Math 248B. Picard functors for curves

1. Motivation

Let S be a locally noetherian scheme, and let $f : X \to S$ be a proper flat map with geometrically integral fibers such that Zariski-locally on S it is projective. In particular, $\mathscr{O}_S \simeq f_*\mathscr{O}_X$ via the natural map. Assume there is given a section $e \in X(S)$. (In what follows the section e can be dropped at the expense of introducing a certain fppf-sheafification. This is very important in practice, since one wants a theory of Jacobians for curves without rational points. But we do not get into that issue here.)

In this handout, we discuss Grothendieck's main results concerning representatibility of the so-called "relative Picard functor" $\mathbf{Pic}_{X/S,e}$, the definition of which we will give shortly. We will then flesh out his conclusions in the special case of S-curves of genus $g > 0$, which is what we need (for $g = 1$).

2. Definition and main theorem

Let's begin by defining the relative Picard functor for $(f : X \to S, e \in X(S))$ as above, and then stating Grothendieck's existence theorem for such functors.

Definition 2.1. The *relative Picard functor* $\mathbf{Pic}_{X/S,e}$ is the contravariant functor on S-schemes that carries any S-scheme S' to the group of isomorphism classes of pairs $(\mathscr{L}', i')$ consisting of an invertible sheaf $\mathscr{L}'$ on $X_{S'}$ and an isomorphism $i' : \mathscr{O}_{S'} \simeq e_{S'}^*\mathscr{L}$ (an "$e_{S'}$-rigidification"). The contravariant functoriality is defined via base change, and (as in the analytic case) this is naturally a functor valued in abelian groups.

Grothendieck also gave a definition without requiring the existence of e, and that is very useful for the study of Jacobians of curves without a rational point, but we are ultimately going to apply the theory to elliptic curves and so will not dwell on the avoidance of e here. The fundamental result is this:

Theorem 2.2 (Grothendieck)**.** *The functor $\mathbf{Pic}_{X/S,e}$ is represented by an S-group scheme $\mathrm{Pic}_{X/S,e}$ locally of finite type.*

When a projective embedding $X \hookrightarrow \mathbf{P}^N_S$ is given, then $\mathrm{Pic}_{X/S,e} = \coprod_\Phi \mathrm{Pic}^\Phi_{X/S,e}$ where $\Phi \in \mathbf{Q}[t]$ varies through the "numeric polynomials" (i.e., $\Phi(\mathbf{Z}) \subseteq \mathbf{Z}$) and $\mathrm{Pic}^\Phi_{X/S,e}$ is a quasi-projective S-scheme representing the subfunctor $\mathbf{Pic}^\Phi_{X/S,e}$ assigning to any S-scheme S' the set of isomorphism classes of pairs $(\mathscr{L}', i')$ over $X_{S'}$ such that for all $s' \in S'$ the line bundle $\mathscr{L}'_{s'}$ on $X_{s'} \subseteq \mathbf{P}^N_{k(s')}$ has Hilbert polynomial Φ; that is,

$$\chi(\mathscr{L}'_{s'}(n)) := \sum_{j\geq 0} (-1)^j \dim_{k(s')} \mathrm{H}^j(X_{s'}, \mathscr{L}'_{s'}(n)) = \Phi(n)$$

for all $n \in \mathbf{Z}$.

Concretely, over $X \times_S \mathrm{Pic}_{X/S,e}$ there is a line bundle $\mathscr{L}^{\mathrm{univ}}$ equipped with a trivialization ι^{univ} along $e_{\mathrm{Pic}_{X/S,e}}$ that makes the pair $(\mathscr{L}^{\mathrm{univ}}, \iota^{\mathrm{univ}})$ universal. We emphasize that even for curves over an algebraically closed field, Grothendieck's Picard scheme is gigantic. It has infinitely many connected components, parameterized by a discrete invariant of a line bundle: the degree. (This will be deduced later from the above abstract general theorem.)

Grothendieck's construction sheds very little light on the geometric of $\mathrm{Pic}_{X/S,e}$, and even for the Picard scheme of a smooth projective variety it is a highly nontrivial matter to understand which geometric points lie in the identity component, and so on. In positive characteristic, already for smooth projective surfaces the problem of determining the smoothness or not of the Picard scheme is a subtle issue; Mumford's book "Lectures on curves on an algebraic surface" is more or less about how to solve this problem.

