Math 248B. Vector bundles modulo relative lattices

1. Motivation

Let M be a complex manifold, $V \to M$ a holomorphic vector bundle of rank $g > 0$, and $L \to M$ a local system of finite free $\mathbf{Z}$-modules of rank $2g$ (see Exercise 2 in HW1). Suppose there is given a map of M-groups $j : L \to V$. We are interested in situations for which each fibral map $\mathbf{Z}^{2g} \simeq L_m \to V_m \simeq \mathbf{C}^g$ is a co-compact lattice (with inclusion depending on $m \in M$), and gluing the quotients V_m/L_m into a "total space" over M in a nice way. The case of most interest to us is $g = 1$, and in effect this is all a vast generalization of the Weierstrass construction

$$\mathbf{Z}^2 \times (\mathbf{C} - \mathbf{R}) \to \mathbf{C} \times (\mathbf{C} - \mathbf{R})$$

over $\mathbf{C} - \mathbf{R}$ defined by $((m, n), \tau) \mapsto (m\tau + n, \tau)$. As a warm-up, we present the higher-rank generalization.

Fix $i = \sqrt{-1} \in \mathbf{C}$ (to define "imaginary part"), and let $\mathfrak{h}_{g,i} \subset \mathrm{Mat}_{g\times g}(\mathbf{C})$ be the subset of $g \times g$ matrices Z that are symmetric and have imaginary part $\mathrm{im}(Z)$ which is positive-definite (in the sense that the corresponding symmetric bilinear form over $\mathbf{R}$ is positive-definite). This is the the *Siegel upper half-space*; for $g = 1$ it is the usual upper half-plane (the connected component of $\mathbf{C} - \mathbf{R}$ containing the chosen i). Obviously $\mathfrak{h}_{g,i}$ is an open locus in the vector space of symmetric $g \times g$ matrices, so it is naturally a complex manifold. Less obvious is the fact (which is trivial for $g = 1$) that $\mathfrak{h}_{g,i}$ is connected; we will not need it, so we do not prove it.

Remark 1.1. In general, $\mathfrak{h}_{g,i} \subset \mathrm{GL}_g(\mathbf{C})$. For $g = 1$ this is the obvious assertion that the half-planes in $\mathbf{C}-\mathbf{R}$ lie in $\mathbf{C}^\times$. In general we argue as follows. Pick $Z \in \mathfrak{h}_{g,i}$. We aim to prove that $\ker Z = 0$ (so Z is invertible). Using the standard hermitian form $\langle \vec{z}, \vec{w}\rangle = \sum z_j \overline{w}_j$ on $\mathbf{C}^g$,

$$\langle (Z - \overline{Z})(v), v\rangle = \langle Zv, v\rangle - \langle \overline{Z}v, v\rangle = \langle Zv, v\rangle - \langle v, Zv\rangle,$$

where the final equality uses the symmetry of Z. Thus, if $Zv = 0$ then the sesquilinear form $B(x, y) = \langle (Z - \overline{Z})(x), y\rangle$ on $\mathbf{C}^g$ satisfies $B(v, v) = 0$. But $Z - \overline{Z} = 2i\mathrm{im}(Z)$ with $\mathrm{im}(Z)$ positive-definite, so $B = 2iH$ where the sesquilinear form H arises from an inner product on $\mathbf{R}^g$. In particular, H is hermitian, so the vanishing of $H(v, v)$ forces $v = 0$.

Consider the map of $\mathfrak{h}_{g,i}$-groups $j : \mathbf{Z}^{2g} \times \mathfrak{h}_{g,i} \to \mathbf{C}^g \times \mathfrak{h}_{g,i}$ defined by $(\vec{m}, Z) \mapsto ((Z\ \ 1_g)\vec{m}, Z)$. (For $g = 1$, writing $\vec{m} = \binom{m}{n}$ recovers the Weierstrass construction.) We claim that for every $Z \in \mathfrak{h}_{g,i}$, the map on Z-fibers is a co-compact lattice inclusion. In other words, we claim that $j_Z : Z\mathbf{Z}^g \oplus \mathbf{Z}^g \to \mathbf{C}^g$ is injective and a co-compact lattice. To see this, the following lemma is convenient:

Lemma 1.2. *A homomorphism $j : L_0 \to V_0$ from a finite free $\mathbf{Z}$-module of rank $2g$ to a g-dimensional $\mathbf{C}$-vector space is a co-compact lattice inclusion if and only if the natural $\mathbf{C}$-linear map of $\mathbf{C}$-vector spaces $\mathbf{C} \otimes_{\mathbf{Z}} L_0 \to V_0 \oplus \overline{V}_0$ defined by $c \otimes \lambda \mapsto (cj(\lambda), c\overline{j(\lambda)})$ is an isomorphism, where $\overline{V}_0 = \mathbf{C} \otimes_{\sigma,\mathbf{C}} V_0$ is the conjugate space (scalar extension by complex conjugation σ) and $\overline{v} := 1 \otimes v$ for $v \in V_0$. Equivalently, upon choosing bases $L_0 \simeq \mathbf{Z}^{2g}$ and $V_0 \simeq \mathbf{C}^g$ to identify j with a $g \times 2g$ matrix $(A\ \ B)$ for $A, B \in \mathrm{Mat}_{g\times g}(\mathbf{C})$, the necessary and sufficient condition is that the matrix*

$$\begin{pmatrix} A & B \\ \overline{A} & \overline{B} \end{pmatrix} \in \mathrm{Mat}_{2g\times 2g}(\mathbf{C})$$

is invertible.

Beware that an abstract $\mathbf{C}$-vector space does not have an intrinsic "complex conjugation" operator (semi-linear over complex conjugation on $\mathbf{C}$), so V_0 and $\overline{V}_0$ are not naturally $\mathbf{C}$-linearly isomorphic. The natural map $V_0 \to \overline{V}_0$ defined by $v \mapsto \overline{v}$ is merely semilinear over the complex conjugation on $\mathbf{C}$. Likewise, if we choose a $\mathbf{C}$-basis $\{v_k\}$ of V_0 and use $\{\overline{v}_k\}$ as a $\mathbf{C}$-basis of $\overline{V}_0$ then the composition $C^g = V_0 \to \overline{V}_0 = \mathbf{C}^g$ is coordinate-wise complex conjugation.

Proof. The co-compact lattice condition is precisely that the $\mathbf{R}$-linear map $j_{\mathbf{R}} : \mathbf{R} \otimes_{\mathbf{Z}} L_0 \to V_0$ is an isomorphism, and this is equivalent to the isomorphism condition after applying scalar extension by $\mathbf{R} \to \mathbf{C}$. But $\mathbf{C} \otimes_{\mathbf{R}} V_0 = (\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}) \otimes_{\mathbf{C}} V_0$ with $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C} \times \mathbf{C}$ as $\mathbf{C}$-algebras via $a \otimes b \mapsto (ab, a\overline{b})$ (we view $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ as a $\mathbf{C}$-algebra using the left tensor structure). Hence, $\mathbf{C} \otimes_{\mathbf{R}} V_0$ is identified as a $\mathbf{C}$-vector space with $V_0 \oplus \overline{V}_0$, and in this way the $\mathbf{C}$-linear scalar extension of $j_{\mathbf{R}}$ is identified with the map introduced in the statement of the lemma. The matrix interpretation is immediate upon identifying the $2g \times 2g$ matrix as computing this $\mathbf{C}$-linear map relative to the $\mathbf{C}$-basis of $\mathbf{C} \otimes_{\mathbf{Z}} L_0$ coming from the chosen $\mathbf{Z}$-basis of L_0 and the C-basis of $V_0 \oplus \overline{V}_0$ coming from the chosen basis of V_0 and the corresponding conjugate basis of $\overline{V}_0$. ■

