) = e(S) \coprod (\text{blah})$ with subspace topology coming from X, so we can shrink U to meet $\varphi^{-1}(0)$ in $e(S)$, and then $t \cdot \mathscr{O}_U = \mathscr{I}_e|_U$. In particular the latter is invertible. ■

Remark 14.3. The element t is unique up to $\mathscr{O}(U)^\times$ (it's not a zero divisor) so any generator of $\mathscr{I}_e$ near $e(S)$ shares the properties of t!

Definition 14.4. We set $\mathscr{O}(ne) = \mathscr{I}_e^{-n} \supset \mathscr{O}_X$ for $n \geq 0$. This is equal to $\mathscr{O}_U \cdot t^{-n}$ on U, and $\mathscr{O}$ on $X \setminus e(S)$.

Since $e : S \to X$ is a closed immersion with $\mathscr{I}_e$ the sheaf of ideals cutting out S, the functor e_* from quasi-coherent $\mathscr{O}_S$ to quasi-coherent $\mathscr{O}_X$-modules is exact and fully faithful with essential image the quasi-coherent $\mathscr{O}_X$ modules that are killed by $\mathscr{I}_e$. If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module such that $\mathscr{I}_e\mathscr{F} = 0$, we therefore can and will identify $\mathscr{F}$ with a sheaf on $\mathscr{O}_S$. If $\mathscr{F}$ is any $\mathscr{O}_X$-module then the adjunction map $\mathscr{F} \to e_*e^*\mathscr{F}$ induces an isomorphism $\mathscr{F}/\mathscr{I}_e\mathscr{F} \cong e_*e^*\mathscr{F}$; combined with the previous part of the paragraph, we see that $e^*\mathscr{F}$ is the sheaf that gets identified with $\mathscr{F}/\mathscr{I}_e\mathscr{F}$. (See tag [04CI] of the Stacks project, for example.)

The "leading coefficient along e" corresponds to the $\mathscr{O}_S$-linear map

$$\mathrm{lead}_n : f_*\mathscr{O}(ne) \to e^*(\mathscr{I}_e^{-n}) = \mathscr{I}_e^{-n}/\mathscr{I}_e^{1-n} \cong (\mathscr{I}_e^{-1}/\mathscr{O}_X)^{\otimes n} \overset{t^n}{\dashrightarrow} \mathscr{O}_S$$

there the dashed arrow exists when there exists t around $e(S)$, in which case the composite map sends $h \in \Gamma(X_W, \mathscr{O}(ne))$ to $(ht^n)(e) \in \Gamma(W, \mathscr{O}_S)$.

The surjection $\mathscr{I}_e/\mathscr{I}_e^2 \twoheadrightarrow e^*(\Omega^1_{X/S})$ coming from the conormal right-exact sequence for $S \xrightarrow{e} X \xrightarrow{f} S$ is an isomorphism. Note that the line bundles $\mathscr{I}_e^{-1}/\mathscr{O}_X$ and $\mathscr{I}_e/\mathscr{I}_e^2 = e^*(\Omega^1_{X/S})$ over S are naturally $\mathscr{O}_S$-dual under $\mathscr{I}_e^{-1}/\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{I}_e/\mathscr{I}_e^2 \twoheadrightarrow \mathscr{O}_S$.

Theorem 14.5 (Riemann–Roch theorem)**.** *Let $f : E \to S$ be an S-elliptic curve and choose $n \geq 1$.*

(1) *The sheaf $f_*\mathscr{O}(ne)$ is locally free of rank n, and $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}(e)$.*
(2) *The formation of $f_*\mathscr{O}(ne)$ commutes with base change on S. For instance, for $S = \mathrm{Spec}(R)$ we have $R' \otimes_R H^0(E, \mathscr{O}(ne)) \xrightarrow{\sim} H^0(E_{R'}, \mathscr{O}(ne_{R'})))$.*
(3) *The sequence*

$$0 \to f_*\mathscr{O}((n-1)e) \to f_*\mathscr{O}(ne) \xrightarrow{\mathrm{lead}_n} \mathscr{I}_e^{-n}/\mathscr{I}_e^{1-n} = e^*(\Omega^1_{E/S})^{\otimes(-n)} \to 0$$

is short exact.

The proof of the theorem rests on the following application of Grothendieck's "cohomology and base change" theorem (see the handout of the same name).

Proposition 14.6. *Let $f : X \to S$ be a proper morphism of schemes with S locally noetherian, and let $\mathscr{F}$ be an S-flat coherent sheaf on X. If $H^i(X_s, \mathscr{F}_s) = 0$ for some $s \in S$ then* (i) *the same holds for all s' near s,* (ii) $\mathrm{R}^i f_*(\mathscr{F})$ *vanishes near s, and* (iii) *$\varphi_{s'}^{i-1}$ is an isomorphism for s' near s.*

In the case $i = 1$, $f_*\mathscr{F}$ is locally free near s and $\varphi^0_{s'} : f_*(\mathscr{F})_{s'} \otimes_{\mathscr{O}_{S,s'}} k(s') \to H^0(X_{s'}, \mathscr{F}_{s'})$ is an isomorphism for all s' near s.

Proof of Theorem 14.5. Since the genus is 1, by Riemann–Roch and Serre duality the fibral cohomologies $H^1(E_s, \mathscr{O}(ne(s)))$ all vanish (for $n \geq 1$), and $H^0(E_s, \mathscr{O}(ne(s)))$ is n-dimensional. By Proposition 14.6 we get that $f_*(\mathscr{O}(ne))$ is locally free and its formation commutes with base change. The base change to fibers then implies that its rank is n everywhere.

To check that the natural map $\mathscr{O}_S \to f_*(\mathscr{O}(e))$ between invertible sheaves is an isomorphism, it suffices to check after passing to stalks and reducing modulo the maximal ideal. But by the base change compatibility, the resulting map is identified with the natural map $k(s) \to H^0(E_s, \mathscr{O}(e(s)))$, and by the classical theory on fibers (or even geometric fibers) this is an isomorphism: it says that the only rational functions on E_s with at worst a simple pole at the origin are the constant functions.

For the final part, we can pass to geometric fibers (why is this OK?) and use the classical theory. ■

Reading 14.7. We mention another consequence of the cohomology and base change theorem, namely that $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_X$ (just as in the $\mathbf{C}$-analytic theory); see Corollary 1.3 on the handout. The proof is similar to the last paragraph of the proof of Theorem 14.5. Namely: $f_*\mathscr{O}_X$ is locally free and there is a natural map $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_X$, which one checks is an isomorphism by checking on fibers.

Now we can establish the local existence of Weierstrass models.

Theorem 14.8. *Zariski locally on S, there exists $E \hookrightarrow \mathbf{P}^2_S$ as a Weierstrass cubic with $e \mapsto [0:1:0]$ (so $E \to S$ is projective Zariski-locally on S).*

Proof. Shrink S so that $S = \mathrm{Spec}(R)$ and $e^*(\Omega^1_{X/S}) \simeq \mathscr{O}_S$. Then the short exact sequence from 14.5(3) has $\mathscr{O}_S$ on the right hand side and is short exact on global sections.

It follows that:

- $H^0(E, \mathscr{O}(2e))$ has basis $1, x$ such that $\mathrm{lead}_2(x) = 1$,
- $H^0(E, \mathscr{O}(3e))$ has basis $1, x, y$ such that $\mathrm{lead}_3(y) = 1$.

Then $\mathrm{lead}_4(x^2) = 1$, $\mathrm{lead}_5(xy) = 1$, $\mathrm{lead}_6(y^2) = \mathrm{lead}_6(x^3) = 1$, and we deduce that

- $H^0(E, \mathscr{O}(4e))$ has basis $1, x, y, x^2$,
- $H^0(E, \mathscr{O}(5e))$ has basis $1, x, y, x^2, xy$,

and $\mathrm{lead}_6(y^2 - x^3) = 0$ so that we get a relation

$$\mathcal{W} : y^2 - x^3 = -a_1xy + a_2x^2 - a_3y + a_4x + a_6$$

for unique $a_i \in R$. Our basis for $\mathscr{O}(3e)$ defines $\varphi : E \to \mathbf{P}^2_S$ landing in $\mathcal{W}$ and sending $e \mapsto [0:1:0]$. By construction, formation of φ commutes with base change, and by the classical theory $\varphi_{\overline{s}} : E_{\overline{s}} \xrightarrow{\sim} \mathcal{W}_{\overline{s}}$ for all $\overline{s}$ (why?). But $E, \mathcal{W}$ are both S-flat (why?), so the fibral isomorphism criterion on HW8 #2 implies that φ is an isomorphism. ∎

Corollary 14.9. *$\omega_{E/S} := f_*\Omega^1_{E/S}$ is a line bundle and satisfies $\omega_{E/S} \xrightarrow{\sim} e^*(\Omega^1_{X/S})$ (via the map $\eta \mapsto e^*(\eta)$), so formation of $\omega_{E/S}$ commutes with base change in S and $f^*\omega_{E/S} \xrightarrow{\sim} \Omega^1_{E/S}$.*

Proof. The isomorphism $\omega_{E/S} \xrightarrow{\sim} e^*(\Omega^1_{X/S})$ comes from applying f_* to the adjunction $\Omega^1_{X/S} \xrightarrow{\sim} e_*e^*\Omega^1_{X/S}$. Work Zariski-locally on S, and use

$$\frac{-dx}{2y + a_1x + a_3} = \frac{-dy}{3x^2 - a_1y + 2a_2x + a_4}$$

to trivialize $\Omega^1_{E/S}$. Use $\mathscr{O}_S = f_*\mathscr{O}_E$ (see 14.7 above) to deduce that $\omega_{E/S}$ is a line bundle. ∎

Corollary 14.10. *For affine S, there exists a Weierstrass cubic model over S if and only if $\omega_{E/S} \in \mathrm{Pic}(S)$ is* trivial.