In this handout, we focus on the case of relative smooth curves, for which things work out very nicely. But we emphasize that even for curves, Grothendieck's result has some major defects: for the study of families of curves we really want to allow degeneration away from smoothness (such as nodal singularities, as in the semistable reduction theorem), but the geometric integrality conditions in Grothendieck's theorem rule out most such degeneration. Artin's theory of algebraic spaces is required in order to get a truly satisfactory geometric understanding of Picard functors in a generality which is well-suited to the way families of curves are used in practice.

3. Dimension and smoothness

Before we turn our attention to curves, we focus on the problem of smoothness and relative dimension for Picard schemes. As a warm-up, we compute the tangent space.

Proposition 3.1. *Let (X, e) be a projective geometrically integral scheme over a field k equipped with a rational point. Then $\mathrm{Pic}_{X/k,e}$ has tangent space at the identity naturally identified with $\mathrm{H}^1(X, \mathscr{O}_X)$ as a k-vector space.*

The projectivity and geometric integrality hypotheses play no role except through their necessity in Grothendieck's existence theorem for Picard schemes. Once one has Artin's more general results via algebraic spaces (which only require the fibers to be geometrically connected and geometrically reduced), the argument below carries over verbatim to give the same results in that setting.

Proof. For a *local* k-algebra R, every line bundle $\mathscr{L}$ on X_R has $e_R^*(\mathscr{L})$ trivial, since R has no nontrivial line bundles. All such trivializations are related through $R^\times$-scaling, so in fact $\mathrm{Pic}_{X/k,e}(R) = \mathrm{Pic}(X_R)$ as groups (functorially in such R). For any locally finite type k-scheme Z and $z \in Z(k)$, the set $\mathrm{Tan}_z(Z)$ is identified with the fiber of $Z(k[\epsilon]) \to Z(k)$ over z, and the k-action is induced by k-scaling on ϵ. In the special case that Z is a k-group and z is the identity point, the resulting identification of $\mathrm{Tan}_z(Z)$ with the kernel of the group homomorphism $Z(k[\epsilon]) \to Z(k)$ is in fact the additive structure on the tangent space. (Why?) Thus, we conclude that

$$\mathrm{Tan}_0(\mathrm{Pic}_{X/k,e}) = \ker(\mathrm{Pic}(X_{k[\epsilon]}) \to \mathrm{Pic}(X))$$

as additive groups, and the k-action is encoded in k-scaling on ϵ. We have therefore entirely encoded our problem in terms of Picard groups.

Since $X_{k[\epsilon]}$ has the same underlying space as X, we naturally identify $\mathscr{O}_{X_{k[\epsilon]}}$ with $\mathscr{O}_X[\epsilon]$, so the exact sequence of abelian sheaves

$$0 \to \mathscr{O}_X \stackrel{1+\epsilon s}{\to} \mathscr{O}_X[\epsilon]^\times \to \mathscr{O}_X^\times \to 1$$

("algebraic exponential sequence") induces an exact sequence of groups

$$\to k \to k[\epsilon]^\times \to k^\times \to \mathrm{H}^1(X, \mathscr{O}_X) \to \mathrm{Pic}(X_{k[\epsilon]}) \to \mathrm{Pic}(X).$$

But the initial three terms form a short exact sequence, so the kernel of interest to us is naturally identified with the underlying additive group of $\mathrm{H}^1(X, \mathscr{O}_X)$. It remains to check that the k-action is compatible. In terms of Čech theory, if $\mathfrak{U} = \{U_i\}$ is a finite open cover of X and $\{s_{ij}\}$ is a 1-cocycle in $Z^1(\mathfrak{U}, \mathscr{O}_X)$, the corresponding class in $\mathrm{Pic}(X_{k[\epsilon]})$ is the line bundle arising from the 1-cocycle $\{1 + \epsilon s_{ij}\}$. The k-action on $\mathrm{H}^1(X, \mathscr{O}_X)$ is induced by the k-action on $\mathscr{O}_X$ (why?), and for any $c \in k$ clearly $1 + \epsilon(cs_{ij}) = 1 + (c\epsilon)s_{ij}$. Thus, the compatibility of k-actions is proved. ■

In general, if Z is a locally finite type k-scheme and $z \in Z(k)$, then $\dim_k \mathrm{Tan}_z(Z) \geq \dim \mathscr{O}_{Z,z}$, with equality if and only if Z is smooth at z (since regularity at a rational point is equivalent to smoothness at such a point, due to the structure of the completed local ring at such a point in the regular case). Thus, if Z is a k-group scheme then $\dim_k \mathrm{Tan}_e(Z) \geq \dim Z$ with equality if and only if Z is smooth. This proves:

Corollary 3.2. *Under the above hypotheses,* $\dim \mathrm{Pic}_{X/k,e} \leq \dim_k \mathrm{H}^1(X, \mathscr{O}_X)$ *with equality if and only if* $\mathrm{Pic}_{X/k,e}$ *is smooth.*

In general, a locally finite type k-group is smooth when $\mathrm{char}(k) = 0$ (Cartier's theorem), but if $\mathrm{char}(k) > 0$ then smoothness of the Picard scheme is a delicate matter. For curves, smoothness always works out, as we explain in the next section.

4. Curves

Now let $f : X \to S$ be an S-curve, with fibers of genus $g > 0$. Suppose there is given a section $e \in X(S)$. By the handout on base change and cohomology, Zariski-locally on S the map f is projective. Hence, $\mathrm{Pic}_{X/S,e}$ always exists. The subfunctors $\mathbf{Pic}^\Phi_{X/S,e}$ in Theorem 2.2 have a concrete interpretation for curves, as follows. Consider a smooth proper connected curve C of genus g over an algebraically closed field k. For any line bundle $\mathscr{L}$ on C, what can we say about the Hilbert polynomial of $\mathscr{L}$ relative to a projective embedding $C \hookrightarrow \mathbf{P}^N_k$? That is, what is the linear polynomial $\chi(\mathscr{L}(n))$? by Riemann–Roch, this is $\deg(\mathscr{L}(n)) + 1 - g = \deg(\mathscr{L}) + \deg(\mathscr{O}_C(1))n + 1 - g = Dn + (\deg(\mathscr{L}) + 1 - g)$ where D is the degree of C as a curve in $\mathbf{P}^N_k$. Since D has nothing to do with $\mathscr{L}$ and involves just the fixed degree of the projective embedding, as we vary through the Hilbert polynomials Φ we are really just varying through the possibilities for the degree $\deg(\mathscr{L})$ of $\mathscr{L}$.

In other words, the open and closed subschemes $\mathrm{Pic}^{\Phi}_{X/S,e}$ have a global meaning (even without reference to the projective embedding): we can relabel these as $\mathrm{Pic}^{d}_{X/S,e}$ according to the degree $d = \deg_{X_s}(\mathscr{L}_s)$ on the fibers of $X \to S$. Intrinsically, this degree is the linear coefficient in the degree-1 polynomial function $\chi(\mathscr{L}_s^{\otimes n})$ of n, and so we obtain most of:

Proposition 4.1. *For any integer d, let $\mathbf{Pic}^d_{X/S,e}$ denote the subfunctor of $\mathbf{Pic}_{X/S,e}$ consisting of pairs $(\mathscr{L}, i)$ such that $\mathscr{L}$ has degree d on all geometric fibers. Then each of these is represented by a finite type S-scheme $\mathrm{Pic}^d_{X/S,e}$ that is quasi-projective Zariski-locally on S, and $\coprod \mathrm{Pic}^d_{X/S,e}$ represents $\mathbf{Pic}_{X/S,e}$.*

The open and closed S-subgroup $\mathrm{Pic}^0_{X/S,e}$ has as its geometric points exactly those of degree 0.

Proof. We just have to prove the final assertion, for which we can pass to the case $S = \mathrm{Spec}\, k$ for an algebraically closed field k and just have to show that all degree-0 points of $\mathrm{Pic}_{X/k,e}(k)$ lie in the identity component. The classical Riemann-Roch argument shows that every degree-0 divisor on X has the form $D - ge$ for an effective divisor D of degree g. Hence, under the k-scheme map

$$X^g \to \mathrm{Pic}_{X/k,e}$$

defined functorially by $(x_1, \dots, x_g) \mapsto \mathscr{I}_{x_1}^{-1} \otimes \cdots \otimes \mathscr{I}_{x_g}^{-1} \otimes \mathscr{I}_e^{n}$ hits exactly the degree-0 locus. But X^g is connected and so all degree-0 points lie in the identity component. Since we know that conversely the identity component consists entirely of degree-0 points (in view of the general decomposition of $\mathrm{Pic}_{X/k,e}$ into a disjoint union according to degree), we are done. ■

Remark 4.2. In view of the description of the identity component of the Picard scheme, the notation $\mathrm{Pic}^0_{X/S,e}$ is unambiguous in the case of S-curves.