By this lemma, to prove that j_Z is a co-compact lattice inclusion it is equivalent to prove invertibility of the $2g \times 2g$ matrix

$$\begin{pmatrix} Z & 1_g \\ \overline{Z} & 1_g \end{pmatrix}.$$

Subtracting the bottom $g \times 2g$ block from the upper one gives

$$\begin{pmatrix} Z - \overline{Z} & 0 \\ \overline{Z} & 1_g \end{pmatrix},$$

so its invertibility is equivalent to that of the $g \times g$ matrix $Z - \overline{Z}$. This is exactly $2i\mathrm{im}(Z)$, and by definition of $\mathfrak{h}_{g,i}$ the real matrix $\mathrm{im}(Z)$ corresponds to a positive-definite symmetric bilinear form over $\mathbf{R}$. Such bilinear forms are non-degenerate, so their associated matrix is always invertible (over $\mathbf{R}$ or over $\mathbf{C}$, which come to the same thing). This proves that the $\mathfrak{h}_{g,i}$-group map j is a co-compact lattice inclusion on all fibers over $\mathfrak{h}_{g,i}$.

2. A quotient construction: C^∞-aspects

Now consider a general map of M-groups $j : L \to V$ where V is a rank-g vector bundle over a complex manifold M and L is a local system of rank-$2g$ finite free $\mathbf{Z}$-modules. We are interested in the cases when the fibral maps $j_m : L_m \to V_m$ are all co-compact lattice inclusions, but we first note that this property of the fibral map is open on the base:

Lemma 2.1. *If j_{m_0} is a co-compact lattice inclusion for some $m_0 \in M$, then the same holds for j_m for all m in an open neighborhood of m_0 in M.*

Proof. The problem is local around m_0, so we may shrink to acquire local frames for L and V. That is, we may arrange that the local system L is split and the vector bundle V is free, which is to say that there is an M-group isomorphism $L \simeq \mathbf{Z}^{2g} \times M$ and a vector bundle isomorphism $V \simeq \mathbf{C}^g \times M$ over M. Let $e_1, \dots, e_{2g} \in L(M)$ correspond to the splitting of L (i.e., $\{e_k(m)\}$ is the resulting basis of the fiber L_m for each $m \in M$), so the M-group map j amounts to the specification of the holomorphic sections $s_k = j \circ e_k : M \to V$. Relative to the chosen global frame $\{v_1, \dots, v_g\}$ in $V(M)$, we have $s_k = \sum a_{hk} v_h$ in $V(M)$ for holomorphic $a_{kh} : M \to \mathbf{C}$. Thus, j corresponds to the $g \times 2g$ matrix $T := (a_{hk})$ of holomorphic functions, and by Lemma 1.2 the hypothesis at m_0 is that the $2g \times 2g$ matrix $\left(\frac{T}{\overline{T}}\right)$ of continuous $\mathbf{C}$-valued functions is invertible at m_0. It is therefore invertible in a neighborhood of m_0. ■

Now suppose that all fibral maps j_m are co-compact lattice inclusions. We impose an equivalence relation on the set V as follows: $v \sim v'$ if v and v' lie over the same point $m \in M$ and if $v - v' \in L_m$ inside of V_m. In other words, we impose the fibral equivalence relation of congruence modulo the lattice L_m in V_m for every $m \in M$. Define the topological space V/L to be the quotient of V modulo $\sim$, so there is are natural continuous map $V \to V/L$ and $V/L \to M$. We then have the following purely topological result:

Proposition 2.2. *The map $\pi : V \to V/L$ is a covering space, and $V/L \to M$ is proper. There is a unique C^∞-structure relative to which π is a local C^∞-isomorphism, and then $V/L \to M$ is a submersion and even a split C^∞ fiber bundle with fiber $(S^1)^g$.*

The map π is a C^∞ quotient in the sense that any C^∞ map $V \to N$ that is $\sim$-invariant uniquely factors through π via a C^∞-map $V/L \to N$.

The careful reader will see that the proof is nothing but a slick generalization of the special case in class for $g = 1$ and the Weierstrass construction over $M = \mathbf{C} - \mathbf{R}$.