In that case, the possible coordinate changes are exactly the classical "Weierstrass transformation formulas" over S.

Proof. For the first part, look back at the construction. For the second part, the claim is that the possible coordinate changes are exactly of the form $(u^2x + r, u^3y + sx + t)$ for $u \in \mathscr{O}_S^\times$. This is because for units v, w if $(vx)^3 - (wy)^2 \in H^0(\mathscr{O}(5e))$ then $v^3 = w^2$, so for $u = w/v$ we have $u^2 = v$, $u^3 = w$. ∎

As an application of the second part of the corollary, if $E \to S$ is an elliptic curve then we obtain $j_{E/S} : S \to \mathbf{A}^1$! (Why?)

15. The group law via Pic

Consider a genus-g S curve $f : X \to S$ with section $e \in X(S)$. As previously remarked (but see Corollary 1.3 on the "cohomology and base change" handout) we have $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_X$ just as in the $\mathbf{C}$-analytic theory, even though we now allow S to be non-reduced!

Definition 15.1. An *e-rigidification* of a line bundle $\mathscr{L}$ on X is $i : \mathscr{O}_S \xrightarrow{\sim} e^*\mathscr{L}$. Pairs $(\mathscr{L}, i)$ have an evident notion of isomorphism, tensor product, and dual.

Remark 15.2. The isomorphsim $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_X$ implies that $\mathrm{Aut}(\mathscr{L}, i) = \{1\}$, just as in the $\mathbf{C}$-analytic theory (with the same proof!).

Definition 15.3. $\underline{\mathrm{Pic}}^0_{X/S,e} : (\mathrm{Sch}/S) \to \mathrm{Ab}$ is the functor

$$S' \mapsto \mathrm{Pic}^0_{e_{S'}}(X_{S'}) := \{(\mathscr{L}', i') \text{ on } X_{S'} :\ \deg(\mathscr{L}'_{s'}) = 0 \text{ for all } s' \in S\}/\cong,$$

contravariant via base change.

Note that there will be no description of E as "V/Λ" as in the $\mathbf{C}$-analytic theory! Grothendeick uses the Hilbert scheme and descent theory to prove the following.

Theorem 15.4 (Grothendieck)**.** *$\underline{\mathrm{Pic}}^0_{X/S,e}$ is represented by an abelian scheme (smooth, proper, with geometrically connected fibers) of relative dimension g, the "relative Jacobian".*

Reading 15.5. See the Pic handout for how to deduce this from Grothendieck's *general* representability theorem for Pic functors that goes *way beyond* curves. Usually Pic schemes are *nasty.* For S-curves they are very nice, but all properties (smoothness, properness, and relative dimension) are proved via *functorial* criteria. The construction itself tells very little.

Remark 15.6. (Important!) $\underline{\mathrm{Pic}}^0_{X/S,e}|_{(\mathrm{Sch}/T)} = \underline{\mathrm{Pic}}^0_{X_T/T,e_T}$.

Now we use the theorem. As in the analytic case we define $\delta_e(\mathscr{L}) = \mathscr{L} \otimes f^*e^*(\mathscr{L}^{-1})$. Note that the second term in the tensor product is trivial for local S since Pic of a local ring is trivial.

Theorem 15.7. *Let (E, e) be an elliptic curve. The map*

$$E \overset{\alpha_{E/S}}{\to} \widehat{E} := \mathrm{Pic}^0_{E/S,e} \tag{15.1}$$

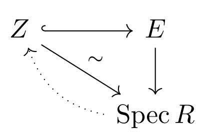
via $E(S') \to \mathrm{Pic}^0_{e_{S'}}(E_{S'})$ sending $x \mapsto \delta_{e_{S'}}(\mathscr{I}_{e_{S'}} \otimes \mathscr{I}_x^{-1}), can.)$ is an isomorphism. In particular (E, e) has a commutative S-group structure commuting with base change. (Functoriality will come later!)

Remark 15.8.
1. On geometric fibers we must recover the classical group law. (Why?)
2. Chapter 2 of Katz–Mazur gives a direct proof that E represents $\underline{\mathrm{Pic}}^0_{E/S,e}$ via (15.1), using cohomological methods akin to the $\mathbf{C}$-analytic case. We'll use the existence of $\widehat{E}$ as a scheme *a priori* to simplify the analysis by reducing isomorphism problems to geometric fibers.

Proof. The formation of $\alpha_{E/S}$ commutes with base change (check!) so by the fibral isomorphism criterion (HW8) we can reduce to the case $S = \mathrm{Spec}(k)$ with k algebraically closed. Then classical Riemann–Roch gives a bijection on k-points.

It remains to show separability: for instance, it suffices to be injective on $k[\epsilon]$-points. Better, we'll show injectivity on R-points for any local R. That is, for $S = \mathrm{Spec}\, R$ any $x \in E(R)$ is determined by $\mathscr{I}_x$ as an abstract $\mathscr{O}_E$-module. (Classically this says that $x \in E(k)$ is determined by the line bundle $\mathscr{O}_E(x)$.)

Observe that the canonical $c_x : \mathscr{I}_x \hookrightarrow \mathscr{O}_E$ corresponds to an element in $\mathrm{Hom}_E(\mathscr{I}_x, \mathscr{O}_E) = H^0(E, \mathscr{I}_x^{-1}) \xleftarrow{\sim} H^0(E, \mathscr{O}_E) = R$ that is an R-module generator, so *all* R-module generators are $R^\times$ multiples of c_x, so all have the same image in $\mathscr{O}_E$, namely $\mathscr{I}_x$. But $\mathrm{Hom}_E(\mathscr{I}_x, \mathscr{O}_E)$ only "knows" $\mathscr{I}_x$ as an $\mathscr{O}_E$-module, so this determines an ideal sheaf in $\mathscr{O}_E$, hence a closed subscheme



such that the dotted arrow followed by the horizontal arrow is x. ∎

Corollary 15.9.
(i) *The S-group law is unique.*
(ii) *The composite*

$$E \overset{\alpha_{E/S}}{\to} \widehat{E} \overset{\alpha_{\widehat{E}/S}}{\to} E^{\wedge\wedge}$$

is functorial. (Get S-homomorphisms by (i) applied to $\widehat{E}$!)
(iii) *Any $(E, e) \to (E', e')$ respects the S-group law.*

Proof. The implication (ii) $\implies$ (iii) is clear. For (i) and (ii), we know these on geometric fibers. (For (ii), see the "Double duality" handout. It amounts to $\widehat{\phi} \circ \phi = [\deg \phi]$ in the classical case.) Now the "killer app" is the following *rigidity lemma.* ∎

Lemma 15.10. *Suppose that G is an S-group and that $f : X \to S$ is closed (e.g. proper) with $\mathscr{O}_S = f_* \mathscr{O}_X$.*

$$\begin{array}{ccc} X & \overset{h}{\underset{h'}{\rightrightarrows}} & G \\ & {}_f\searrow \quad \swarrow & \\ & S & \end{array}$$

such that $h_{\overline{s}} = h'_{\overline{s}}$ for all geometric points $\overline{s}$. Then $h = (g \circ f) \cdot h'$ for $g \in G(S)$. So if *there exists $x_0 \in X(S)$ such that $h \circ x_0 = h' \circ x_0$ in $G(S)$ then $g = e$, so $h = h'$.*

Proof. See §4 of the "Cohomology and base change" handout. ■

This is very powerful to prove identities, even over an artin local ring. To name one application, it implies that on a Weierstrass model we have $-(x, y) = (x, -y - a_1 x - a_3)$, since the same is true on fibers.

16. Level structures

Proposition 16.1. *Choose $N \in \mathbf{Z} - \{0\}$.*

(1) *The map $[N] : E \to E$ is finite locally free of rank N^2. Thus we have a diagram*

$$\begin{array}{ccc} E[N] & \hookrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle [N]} \\ S & \overset{e}{\hookrightarrow} & E \end{array}$$

making $E[N] \to S$ a finite locally free S-group of rank N^2.

If $N \in \mathscr{O}(S)^\times$ then $[N]$ is étale, so $E[N] \to S$ is finite étale. In such cases, there exists an étale cover $S' \to S$ such that $E[N]_{S'} \simeq (\mathbf{Z}/N\mathbf{Z})^2 \times S'$ as S'-groups.

Proof. Recall our running assumption that S is locally noetherian. We have a diagram

$$\begin{array}{ccc} E & \xrightarrow{[N]} & E \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

in which the maps $E \to S$ are flat and finite type. By the fibral flatness criterion (HW8 #2(i)), it's enough to check flatness for $E_{\overline{s}} \to E_{\overline{s}}$, which is classical. The map $[N]$ is also quasi-finite and proper (the latter because $E \to S$ is proper), so finite by Zariski's main theorem. We can then read off the rank from the rank on s-fibers.