Proposition 4.3. *The S-group $\mathrm{Pic}_{X/S,e}$ is smooth, with fibers of dimension g.*

Proof. We appeal to the functorial criterion for smoothness: for a surjective map $A \to A_0$ of artin local rings (with $\mathrm{Spec}\, A$ an S-scheme), we want to prove that the map $\mathrm{Pic}_{X/S,e}(A) \to \mathrm{Pic}_{X/S,e}(A_0)$ is surjective. Since A and A_0 are local, by arguing exactly as in the proof of Proposition 3.1 we can replace S with $\mathrm{Spec}\, A$ and identity the map of interest with the reduction map of groups $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ (where $X_0 = X \otimes_A A_0$). By induction on the length of the kernel I of $A \twoheadrightarrow A_0$, we can assume that I is killed by the maximal ideal of A, so it may be viewed as a vector space of finite dimension over the residue field k of A.

Since X is A-flat, tensoring the A-linear sequence $0 \to I \to A \to A_0 \to 0$ against $\mathscr{O}_X$ yields an exact sequence

$$0 \to I \otimes_k \mathscr{O}_{X_k} \to \mathscr{O}_X \to \mathscr{O}_{X_0} \to 0$$

(where X_k denotes the special fiber. Hence, on unit groups we have an exact sequence

$$0 \to I \otimes_k \mathscr{O}_{X_k} \to \mathscr{O}_X^{\times} \to \mathscr{O}_{X_0}^{\times} \to 1$$

in which the first map is $s \mapsto 1 + s$. Thus, passing to abelian sheaf cohomology gives an exact sequence

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_0) \to \mathrm{H}^2(X, I \otimes_k \mathscr{O}_{X_k}) = \mathrm{H}^2(X_k, I \otimes_k \mathscr{O}_{X_k}) = I \otimes_k \mathrm{H}^2(X_k, \mathscr{O}_{X_k}).$$

We conclude that the Picard scheme is smooth whenever the fibral structure sheaves have vanishing H^2. This certainly happens for curves. In such cases, the dimension is equal to the genus, by Corollary 3.2. ■

We have proved that the S-group scheme $\mathrm{Pic}^0_{X/S,e}$ is smooth, and by construction it is finite type as well as separated (even quasi-projective locally over S). But so far we have just used that the geometric fiber curves are integral, not any smoothness properties. (In fact, singular irreducible curves give rise to non-proper

Pic^0's; the theory of "generalized Jacobians" of Lang–Rosenlicht rests on such examples.) Smoothness of the curves now plays an essential role:

Theorem 4.4. *The S-group $\mathrm{Pic}^0_{X/S,e}$ is proper, so it is an abelian scheme of relative dimension g.*

Proof. This S-group is known to be separated and of finite type (by construction), so by the valuative criterion it remains to check that if R is a discrete valuation ring with fraction field K and if there is given a map $\mathrm{Spec}\, R \to S$ then the map $\mathrm{Pic}^0_{X/S,e}(R) \to \mathrm{Pic}^0_{X/R}(K)$ is surjective. But $\mathrm{Spec}\, R$ is connected, so the disjoint union decomposition of $\mathrm{Pic}_{X/S,e}$ according to degrees of fibral line bundles allows us to remove consideration of the relative identity component: it suffices to prove that $\mathrm{Pic}_{X/S,e}(R) \to \mathrm{Pic}_{X/S,e}(K)$ is surjective. Since R and K are local, by replacing X with its pullback along $\mathrm{Spec}\, R \to S$ we can assume $S = \mathrm{Spec}\, R$ and it suffices to prove that the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_K)$ is surjective.

Now X is smooth over the regular local ring R, so X is regular. Moreover, it is irreducible (due to R-flatness and irreducibility of the generic fiber) and all closed points are in the special fiber (by R-properness), so such points are of codimension-2. It follows that all points distinct from the generic point of the generic fiber and the closed points of the special fiber are codimension-1. For any closed point $\xi \in X_K$, the schematic closure of ξ in X is a closed integral subscheme of codimension 1, and it is Cartier since all height-1 primes in a regular local ring are principal. Using the assignment of line bundles to Cartier divisors, and the fact that every line bundle on X_K arises from divisors of closed points, the desired surjectivity on Picard groups is established. ■