Proof. We may work locally over M, so as in the proof of Lemma 2.1 we can assume that $L = \mathbf{Z}^{2g} \times M$ and $V = \mathbf{C}^g \times M$ with j corresponding to a $g \times 2g$ matrix (a_{hk}) of holomorphic functions $T = (a_{hk})$ such that the $2g \times 2g$ matrix

$$\begin{pmatrix} T(m) \\ \hline \overline{T(m)} \end{pmatrix} \in \mathrm{Mat}_{2g\times 2g}(\mathbf{C})$$

is invertible for each $m \in M$. In particular, the top $g \times 2g$ matrix $T(m)$ is surjective as a linear map $\mathbf{C}^{2g} \to \mathbf{C}^g$, so by equality of row rank and column rank there is an invertible $g \times g$ submatrix. Working locally around some $m_0 \in M$, we may rearrange the order of the trivialization of L so that the right $g \times g$ submatrix of $T(m_0)$ is invertible. By shrinking around m_0 we can then assume that the right $g \times g$ submatrix of $T(m)$ is invertible for all $m \in M$. In other words, $j(e_{g+1}), \dots, j(e_{2g}) \in V(M)$ is a global frame, where $\{e_1, \dots, e_{2g}\}$ in $L(M)$ is the chosen trivialization of $L \to M$.

Writing $T = (A\ \ B)$ with $g \times g$ matrices A and B whose entries are holomorphic functions, we have arranged that B is invertible, so by modifying the initial isomorphism $V \simeq \mathbf{C}^g \times M$ via B^{-1} we can arrange that $T = (Z\ \ 1_g)$ for some holomorphic map $Z : M \to \mathrm{Mat}_{g\times g}(\mathbf{C})$. Consider the C^∞-map $\overline{Z} : M \to \mathrm{Mat}_{g\times g}(\mathbf{C})$. By Lemma 1.2, the fibral lattice condition says exactly that the C^∞-map

$$\begin{pmatrix} Z & 1_g \\ \overline{Z} & 1_g \end{pmatrix} : M \to \mathrm{Mat}_{2g\times 2g}(\mathbf{C})$$

is valued in $\mathrm{GL}_{2g}(\mathbf{C})$, or equivalently (by subtracting the bottom $g \times 2g$ block from the top one) that that $2i\mathrm{im}(Z) = Z - \overline{Z}$ is valued in $\mathrm{GL}_g(\mathbf{C})$. That is, $\mathrm{im}(Z)$ is invertible, which is to say $Z = A + iB$ for continuous maps $A, B : M \rightrightarrows \mathrm{Mat}_g(\mathbf{R})$ with B valued in $\mathrm{GL}_g(\mathbf{R})$. Consider the C^∞-map $\mathbf{C}^g \times M \to \mathbf{C}^g \times M$ over M defined by $(x + iy, m) \mapsto (x + (A(m) + iB(m))y, m) = ((x + A(m)y) + iB(m)y, m)$. This is clearly a C^∞-isomorphism, as the inverse is $(u + iv, m) \mapsto ((u - A(m)B(m)^{-1}v) + iB(m)^{-1}v, m)$. This isomorphism carries $j : \mathbf{Z}^{2g} \times M \to \mathbf{C}^g \times M$ over to the "constant" inclusion $\mathbf{Z}[i]^g \times M \to \mathbf{C}^g \times M$. Thus, topologically we see that V/L over M is precisely $(\mathbf{C}/\mathbf{Z}[i])^g \times M$ with $V \to V/L$ going over to the topological product of M against the g-fold product of the natural quotient map $\mathbf{C} \to \mathbf{C}/\mathbf{Z}[i] = S^1 \times S^1$ that is clearly a covering space map. Likewise, $V/L \to M$ is topologically the projection $(\mathbf{C}/\mathbf{Z}[i])^g \times M \to M$ which is visibly proper.

Using the natural C^∞-structure on $(\mathbf{C}/\mathbf{Z}[i])^g \times M$ then equips V/L with a C^∞-structure relative to which the quotient map π is a local C^∞-isomorphism since $\mathbf{C} \to \mathbf{C}/\mathbf{Z}[i]$ has that property. This C^∞-structure on V/L is uniquely determined by the property that π is a local C^∞-isomorphism, as this latter map is a surjective local homeomorphism. (The real issue is the existence of such a C^∞-structure, which we proved already.) The preceding calculations using C^∞-isomorphisms then also show that V/L admits a C^∞-isomorphism to $(\mathbf{C}/\mathbf{Z}[i])^g \times M = (S^1)^{2g} \times M$ over M.