For (2), the first part is a consequence of the fibral étaleness criterion (HW8 #2(ii)). Every $E[N]_{\overline{s}}$ is a copy of $(\mathbf{Z}/N\mathbf{Z})^2$, so the existence of the desired étale cover $S' \to S$ is a consequence of HW8 #4(iii). ■

Example 16.2. An example of part (2) of the previous theorem is that if $S = \operatorname{Spec} K$ for $\operatorname{char} K \nmid N$, then we can take $S' = \operatorname{Spec} K(E[N])$.

Definition 16.3. For $E \to S$ with S a $\mathbf{Z}[\frac{1}{N}]$-scheme, a *full level-N structure* is an S-group isomorphism $\phi : (\mathbf{Z}/N\mathbf{Z})^2 \times S \xrightarrow{\sim} E[N]$.

Equivalently, a full level-N structure is an ordered pair (P, Q) of elements of $E[N](S)$ such that $\{P(\overline{s}), Q(\overline{s})\}$ is a $\mathbf{Z}/N\mathbf{Z}$-basis of $E_{\overline{s}}[N]$ for all geometric points $\overline{s}$ of S. Indeed, consider the S-map $(\mathbf{Z}/N\mathbf{Z})^2 \times S \to E[N]$ sending $(i, j) \mapsto iP + jQ$. This is a fibral isomorphism (why?) so Zariski locally on S it's an N^2-by-N^2-matrix with $\mathscr{O}$-entries and determinant a unit, hence an isomorphism.

Thus *any* $E \to S$ with S a $\mathbf{Z}[\frac{1}{N}]$-scheme has a full level-N structure étale-locally over S. This is the algebraic analogue of the fact that any elliptic curve $E \to M$ over a $\mathbf{C}$-manifold gets an H_1-trivialization (or a full level-N structure with fixed Weil pairing) locally on M.

An isomorphism of full level-N structures is defined in the evident manner.

Proposition 16.4. *Let $\alpha \in \mathrm{Aut}_S(E,\phi)$. If $N \geq 3$ then $\alpha = \mathrm{id}$. If $N = 2$ then there is a decomposition $S = S_+ \coprod S_-$ such that $\alpha|_{S_\pm} = \pm\mathrm{id}$.*

Proof. For $N \geq 3$, the rigidity lemma reduces us to the same statement on geometric fibers. The case $N = 2$ is trickier. We have $\alpha = 1 + 2\beta$ for some $\beta \in \mathrm{End}(E)$, which implies $\alpha^2 = 1 + 4\beta'$, so α^2 is trivial on $E[4]$. By the case $N = 4$ we have $\alpha^2 = 1$. Consider the commuting endomorphisms $\alpha - 1$ and $\alpha + 1$. Their difference is $[2]$, which is nonzero on all fibers, so they cannot both vanish on any fiber. On the other hand, since their composition is zero they cannot both be isogenies on any fiber either. We can define $S_\pm = \{s \in S : \alpha_s = \pm 1\}$, so that $S_\pm$ are disjoint and cover S. To see that these are both clopen, use Proposition 1.1 from the "Level structures" handouts, which says that for any homomorphism f between two elliptic curves, the locus of points on the base such that f hassome fixed degree d is always open and closed. ∎

17. Construction and properties of $Y(N)$

Theorem 17.1. *For $N = 3, 4$ the moduli problem on* $(\mathrm{Sch}/\mathbf{Z}[\frac{1}{N}])$

$$S \rightsquigarrow \{(E \to S, \phi_N)\}/\simeq$$

is represented by a smooth affine $\mathbf{Z}[\frac{1}{N}]$-scheme $Y(N)$ of pure relative dimension 1.

Remarks 17.2.

1. We'll discuss $N = 3$. The case $N = 4$ is done in §3 of the "Level structures" handout. Note that the proof in Katz–Mazur has a genuine error — see Remark 2.8 in the same handout.
2. Smoothness (hence flatness) and pure relative dimension 1 are seen by inspection of the coordinate rings of $Y(3)$ and $Y(4)$. This is the "wrong proof": one should really use deformation theory. See §5 of the "Level structures" handout.

For $N = 3$, rigorous arguments in the relative setting are in §4 of the handout, so here we'll explain over a field k of characteristic $\neq 3$ why any $(E,(P,Q))_{/k}$ admits a "universal" Weierstrass form with relations on coefficients encoding the level-3 structure (which then gives the coordinate ring of $Y(3)$).

We warn the reader that even on (Sch/k), to prove the theorem we must work over k-algebras, not just fields, so that case is no easier than than thinking about $(\mathrm{Sch}/\mathbf{Z}[\frac{1}{3}])$. On the other hand, over k we can use divisors, talk about ord_P for $P \in E(k)$, etc. So:

Sketch for $N = 3$. Consider

$$E\colon y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with full level-3 structure (P,Q). Since $[3]P = e$, the divisor $3P - 3e$ is the divisor of some function in $\mathscr{O}(3e)$ which genuinely has a pole of order 3 at e, i.e. $3P - 3e = \mathrm{div}(y - ax - b)$ for some unique $a, b \in k$. Rename $y - ax - b$ as y so that now $3P - 3e = \mathrm{div}(y)$. In particular now y vanishes at P, i.e., $y(P) = 0$. Replacing x with $x - x(P)$ we have changed coordinates so that $P = (0,0)$.

Next, recall that $-(x,y) = (x, -y - a_1x - a_3)$, so that $-P = (0, -a_3)$, and since $-P \neq P$ we get $a_3 \neq 0$. We see that $\mathrm{div}(x) \geq P + (-P) - 2e$, but since x vanishes to exact order 2 at e and has no other poles we get $\mathrm{div}(x) = P + (-P) - 2e$.

Now look at the equation of E near P. The order at P of $y^2 + a_1xy + a_3y$ is at least 3, and the order at P of x^3 is 3, so the order at P of $a_2x^2 + a_4x + a_6$ is at least 3. Since x has a simple zero at P, this forces $a_2 = a_4 = a_6 = 0$. That is, we have arranged that E has equation

$$E\colon y(y + a_1x + a_3) = x^3$$

with $P = (0,0)$, and of course $\Delta = (a_1^3 - 27a_3)a_1^3$ must be a unit.

Finally we note that any coordinate change of E must take (x,y) to $(u^2x + r, u^3y + sx + t)$. To preserve the above form r, s, t must be zero (if we had $s \neq 0$ this would ruin the divisor of y at P), i.e., any coordinate change preserving this shape must be (u^2x, u^3y) for some unit u.

Conversely for any E and P of this shape, we can check that P is indeed a point of exact order 3: we have $y + a_1x + a_3$ a unit at P, so $\mathrm{ord}_P(y) \geq 3$, yet the divisor of poles of y is $-3e$, and so $\mathrm{div}(y) = 3P - 3e$. This means $3P = 3e = e$ in the group law. We also have $\mathrm{ord}_P(x) = 1$ and $-P = (0, -a_3) \neq P$, so $\mathrm{div}(x) = (P) + (-P) - 2e$.

In summary, we've proved:

Lemma 17.3. *All pairs* (E, P) *of order* 3 *in* $E(k)$ *have the form*

$$E\colon y(y + a_1x + a_3) = x^3$$

with $P = (0, 0)$ *and* $(a_1^3 - 27a_3)a_3^3 \in k^\times$*, and then*

- $\operatorname{div}(y) = 3P - 3e$ *and* $\operatorname{div}(x) = P + (-P) - 2e$*, and*
- (x, y) *is unique up to a change of variables* (u^2x, u^3y) *with* $u \in k^\times$*.*

Now we have to account for Q. Just as for P we have to have unique A, B so that $y - Ax - B$ has a triple zero at Q. Plugging this into the defining equation for E we find that

$$x^3 - (Ax + B)((A + a_1)x + (B + a_3)) \tag{17.1}$$

has a triple zero at $x(Q)$. This cubic must then be precisely $(x - x(Q))^3$, and comparing quadratic terms we find that $3x(Q) = A(A + a_1)$. Since $x(Q) \neq 0$ (because $Q \notin \langle P\rangle$) we must have A a unit. Taking $u = A$ in our change of coordinates, we reduce to $A = 1$ and now our Weierstrass coordinates are *unique*.

Taking $C = x(Q) \neq 0$, we have $Q = (C, C + B)$ with $C \in k^\times$. Setting the expression (17.1) with $A = 1$ to be $(x - C)^3$ we get the relations

$$a_1 = 3C - 1, \qquad a_3 = -3C^2 - B - 3BC, \qquad B^3 = (B + C)^3.$$

Lemma 17.4. *All elliptic curves with full level-*3 *structure over* k *(of characterstic not equal to* 3*) are* uniquely *of the form*

$$E : y(y + a_1x + a_3) = x^3$$

with

- $P = (0, 0)$*,*
- $Q = (C, B + C)$*,*
- $a_1 = 3C - 1$*,*
- $a_3 = -3C^2 - B - 3BC$

for some B, C *with* $C, \Delta = (a_1^3 - 27a_3)a_3^3 \in k^\times$ *and* $B^3 = (B + C)^3$*.*

Moreover one checks conversely that anything of the above shape is indeed an elliptic curve with full level-3 structure.

Repeating these arguments somewhat more carefully in the relative setting establishes, thus, that

$$Y(3) = \operatorname{Spec} \mathbf{Z}[\frac{1}{3}][B, C][\frac{1}{\Delta}]/(B^3 - (B + C)^3)$$

with Δ given as above. Moreover the uniqueness of the Weierstrass form (and level structure) Zariski-locally means that these Weierstrass models patch together, and so the universal elliptic curve with full level-3 structure over $Y(3)$ even has a global Weierstrass model given by the equation $y(y + a_1x + a_3) = x^3$ with level structure (P, Q) as above. ∎

Remark 17.5. Taking $c_4 = a_1(a_1^2 - 24a_3)$, we even get an explicit $j_{E/S} : Y(3) \to \mathbf{A}^1$. Is it finite? Yes! See §5 of the level structures handout. Uses Néron models to prove proper, yet also quasi-finite.

Now we construct *lots* of modular curves over $\mathbf{Z}[\frac{1}{N}]$ (and we'll check the relation with the $\mathbf{C}$-analytic theory next time) First we create a moduli scheme for level structures on a *fixed* elliptic curve.

Proposition 17.6. *Let* S *be a* $\mathbf{Z}[\frac{1}{N}]$*-scheme for* $N \geq 1$ *and fix* $E \to S$*. The functor*

$$T \rightsquigarrow \operatorname{Isom}_{T-\mathrm{gp}}((\mathbf{Z}/N\mathbf{Z})^2 \times T, E_T[N])$$

on (Sch/S) *is represented by a* finite *étale* cover $I_{E/S,N} \to S$*.*

Proof. We can reformulate the functor as consisting of ordered pairs (P, Q) in $E[N](T)$ such that the induced map of T-groups

$$(\mathbf{Z}/N\mathbf{Z})^2 \times T = \coprod_{(i,j)\in(\mathbf{Z}/N\mathbf{Z})^2} T \to E_T[N] = E[N]_T$$

(carrying the (i, j)-part to $iP + jQ \in E[N]_T(T) = E[N](T)$) is an isomorphism. To give the ordered pair (P, Q) is to give a map $T \to E[N] \times_S E[N]$, so over $M = E[N] \times_S E[N]$ there is a "universal homomorphism" $h : (\mathbf{Z}/N\mathbf{Z})^2 \times M \to E[N]_M$ of M-groups (given by $(i, j, m) \mapsto ip_1(m) + jp_2(m)$). Hence, we are interested in maps $g : T \to M$ such that $g^*(h) : (\mathbf{Z}/N\mathbf{Z})^2 \times T \to E[N]_T$ is an isomorphism. More specifically, we want to show that this isomorphism condition is equivalent to g factoring through some specific clopen subscheme U of M. One this is done, we can take $I_{E/S,N} = U$; this is finite étale over S since M is finite étale over S, and it is surjective onto S since for any geometric point $\overline{s}$ of S the set $I_{E/S,N}(\overline{s}) = \mathrm{Isom}_{\mathrm{gp}}((\mathbf{Z}/N\mathbf{Z})^2, E(\overline{s})[N])$ is clearly non-empty.

Now consider the universal homomorphism $h : (\mathbf{Z}/N\mathbf{Z})^2 \times M \to E[N]_M$. This is a map between finite étale M-schemes of the same rank (N^2), so Zariski-locally on M it is described by a square matrix. The unit condition on the determinant corresponds to the isomorphism property, so this is represented by an open subscheme U of M. To prove that U is closed in M, it is equivalent to show that the complement is open. Hence, it suffices to prove that if $h : G \to G'$ is a homomorphism between finite étale group schemes over a scheme M and if it has a nontrivial kernel on some fiber then it has a nontrivial kernel on fibers over all nearby points in M. But the map h is finite étale since G and G' are finite étale over M, so $\ker h$ is finite étale over M. Hence, the order of the geometric fiber of this kernel is Zariski-locally constant on M, and the order is the same as the number of geometric points in the fiber due to the étaleness. ■

Example 17.7. If in the previous proposition we have $S = \mathrm{Spec}\, K$ and $\mathrm{char} K \nmid N$, then $I_{E/K,N}$ corresponds to the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-set $\mathrm{Isom}_{\mathrm{gp}}((\mathbf{Z}/N\mathbf{Z})^2, E[N](K^{\mathrm{sep}})$ (with Galois acting through the natural action on $E[N](K^{\mathrm{sep}})$. For a finite extension K'/K, then, $I_{E/K,N}(K')$ corresponds to the $\mathrm{Gal}(K^{\mathrm{sep}}/K')$-invariants, i.e., the full level-N structures over K'. In particular this is empty if $\mu_N \not\subset K'$.

There is a left $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-action on $I_{E/S,N}$ via $g \circ \phi = \phi \circ g^t$. This is simply transitive on geometric fibers (see the preceding Example). This leads us to the following.

Definition 17.8. For a finite group G, an *étale G-torsor* over a scheme S is a finite étale cover $X \to S$ equipped with left G-action such that the map $G \times X \to X \times_S X$ taking $(g, x) \mapsto (gx, x)$ is an isomorphism.

The last condition is equivalent to saying that G acts simply transitively on $X(\overline{s})$ for all geometric points $\overline{s}$ of S. (Why?)

Example 17.9. $I_{E/S,N}$ is an étale $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-torsor over S.

Example 17.10. If K'/K is a finite Galois extension and $G := \mathrm{Gal}(K'/K)^{\mathrm{opp}}$, the map $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$ with its G-action is an étale G-torsor. To see this, apply Spec to the map $K' \otimes_K K' \xrightarrow{\sim} \prod_{g\in G} K'$ sending $a \otimes b \mapsto (g(a)b)_g$.

(Warning: observe that the G-action can have "physical" fixed points even if it doesn't have "geometric" ones.)

Lemma 17.11.

(1) *Any G-equivariant map*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

between étale G-torsors is an isomorphism

(2) *Any commutative diagram of G-torsors*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle G} \\ S' & \longrightarrow & S \end{array}$$

is cartesian.

Proof. (1) Check on geometric fibers.
(2) Apply (1) to $X' \to X_S$. ■

Theorem 17.12. *Let $X = \operatorname{Spec} B$ be of finite type over noetherian A, with action over A by a finite group G.*

(1) *The map $X \to X/G := \operatorname{Spec} B^G$ is finite surjective with B^G of finite type over A, and is initial among G-invariant maps from X to* any *scheme.*
(2) *If G acts freely on $X(k)$ for all algebraically closed k, then $X \to X/G$ is a finite étale G-torsor (hence flat, and its formation commutes with* any *base change on A).*

Proof. (1) can be found as exercise 2 on HW9. A reference for (2) is SGA3, Exp. V, Thm. 4.1. See also §6, Thm. and Prop. 2 of Mumford's Abelian Varieties for the case where A is an algebraically closed field. ■

Example 17.13. Let $B = k[x,y]/(xy)$ with the action of $G = \mathbf{Z}/2\mathbf{Z}$ given by $(x,y) \to (-x,-y)$. This action is of course not free at $(0,0)$. The G-invariants are $B^G = k[u,v]/(uv) \hookrightarrow B$ with $u \mapsto x^2$ and $v \mapsto y^2$. The map $B^G \to B$ is not flat: generically it has degree 2, but the fiber over $(0,0)$ has degree 3.

Now we can prove the following.

Theorem 17.14. *For $N \geq 3$, $Y(N)$ exists over $\mathbf{Z}[\frac{1}{N}]$ as a smooth affine scheme of pure relative dimension 1.*

Smoothness and purity of relative dimension 1 will be established by functorial techniques; see the "Level structures" handout for these.

Proof. The cases $N = 3, 4$ are handled explicitly, as explained above and in the "Level structures" handout. We use these as a crutch.

Case 1: Suppose $N = 3^r$ for $r \geq 2$, so that $\mathbf{Z}[\frac{1}{N}] = \mathbf{Z}[\frac{1}{3}]$. From the universal map $E_3 \to Y(3)$, we obtain a map $Y(3) \to I_{E_3/Y(3),3}$ taking a point $T \to Y(3)$ to the universal level-3 structure on $E_3 \times_{Y(3)} T$. We can then define $Y(3^r)$ via the pullback diagram

$$\begin{array}{ccccc} Y(3^r) & \longrightarrow & I_{E_3/Y(3),3^r} & & \\ \big\downarrow & & \big\downarrow{\scriptstyle \text{forget}} & & \\ Y(3) & \longrightarrow & I_{E_3/Y(3),3} & \longrightarrow & Y(3) \end{array}$$

For T a $\mathbf{Z}[\frac{1}{3}]$-scheme, a T-point h of $Y(3)$ is precisely an elliptic curve E_h over T together with a full level-3 structure, obtained by pullback via h from the universal structure on $E_3/Y(3)$. A T-point of $I_{E_3/Y(3),3^r}$ which matches h over $Y(3)$ is a full level-3^r structure on the pullback of $E_3/Y(3)$ by h, i.e., a level-3^r structure on E_h. Thus the T-points of the fiber product are precisely the elliptic curves E_h over T together with a full level-3 structure ϕ_1 and a level 3^r-structure ϕ_r lifting ϕ_1. This is nothing other than an elliptic curve and a level 3^r-structure, and so $Y(3^r)$ represents the functor we want it to.