Finally, we address the C^∞-quotient property of π. This amounts to proving that a C^∞-map $h : \mathbf{C}^g \times M \to N$ that is invariant by the $\mathbf{Z}[i]^g$-translation on $\mathbf{C}^g$ factors through $\pi : \mathbf{C}^g \times M \to (\mathbf{C}/\mathbf{Z}[i])^g \times M$ via a C^∞-map $\overline{h} : (\mathbf{C}/\mathbf{Z}[i])^g \times M \to N$. There is certainly a unique continuous factorization since π is a proper surjective map, and $\overline{h}$ is C^∞ because its composition with the surjective local C^∞-isomorphism π is the map h which is assumed to be C^∞. ■

3. A quotient construction: complex-analytic aspects

Before we enhance V/L to a complex manifold in a useful way, we record a fibral isomorphism criterion:

Lemma 3.1. *Let $X, Y \rightrightarrows S$ be C^∞ submersions between C^∞ manifolds, and let $f : X \to Y$ be a C^∞ map over S such that the induced map $f_s : X_s \to Y_s$ between fibers over each $s \in S$ is a C^∞ isomorphism. Then f is a C^∞ isomorphism. The same holds in the category of complex manifolds, using holomorphic maps.*

Proof. Clearly f is bijective, so it suffices to prove that it is a local isomorphism. By the inverse function theorem, it is equivalent to prove that for each $x \in S$ the derivative map $\mathrm{d}f(x) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ is an isomorphism. If $s \in S$ is that point over which x and $f(x)$ lie, the submersion property for the maps $X, Y \rightrightarrows S$ gives that the maps $\mathrm{T}_x(X), \mathrm{T}_{f(x)}(Y) \rightrightarrows \mathrm{T}_s(S)$ are surjective. Moreover, by the submersion theorem, the respective kernels are identified with $\mathrm{T}_x(X_s)$ and $\mathrm{T}_{f(x)}(Y_s)$. By the functoriality of derivative maps (i.e., the Chain Rule), the map $\mathrm{d}f(x)$ commutes with the quotient maps onto $\mathrm{T}_s(S)$ and carries $\mathrm{T}_x(X_s)$ to $\mathrm{T}_{f(x)}(Y_s)$ via $\mathrm{d}(f_s)(x)$. That is, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}_x(X_s) & \longrightarrow & \mathrm{T}_x(X) & \longrightarrow & \mathrm{T}_s(S) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{d}(f_s)(x)} & & \downarrow{\scriptstyle \mathrm{d}f(x)} & & \| & & \\ 0 & \longrightarrow & \mathrm{T}_{f(x)}(Y_s) & \longrightarrow & \mathrm{T}_{f(x)}(Y) & \longrightarrow & \mathrm{T}_s(S) & \longrightarrow & 0 \end{array}$$

But f_s is an isomorphism by hypothesis, so the left arrow is an isomorphism and hence so is the middle arrow. ■

Theorem 3.2. *There is a unique complex manifold structure on V/L relative to which $\pi : V \to V/L$ is a local analytic isomorphism, and this makes $V/L \to M$ a submersion.*

The map π is an analytic quotient in the sense that any holomorphic map $V \to N$ that is $\sim$-invariant uniquely factors through π via a holomorphic map $V/L \to N$, and if $h : M' \to M$ is a holomorphic map with $j' : L' \to V'$ the base change of $j : L \to V$ by h then the natural holomorphic map $V'/L' \to (V/L) \times_M M'$ over M' is an isomorphism. Moreover, if $\Lambda \to W$ is another such map over M then the resulting natural holomorphic map $(V \oplus W)/(L \oplus \Lambda) \to (V/L) \times_M (W/\Lambda)$ over M is an isomorphism.

Most of the proof of this theorem is formal nonsense. Only at the end of the argument does a real idea emerge. In effect, the main challenge of the argument is to be rigorous.