Case 2: If $3 \mid N$, write $N = 3^r d$ with $3 \nmid d$. Then $E[N] \simeq E[3^r] \times E[d]$, so $Y(N)$ is equal to $I_{E_{3^r}/Y(3^r),d}$.

Case 3: If $4 \mid N$, we proceed similarly to cases 1 and 2 using $N = 2^r \cdot (\text{odd})$.

Case 4: Suppose $3 \nmid N$ and $\operatorname{ord}_2(N) \leq 1$. Then $N = d$ or $N = 2d$ with $d \geq 5$ and $\gcd(6,d) = 1$. First work over $\mathbf{Z}[\frac{1}{3N}]$, the idea being that we go up to $3N$ and then descend back down. We have an action of $G = \mathrm{GL}_2(\mathbf{Z}/3\mathbf{Z})$ on $Y(3N) = I_{E_3/Y(3),N}$ which is free because $N > 2$, and this lets us make

$Y(N) := Y(3N)/G$ over $\mathbf{Z}[\frac{1}{3N}]$. The map $Y(3N) \to Y(N)$ is a G-torsor by the SGA3 result quoted earlier (Theorem 17.12(2)).

To see that $Y(N)$ has the right universal property, consider an elliptic curve $(E,\phi) \to S$ with full level N-structure. We get a map $I_{E/S,3} \to I_{E_3/Y(3),N}$ taking a T-point to its enrichment by ϕ_T. Since $I_{E/S,3} \to S$ is a G-torsor, the intialness of $I_{E/S,3} \to S$ among G-invariant maps out of $I_{E/S}$ means that we can complete the diagram

$$\begin{array}{ccc} I_{E/S,3} & \longrightarrow & I_{E_3/Y(3),N} = Y(3N) \\ {\scriptstyle G-\text{torsor}}\downarrow & & \downarrow{\scriptstyle G-\text{torsor}} \\ S & \overset{\exists \text{ unique}}{\dashrightarrow} & Y(3N)/G =: Y \end{array}$$

to get a unique point of $Y(N)$, as desired.

For $N = d$ odd, we can do the same over $\mathbf{Z}[\frac{1}{2N}]$ using an auxiliary full level-4 structure and $\mathrm{GL}_2(\mathbf{Z}/4\mathbf{Z})$-action. Then we can glue over $\mathbf{Z}[\frac{1}{2N}]$ and $\mathbf{Z}[\frac{1}{3N}]$ to obtain $Y(N)$ over $\mathbf{Z}[\frac{1}{N}]$.

It remains to handle the case $N = 2d$ for odd $d \geq 5$, but now we can just take $Y(2d) = I_{E_d/Y(d),2}$ over $\mathbf{Z}[\frac{1}{N}]$. ■

Remark 17.15. For $N \geq 3$, the j-map $j_{E_N/Y(N)} : Y(N) \to \mathbf{A}^1_{\mathbf{Z}[\frac{1}{N}]}$ is finite flat and surjective due to the valuative criterion for properness and the fibral flatness criterion over $\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]$. See §5 of the "Level structures" handout. (Concretely we get finite flat $\mathbf{Z}[\frac{1}{N}][j] \hookrightarrow \mathscr{O}(Y(N))$.)

Definition 17.16. Le For $N \geq 1$ and a subgroup $G \subset \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$, a G-structure on $E \to S$ (with S a $\mathbf{Z}[\frac{1}{N}]$-scheme) is an element $\alpha \in (G\backslash I_{E/S,N})(S)$. Let $I_{E/S,G}$ be the quotient $G\backslash I_{E/S,N}$, the scheme of G-structures on $E \to S$, which is finite étale over S by Theorem 17.12(2).

Example 17.17. If $G = G\Gamma_1(N) = \left\{\begin{pmatrix} * & * \\ 0 & 1\end{pmatrix}\right\}$ then a G-structure is $(\mathbf{Z}/N\mathbf{Z}) \times S \hookrightarrow E[N]$. If $G = G\Gamma_0(N) = \left\{\begin{pmatrix} * & * \\ 0 & *\end{pmatrix}\right\}$ then a G-structure is $C \hookrightarrow E[N]$ with all $C_{\overline{s}}$ cyclic of order N. Call these $\Gamma_1(N)$-structures and $\Gamma_0(N)$-structures, respectively.

We can always increase N, due to the following lemma.

Lemma 17.18. *For $N \mid N'$ and $G' \subset \mathrm{GL}_2(\mathbf{Z}/N'\mathbf{Z})$ the preimage of G, over $\mathbf{Z}[\frac{1}{N'}]$ we have $I_{E/S,G'} \xrightarrow{\sim} I_{E/S,G}$ (over S).*

Proof. By the fibral isomorphism criterion, this reduces to the fact that the map $G'\backslash \mathrm{GL}_2(\mathbf{Z}/N'\mathbf{Z}) \to G\backslash \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ is bijective. ■

Variant: For $N \geq 1$ and a prime $p \nmid N$, a $\Gamma_1(N,p)$-structure on $E \to S$ (with S a $\mathbf{Z}[\frac{1}{N}]$-scheme) is $(\mathbf{Z}/N\mathbf{Z} \times S) \times_S C \hookrightarrow E[N] \times_S E[p] = E[Np]$ where $C \to S$ is order p. These are rigid when $\Gamma_1(N)$-structures are rigid (why?).

Theorem 17.19.

(1) *G-structures are rigid when the preimage of G in $\mathrm{SL}_2(\widehat{\mathbf{Z}})$ is torsion-free (e.g. for $G\Gamma_1(N)$ with $N \geq 4$).*
(2) *$F_G : S \rightsquigarrow \{(E/S,\alpha)\}/\cong$ on $(\mathrm{Sch}/\mathbf{Z}[\frac{1}{N}])$, with α a G-structure, admits a* coarse *moduli scheme Y_G over $\mathbf{Z}[\frac{1}{N}]$, akin to §5 of the "classical modular curves" handout except that we demand $F_G(k) \xrightarrow{\sim} Y_G(k)$ for* all *algebraically closed k.*

   The scheme Y_G is affine, finite type, normal, flat of pure relative dimension 1 over $\mathbf{Z}[\frac{1}{N}]$. In the rigid case, Y_G is fine and smooth over $\mathbf{Z}[\frac{1}{N}]$.
(3) *For $N \geq 1$ and $p \nmid N$, there exists $Y_1(N,p) \to \operatorname{Spec} \mathbf{Z}[\frac{1}{N}]$ as in (2), except that for $N \geq 4$ it's merely regular (and not smooth) at (supersingular) points in characteristic p.*

Proof. (1) is just as in the **C**-analytic case. For (2), use $G\backslash Y(N)$; cf. Theorem 5.5 in the "Classical modular curves" handout for the **C**-analytic version, which is similar, except for smoothness in the rigid case for which we use the SGA3 result (Theorem 17.12(2)).

(3) If $N \geq 4$, for universal $E \to Y = Y_1(N)$ we take $Y_1(N,p)$ to be $\mathrm{Ord}^p_{E[p]/Y}$, as defined in HW9. See HW9 and the Katz–Mazur handout for structural properties, especially that it is *regular* and $\mathbf{Z}[\frac{1}{N}]$-flat (iff points in characteristic p lift to characteristic 0). This uses p-divisible groups and deformation theory.

For $N \leq 3$, adjoin a $\Gamma(\ell)$-structure over $\mathbf{Z}[\frac{1}{N\ell}]$ for primes $\ell \nmid N$, $\ell \geq 3$, etc. ■

Corollary 17.20. *For Y as above, $j : Y \to \mathbf{A}^1_{\mathbf{Z}[\frac{1}{N}]}$ is finite flat, and is an isomorphism for $Y = Y(1)$.*

Proof. Finiteness is inherited from the $Y(N)$-case, and for flatness use fibral flatness. For $Y(1)$, compute the degree using $\overline{\mathbf{Q}}$-points. ■

Theorem 17.21. (1) *For Y as above, and M the corresponding* **C***-analytic (coarse) moduli space, the map $Y(\mathbf{C}) \to |M|$ from algebraic isomorphism classes over* **C** *to* **C***-analytic isomorphism classes, which is bijective by GAGA, is a holomorphic isomorphism.*

(2) *When Y is a fine moduli space, the map $E^{\mathrm{an}}_Y \to Y^{\mathrm{an}}$ is universal in the* **C***-analytic category, i.e., the holomorphic map $Y^{\mathrm{an}} \to M$ is an isomorphism.*

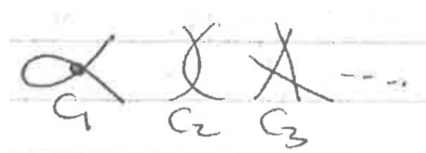
Remark 17.22. The map is *not* built in the direction $M \to Y^{\mathrm{an}}$! That is, there is no $\mathfrak{h} \to Y^{\mathrm{an}}$....

Proof. Analytification commutes with quotients by finite group actions, so it's enough to treat the fine moduli cases, or indeed to treat $Y(N)$ for $N \geq 3$. Then we have a holomorphic map $Y^{\mathrm{an}}_{\mathbf{C}} \to M$ between *pure dimension* 1 **C**-manifolds, bijective by GAGA, so must be an analytic isomorphism. ■

18. Generalized elliptic curves

Example 18.1. The curve $Y_1(N)_{\overline{\mathbf{Q}}}$ is geometrically connected. What about $Y_1(N)_{\overline{\mathbf{F}}_p}$ for $p \nmid N$? For this and for q-expansions, we need to consider generalized elliptic curves. References for this discussion include Deligne–Rapoport, and §§2.1, 2.2, 2.5 of Brian's paper "Arithmetic moduli of generalized elliptic curves".

Definition 18.2. For an algebraically closed field k, a *stable genus*-1 *curve* over k is a proper connected reduced curve C over k that is either smooth of genus 1, or else the Néron polygon C_n for $n \geq 1$, obtained from $\mathbf{P}^1_k \times \mathbf{Z}/n\mathbf{Z}$ by gluing the ∞-section on the ith copy of $\mathbf{P}^1_k$ to the 0-section on the $i+1$th. (In any case a stable genus-1 curve has arithmetic genus 1.)



Note that for $d \mid n$ we have $C_n/(\mathbf{Z}/d\mathbf{Z}) \xrightarrow{\sim} C_{n/d}$. The above definition is "ad hoc", and for instance is too weak to answer the following question: if the map $C \to S$ is proper, flat, and of finite presentation, is $\{s \in S \colon C_{\overline{s}} \text{ is stable genus } 1\}$ open in S?

Definition 18.3. A *stable genus*-1 *curve* $C \to S$ is proper, flat, finite-presentation such that all $C_{\overline{s}}$ are either smooth genus 1 or a Néron polygon.

Example 18.4. A Weierstrass cubic $\mathcal{W} \subset \mathbf{P}^2_S$ with at worst nodal geometric fibers.

By the "Riemann–Roch" method, every stable genus-1 $C \to S$ with $e \in C^{\mathrm{sm}}(S)$ and all $C_{\overline{s}}$ *irreducible* has a Weierstrass form (with $e = [0:1:0]$) Zariski-locally in S.

Example 18.5. Let R be a discrete valuation ring with fraction field K with residue field k, and let E be an elliptic curve over K. Typically E will not extend to an elliptic curve over R (that is, E will typically not have good reduction), and then there are several models of E over R (i.e., R-schemes with generic fiber E) that one often considers.

Any Weierstrass model W for E over R is proper and flat with geometrically reduced and irreducible special fibers, and even normal. Being proper, we have $W(R) = W(K) = E(K)$. Among Weierstrass models there is always a unique minimal one (e.g. whose discriminant has minimal valuation). Its smooth locus W^{sm} is a group scheme over R, and the K-points of E that extend to W^{sm} define the finite-index subgroup $E_0(K)$.

Any smooth K-curve C of positive genus has a *minimal regular proper model* C^{reg}, i.e., a regular proper R-model with the universal property that for any regular proper R-model $\mathscr{C}$ of C there is a unique (dominant) map $\mathscr{C} \to C^{\mathrm{reg}}$. Every R-point of C^{reg} meets each fiber at a non-singular point, and therefore lies in the smooth locus of C. The special fiber of C^{reg} is automatically a proper, geometrically connected, 1-dimensional k-scheme. One constructs a regular proper model by repeated blowing up and normalization of a normal proper model, and then one blows down certain components of the special fiber to obtain the minimal regular proper model.

If W is any Weierstrass model of E over R, then its minimal regular resolution $\mathscr{W} \to W$ coincides with E^{reg} if and only if W is the minimal Weierstrass model.

Finally we have the *Néron model* of an abelian variety A over K, a smooth R-model $\mathscr{A}$ satisfying the Néron mapping property: for any smooth R-scheme Z, each map $Z_K \to E$ extends uniquely to a map $Z \to \mathscr{A}$ (so in particular $\mathscr{A}(R) \to A(K)$ is a bijection). It is automatically an R-group scheme, separated, finite type, and quasi-projective. The Néron model of an abelian variety is proper over R (hence an abelian scheme over R) if and only if the identity component of the special fiber is proper, and any abelian R-scheme is the Néron model of its generic fiber.

Now let us consider the relationship between these various models in the case of an elliptic curve. The smooth locus $(E^{\mathrm{reg}})^{\mathrm{sm}}$ in the minimal regular proper model of E is a smooth model of E, and so there is a canonical map $(E^{\mathrm{reg}})^{\mathrm{sm}} \to \mathscr{E}$ to the Néron model. It is a theorem that this is an isomorphism. In particular if W is a minimal Weierstrass model, then the Néron model is the R-smooth locus in the minimal regular resolution of W. Furthermore, the canonical map $W^{\mathrm{sm}} \to \mathscr{E}$ turns out to be an isomorphism onto the open R-subgroup of $\mathscr{E}$ obtained by removing the non-identity components in the special fiber. Thus $E_0(K)$ is the subgroup of points which extend to the identity comopnent of the special fiber of $\mathscr{E}$.

See the notes “Minimal models for elliptic curves” on Brian's website.

Definition 18.6. A *generalized elliptic curve* is a map $E \to S$ together with a section $e \in E^{\mathrm{sm}}(S)$, along with a map $+ : E^{\mathrm{sm}} \times E \to E$, such that:

- $E \to S$ is stable genus 1, and
- $+$ defines a commutative group law on (E^{sm}, e) making the geometric component group cyclic.

Example 18.7. The Néron polygon C_n over S is a generalized elliptic curve. Its smooth locus is $C_n^{\mathrm{sm}} = \mathbf{G}_m \times \mathbf{Z}/n\mathbf{Z}$, with action on C_n via $\mathbf{G}_m$-scaling and $\mathbf{Z}/n\mathbf{Z}$-rotation.

One can compute that $\underline{\mathrm{Aut}}(C_n, +) = \mu_n \rtimes \mathbf{Z}/2\mathbf{Z}$ in which $\zeta \in \mu_n$ acts via $\zeta \cdot (t, i) = (\zeta^i t, i)$ and the generator of $\mathbf{Z}/2\mathbf{Z}$ is inversion. This fact is useful e.g. for computing the field of rationality of cusps.

Theorem 18.8. *Any stable genus-*1 *curve $C \to S$ with $e \in C^{\mathrm{sm}}(S)$ and all $C_{\overline{s}}$* irreducible *admits a unique $+$ as above. (This is* false *if you allow reducible fibers — see Example 2.1.11 of Brian's paper.)*

Example 18.9. Any Weierstrass curve $\mathcal{W} \subset \mathbf{P}^2_S$ with at worst nodal fibers satisfies the hypotheses of the Theorem (with $e = [0 : 1 : 0]$).

19. Tate curves

Let Tate(q) be the Tate curve $\{y^2 + xy = x^3 + a_4(q)x + a_6(q)\} \subset \mathbf{P}^2_{\mathbf{Z}[[q]]}$ with a_4, a_6 having their usual meanings in the theory of the Tate curve, i.e.,

$$a_4 = -s_3, \qquad a_6 = -(5s_3 + 7s_5)/12, \qquad \text{with } s_k = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n}.$$

This has discriminant $\Delta = q\prod_{n\geq 1}(1-q^n)^{24}$ and j-invariant $j(q) = \frac{1}{q} + 744 + 196884q + \cdots$. The Tate curve is regular because near $(0,0)$ along $q = 0$ we have lowest order part $a_6(q) = -q \times$ (1-unit). By Theorem 18.8 the Tate curve is a generalized elliptic curve.

We briefly recall the uniformization property of the Tate curve. If K is a complete non-archimedean field with integer ring A and $\phi : \mathbf{Z}[[q]] \to A$ is any map (so that $|\phi(q)| < 1$), then the series a_4 and a_6 converge at $\phi(q)$ and we let E_ϕ denote the curve $\mathrm{Tate}(q) \times_\phi A$. Then there is a Galois-equivariant homomorphism $\overline{K}^\times \to E_\phi(\overline{K})$ with kernel $\phi(q)^{\mathbf{Z}}$. The curve E_ϕ is a model (with special fiber C_1 over k, the residue field of A) of an elliptic curve with split multiplicative reduction. Any elliptic curve over K with split multiplicative reduction arises in this manner, and indeed one can prove that over $A[[q]]$ the Tate curve is the universal deformation of $(C_1, 1, +)_k$ for complete local noetherian A-algebras.

Proposition 19.1. *The elliptic curve* $\mathrm{Tate}(q^n)|_{\mathbf{Z}((q))}$ *extends uniquely to a generalized elliptic curve over* $\mathbf{Z}[[q]]$ *with n-gon geometric fibers over $q = 0$.*

To explain this we describe another construction of the Tate curve via formal schemes and algebraization, following the treatment in Brian's paper "Arithmetic moduli of generalized elliptic curves". Fix an integer $n > 1$, and define formal annuli $\Delta_i = \mathrm{Spf}\, Z[[q]]\{\{X_i, Y_i\}\}/(X_iY_i - q)$ for $i \in \mathbf{Z}/n\mathbf{Z}$ (here the curly braces denote restricted power series, i.e., power series whose coefficients tend to 0 in the total degree). The formal n-gon Tate curve $\widehat{\underline{\mathrm{Tate}}}_n$ over $\mathbf{Z}[[q]]$ is defined by gluing the subschemes $\Delta_i\{1/Y_i\}$ and $\Delta_{i+1}\{1/X_{i+1}\}$ along $Y_i = X_{i+1}$ and $X_i = Y_{i+1}$. There is an action of $\mathbf{Z}/n\mathbf{Z}$ on $\widehat{\underline{\mathrm{Tate}}}_n$ by "rotation", and $\widehat{\underline{\mathrm{Tate}}}_1$ is defined to be the quotient of $\widehat{\underline{\mathrm{Tate}}}_n$ by this action (this is independent of n).

This action of $\mathbf{Z}/n\mathbf{Z}$ also guarantees that any infinitesimal neighborhood of $\widehat{\underline{\mathrm{Tate}}}_n$ has the structure of a generalized elliptic curve. This is because of the following (see Corollary 2.2.3 of Brian's paper).

Proposition 19.2. *Let $C \to S$ be a stable genus-1 curve, and let $D \hookrightarrow C$ be an S-ample relative effective Cartier divisor supported in C^{sm}. Assume that D is a commutative S-group scheme and that there is given an action of D on C that extends the group scheme structure on D.*

Suppose furthermore that on geometric fibers the action of $D_{\overline{s}}$ on $C_{\overline{s}}$ extends to the structure of a generalized elliptic curve. Then the action of D on C extends to a generalized elliptic curve structure on C.

Now the infinitesimal neighborhoods $\widehat{\underline{\mathrm{Tate}}}_n \times \mathbf{Z}[[q]]/q^i$ form a compatible system of generalized elliptic curves (with a compatible collection of embeddings of $\mathbf{Z}/n\mathbf{Z}$, which yield relatively ample effective Cartier divisors). By Grothendieck's formal GAGA and existence theorems this setup algebraizes to a proper flat $\mathbf{Z}[[q]]$-scheme $\underline{\mathrm{Tate}}_n$ with n-gon special fibers, and carrying a compatible action of $\mathbf{Z}/n\mathbf{Z}$. As in the Proposition the action once again extends to a generalized elliptic curve structure on $\underline{\mathrm{Tate}}_n$.

The main point now is to check that $\underline{\mathrm{Tate}}_n$ restricts over $\mathrm{Spec}\,\mathbf{Z}((q))$ to $\underline{\mathrm{Tate}}_1 \otimes_{\mathbf{Z}[[q^n]]} \mathbf{Z}((q))$. (That this restriction has at most one extension with n-gon geometric fibers follows from a somewhat general principle.) This follows again by formal GAGA and considerations involving the action of $\mathbf{Z}/n\mathbf{Z}$.

We mention two further facts.

- There exists a unique map $\mu_N \hookrightarrow \mathrm{Tate}(q)^{\mathrm{sm}}$ lifting $\mu_N \hookrightarrow C_1^{\mathrm{sm}}$.
- $\mathrm{Tate}(q^n)|_{\mathbf{Q}((q))}$ has minimal regular proper model over $\mathbf{Q}[[q]]$ with n-gon fiber.

Now let us turn to differentials on Tate curves. There exist unique $X(t,q) \in \frac{t}{1-t^2} + \mathbf{Z}[t + t^{-1}][[q]]$ and $Y(t,q) \in \frac{t}{1-t^3} + \mathbf{Z}[t+t^{-1}][[q]]$ so that $[X : Y : 1]$ (along with $1 \mapsto [0:1:0]$) defines a map $\mathbf{G}_m \to \mathrm{Tate}(q) \subset \mathbf{P}^2_{\mathbf{Z}[[q]]}$. This map induces an isomorphism of formal groups over $\mathbf{Z}[[q]]$:

$$\widehat{\mathbf{G}}_m \xrightarrow{\sim} \mathrm{Tate}(q)^\wedge_0$$

sending $\frac{dt}{t} \mapsto \frac{dX}{2Y+X}$. (These are both invariant differentials and they agree at the identity! We remark that in the construction of X, Y, one can think of $X + \frac{1}{12} = \frac{1}{(2\pi i)^2}\wp$ and $2Y + X = \frac{1}{(2\pi i)^3}\wp'$, so that morally speaking "$\frac{dX}{2Y+X} = 2\pi i dz$.") But does $\frac{dX}{2Y+X} \in \Omega^1(\mathrm{Tate}(q)^{\mathrm{sm}})$ trivialize some line bundle over all of $\mathrm{Tate}(q)$?

To address this last question we need a digression about duality. For a semistable curve $C \to S$ (that is, a finite presentation and flat map such that all $C_{\overline{s}}$ are reduced of dimension 1 with formal singularities isomorphic to $k[[u,v]]/(uv)$) there's an associated *invertible* "dualizing sheaf" $K_{C/S}$ such that:

(i) $K_{C/S}|_{C^{\mathrm{sm}}} \cong \Omega^1_{C^{\mathrm{sm}}/S}$ canonically,
(ii) Its formation commutes with base change on S, in a manner such that étale locally on C it s compatible with the isomorphism in (i), and
(iii) For $S = \operatorname{Spec} k$ with k algebraically closed, and a normalization $\pi : \widetilde{C} \to C$, we have

$$K_{C/S}(U) = \{\eta \in \Omega^1(U^{\mathrm{sm}}) \colon \forall\, x \in U^{\mathrm{sing}} \text{ with } \pi^{-1}(x) = \{\widetilde{x}_1, \widetilde{x}_2\} \text{ we have } \operatorname{Res}_{\widetilde{x}_1}(\eta) + \operatorname{Res}_{\widetilde{x}_2}(\eta) = 0\}.$$

Theorem 19.3. *If $C \to \operatorname{Spec} k$ is proper, semistable, and connected, then it's stable genus 1 if and only if $K_{C/k} \cong \mathscr{O}_C$.*

This theorem underlies why "stable genus-1" is an open condition on the base of a proper flat family, and also the proof of the following.

Theorem 19.4. *For a generalized elliptic curve $f : E \to S$, the sheaf $\omega_{E/S} := f_*(K_{E/S})$ is an invertible sheaf whose formation commutes with base change, and satisfies*

- $f^*\omega_{E/S} \xrightarrow{\sim} K_{E/S}$,
- $\omega_{E/S} \xrightarrow{\sim} e^*K_{E/S} = e^*(\Omega^1_{E^{\mathrm{sm}}/S})$.

Continuing our discussion of the Tate curve: the sheaf $\omega_{\mathrm{Tate}(q)/\mathbf{Z}[[q]]}$ has a unique trivializing section inducing "$\frac{dt}{t}$" along the identity in $\widehat{\mathbf{G}}_m \cong \mathrm{Tate}_0^{\wedge}$. Over $\mathbf{Z}((q))$ it's $\frac{dX}{2Y+X}$, so ditto over $\mathrm{Tate}^{\mathrm{sm}}$. We write $\frac{dt}{t}$ to denote this global section over $\mathrm{Tate}(q)$.

20. Moduli of generalized elliptic curves

Deligne–Rapoport define versions of all of our moduli problems for generalized elliptic curves (over $\mathrm{Sch}(\mathbf{Z}[\frac{1}{n}])$), and make moduli spaces X as algebraic spaces of finite type over $\mathbf{Z}[\frac{1}{n}]$ such that:

(1) Rigidity for $\Gamma_1(N)$ needs $N \geq 5$ (because a 2-gon with $\Gamma_1(4)$-structure has a nontrivial automorphism).
(2) Artin's method is used to show that $X \to \operatorname{Spec} \mathbf{Z}[\frac{1}{n}]$ has the same properties as the corresponding Y, except proper rather than affine. Properness is proved by checking the valuative criterion, which comes down to the semistable reduction theorem.
(3) The open $Y \hookrightarrow X$ is the complement of the zero locus of an ideal $\mathscr{I}_\infty$ whose local generator is a non-zerodivisor on fibers over $\operatorname{Spec} \mathbf{Z}[\frac{1}{n}]$, with $Z(\mathscr{I}_\infty) \to \operatorname{Spec} \mathbf{Z}[\frac{1}{n}]$ having *reduced* geometric fibers. This is proved via deformation theory, and implies that $Y_{\overline{\mathbf{F}}_p}$ is connected if $X_{\overline{\mathbf{F}}_p}$ is connected (for $p \nmid N$).
(4) The j-invariant map $j : Y \to \mathbf{A}^1$ extends to $j : X \to \mathbf{P}^1_{\mathbf{Z}[\frac{1}{n}]}$ quasi-finite and flat, hence finite by Zariski's main theorem for algebraic spaces, and this implies that X is a scheme.

If M is the analytic moduli space corresponding to Y, there is a diagram

$$\begin{array}{ccc} Y^{\mathrm{an}}_{\mathbf{C}} & \xrightarrow{\sim} & M \\ \big\downarrow & & \big\downarrow \\ X^{\mathrm{an}}_{\mathbf{C}} & \xrightarrow{\sim} & \overline{M} \end{array}$$

where the left-hand inclusion has finite complement and the right-hand inclusion is the classical compactification.

Example 20.1. $X_1(N)_{\mathbf{C}}$ is connected, so the normal proper flat map $X_1(N) \to \operatorname{Spec} \mathbf{Z}[\frac{1}{N}]$ is its own Stein factorization. It follows that all $X_1(N)_{\overline{\mathbf{F}}_p}$ are connected, and therefore all the $Y_1(N)_{\overline{\mathbf{F}}_p}$ are also connected.

Definition 20.2. Choose $N \geq 5$, and a field $K \supset \mathbf{Q}(\zeta_N)$ (e.g. $K = \mathbf{C}$ and $\zeta_N = e^{2\pi i/N}$. Then the Néron 1-gon C_1 together with the point $\zeta_N \in \mu_N$ defines a point $\infty : \operatorname{Spec}(K) \to X_1(N)$, or equivalently $\infty \in X_1(N)_K(K)$. This lifts to $(\mathrm{Tate}(q), \zeta_N) : \operatorname{Spec} K[[q]] \to X_1(N)_K$.

Let $X = X_1(N)$, $N \geq 5$, and write $\mathbb{E} \to X$ for the universal structure.

Theorem 20.3. *The K-algebra map $\mathscr{O}^{\wedge}_{X_K,\infty} \to K[[q]]$ is an isomorphism.*

Proof. Both sides are complete discrete valuation rings with residue field K, so it is enough to check surjectivity mod q^2. If this were to fail, Tate mod q^2 would be the trivial deformation of $(C_1, +)$ (why?), and this is a contradiction. ■

Remark 20.4. The actual residue field at $\infty \in X_{\mathbf{Q}}$ is $\mathbf{Q}(\zeta_N)^+$, and $\mathbb{E}$ over $\mathscr{O}^{\wedge}_{X_{\mathbf{Q}},\infty}$ is a "twisted" Tate curve over $\mathbf{Q}(\zeta_N)^+[[q]]$.

Fix $i = \sqrt{-1}$ and set $\zeta_N = e^{2\pi i/N} \in \mu_N(\mathbf{C})$. This fixes $\mathbf{Q}(\zeta_N) \hookrightarrow \mathbf{C}$, and gives a point $\infty \in X(\mathbf{C})$. For the line bundle $\omega := \omega_{\mathbb{E}/X}$ on X we have, via $\operatorname{Spec} \mathbf{C}[[q]] \to X_{\mathbf{C}}$:

$$(\omega^{\otimes k}_{\mathbf{C},\infty})^{\wedge} \simeq \mathscr{O}^{\wedge}_{X_{\mathbf{C}},\infty} \left(\frac{dt}{t}\right)^{\otimes k} = \mathbf{C}[[q]] \left(\frac{dt}{t}\right)^{\otimes k} \tag{20.1}$$

sending (the image of) $f \in \Gamma(X_{\mathbf{C}}, \omega^{\otimes k}_{\mathbf{C}})$ to $f_\infty \left(\frac{dt}{t}\right)^{\otimes k}$.

How does this formal algebraic identification (20.1) relate to the $\mathbf{C}$-analytic theory? Via the *canonical* isomorphism of $Y^{\mathrm{an}}_{\mathbf{C}}$ with the analytic moduli space, we get

$$\begin{array}{ccccc} \mathrm{Tate}^{\mathrm{an}} = (\mathbf{C}^\times \times \Delta^*)/(q^{\mathbf{Z}} \times \Delta^*) & \longrightarrow & & & \mathbb{E}^{\mathrm{an}}_{\mathbf{C}} \\ \downarrow & & & & \downarrow \\ \Delta^* & \xrightarrow{\ f\ } & Y^{\mathrm{an}}_{\mathbf{C}} & \hookrightarrow & X^{\mathrm{an}}_{\mathbf{C}} \end{array}$$

But f is injective, so is an open immersion, and we know it lands in an explicit bounded part of $Y^{\mathrm{an}}_{\mathbf{C}}$. So it has no singularities, and therefore extends to $\overline{f} : \Delta \hookrightarrow X^{\mathrm{an}}_{\mathbf{C}}$. There is a separated smooth Δ-group $\overline{\mathrm{Tate}}^{\mathrm{an}} = (\mathbf{C}^\times \times \Delta)/(q^{\mathbf{Z}} \times \Delta^\times)$ with $\mathbf{C}^\times$-fiber at the origin, and then we have the following two key propositions.

Proposition 20.5. *We have $\overline{f}(0) = \infty$, and there exists a unique Δ-group isomorphism $(\mathbb{E}^{\mathrm{sm}}_{\mathbf{C}})^{\mathrm{an}} \simeq \overline{\mathrm{Tate}}^{\mathrm{an}}$ extending the one we have over Δ^*.*

This carries dw/w on the right-hand side to a trivializing section of $\omega^{\mathrm{an}}_{\mathbf{C}} = (e^*\Omega^1_{\mathbb{E}^{\mathrm{sm}}/X})^{\mathrm{an}}$ inducing dt/t in $(\omega^{\mathrm{an}}_{\mathbf{C},\infty})^{\wedge}$ via (20.1) for $k = 1$.

Proof. We have $\overline{f}(0) = \infty$ via continuity by inspection on $X^{\mathrm{an}}_{\mathbf{C}}$. To extend this to a Δ-group isomorphism, use "relative exp uniformization with multiplicative degeneration". To check that dt/t corresponds to dw/w, it's enough to show that the composite map of formal $\mathbf{C}[[q]]$-groups

$$\begin{array}{ccccccccc} \widehat{\mathbf{G}}_m & \xrightarrow[X(t,q)\ldots]{\sim} & \mathrm{Tate}(q)^{\wedge}_0 & \xrightarrow[\text{def thy}]{\sim} & (\mathbb{E}^{\mathrm{sm}}_{\mathbf{C}})^{\wedge}_{e(\infty)} & \xrightarrow[\varprojlim \bmod q^n]{\sim} & (\overline{\mathrm{Tate}}^{\mathrm{an}})^{\wedge}_0 & \xrightarrow[\exp]{\sim} & \widehat{\mathbf{G}}_m \\ \frac{dt}{t} & \longmapsto & \frac{dX}{2Y+X} & \longmapsto & \text{“}\frac{dt}{t}\text{”} & \overset{?}{\Longleftrightarrow} & \frac{dw}{w} & \longleftarrow\!\shortmid & \frac{dw}{w} \end{array}$$

is the identity map. Modulo each q^n, the map arises from an automorphism of $(\mathbf{G}_m \bmod q^n)^{\mathrm{an}}$, so this is a *sign* problem. But it respects ζ_N's via the universal $\Gamma_1(N)$-structure! ■

Proposition 20.6. *The map*

$$\mathbf{C}[[q]] \xleftarrow[(20.3)]{\sim} \mathscr{O}^{\wedge}_{X_{\mathbf{C}},\infty} \xrightarrow[\text{can.}]{\sim} \mathscr{O}^{\wedge}_{X^{\mathrm{an}}_{\mathbf{C},\infty}} \xleftarrow[\overline{f}]{\sim} \mathscr{O}^{\wedge}_{\Delta,0} = \mathbf{C}[[q]]$$

is the identity.

Proof. The Kodaira–Spencer map has $\mathrm{KS}\left(\left(\frac{dt}{t}\right)^{\otimes 2}\right) = \frac{dq}{q}$, so if $q \mapsto q'$ then Proposition 20.5 gives $\frac{dq'}{q'} = \frac{dq}{q}$, and therefore $q' = cq$ for some $c \in \mathbf{C}^\times$. Why does c equal 1? Use that $\mathrm{Tate}(q)$ mod q^2 is the universal first-order deformation. ■

Define $M_k(\Gamma_1(N))$ to be $\Gamma(X_{\mathbf{C}}, (\omega_{\mathbf{C}})^{\otimes k})$. Using a version of the above at all cusps, we get the following.

Corollary 20.7.

(1) $\Gamma(X_{\mathbf{C}}^{\mathrm{an}}, (\omega_{\mathbf{C}}^{\mathrm{an}})^{\otimes k}) = M_k(\Gamma_1(N))$ *inside* $\Gamma(Y_{\mathbf{C}}^{\mathrm{an}}, (\omega_{\mathbf{C}}^{\mathrm{an}})^{\otimes k}) = \{$*modular forms without cusp conditions*$\}$.

(2) *There is a commutative diagram*

$$\begin{array}{ccc} \Gamma(X_{\mathbf{C}}, \omega_{\mathbf{C}}^{\otimes k}) & \xrightarrow[\text{GAGA}]{\sim} & \Gamma(X_{\mathbf{C}}^{\mathrm{an}}, (\omega_{\mathbf{C}}^{\mathrm{an}})^{\otimes k}) \\ {\scriptstyle \left(\frac{dt}{t}\right)^{\otimes k}-\mathit{coeff.}} \searrow & & \swarrow {\scriptstyle \text{classical } q-\text{exp}} \\ & \mathbf{C}[[q]] & \end{array}$$

via (20.1)*, so we have* algebraic *q-expansions over* $\mathbf{C}$*.*

Reading 20.8. Now see the "Algebraic q-expansions" handout for a proof of the q-expansion principle for $M_k(\Gamma_1(N), A)$ with A any $\mathbf{Z}[\frac{1}{N}, \zeta_N]$-algebra, as well as a discussion of the bounded denominators problem that we posed in the first lecture.