Proof. The analogous result has already been proved in the C^∞ setting, so in particular the underlying C^∞-structure from any such hypothetical complex-analytic structure on V/L would have to recover the one we have already built. Since the C^∞ uniqueness argument carries over to show that the analytic structure on V/L (if it exists!) would be uniquely characterized by requiring π to be a local analytic isomorphism, and the proof of the quotient mapping property carries over once we have found a complex-analytic structure on V/L making π a local analytic isomorphism, our problem reduces to enhancing the C^∞ manifold V/L to a complex manifold such that (i) π is holomorphic and even a local analytic isomorphism (so$V/L \to M$ is necessarily holomorphic, as its composition with the surjective local analytic isomorphism π is the map $V \to M$ that is holomorphic), (ii) the formation of this analytic structure is compatible base change on M and with fiber products over M as at the end of the statement of the theorem. (The submersion property can be verified on the underlying C^∞ manifolds, due to the consistency of C^∞ and complex-analytic tangent spaces.)

Let's grant (i) and prove (ii) conditional on this. Consider any holomorphic map $h : M' \to M$ and the natural holomorphic map $\phi : V'/L' \to (V/L) \times_M M'$ over M'. By Lemma 3.1, to prove this latter map is an analytic isomorphism it suffices to prove that the induced analytic map between fibers over each $m' \in M'$ is an isomorphism. The map ϕ respects the local analytic isomorphisms from $V' = V \times_M M'$ onto both sides, so for any $m' \in M'$ the map $\phi_m : (V'/L')_{m'} \to ((V/L) \times_M M')_{m'} = (V/L)_{h(m')}$ respects the surjective local analytic maps from $V'_{m'} = V_{h(m')}$ onto both sides. It follows that $\phi_{m'}$ is a local analytic isomorphism. But it is clearly bijective, and so it is an analytic isomorphism. Thus, ϕ is an analytic isomorphism. Similarly, to prove that the analytic map

$$(V \oplus W)/(L \oplus \Lambda) \to (V/L) \times_M (W/\Lambda)$$

over M is an isomorphism, it suffices to prove that the map between fibers over each $m \in M$ is an isomorphism. This is an analytic map

$$((V \oplus W)/(L \oplus \Lambda))_m \to ((V/L) \times_M (W/\Lambda))_m = (V/L)_m \times (W/\Lambda)_m$$

that respects the surjective local analytic isomorphisms from $(V \oplus W)_m = V_m \times W_m$ onto both sides, so this fibral map is a local analytic isomorphism. But it is clearly bijective, so it is an analytic isomorphism. This completes the proof of (ii), conditional on (i).

In view of the established *uniqueness* of the desired complex structure (if it exists!), to construct it we may work *locally* over M (as the complex structures built locally over M must agree on overlaps, due to the uniqueness). Since π is a covering map, we can cover V/L by connected open sets U_k such that $\pi^{-1}(U_k) \to U_k$

is a split covering map; i.e., $\pi^{-1}(U_k) = \Sigma_k \times U_k$ for a discrete space Σ_k. In other words, each connected component of $\pi^{-1}(U_k)$ maps homeomorphically onto U_k. Consider an arbitrary connected open set $U \subset V/L$ such that $\pi^{-1}(U) \to U$ is a split covering, so every connected component of $\pi^{-1}(U)$ maps homeomorphically onto U. These components are open in V and so inherit a complex structure from V. Assume (as we will prove below) that these all endow the open $U \subset V/L$ with the *same* complex structure (i.e., for any two connected components C and C' of $\pi^{-1}(U)$, the composite homeomorphism $C \simeq U \simeq C'$ is a holomorphic isomorphism). By giving U that common complex structure, the covering map $\pi^{-1}(U) \to U$ becomes a local analytic isomorphism. This is clearly the only complex structure on U with that property. By first letting U vary through the U_k's and then vary through the *connected components* of each $U_k \cap U_{k'}$, it follows from the uniqueness that these complex structures on the connected components of $U_k \cap U_{k'}$ arising from U_k and $U_{k'}$ separately must *coincide*. In other words, the complex structures would necessarily agree on the entire overlaps $U_k \cap U_k$ and thus globalize to a complex structure on V/L making π a holomorphic map and local analytic isomorphism, as desired.

So we finally come to the non-formal part of the argument, where we have to prove something specific about the complex structure on V in relation to L: if $U \subset V/L$ is a connected open set such that $\pi^{-1}(U) \to U$ is a split covering space, then we claim that for any two connected components C and C' of $\pi^{-1}(U)$, the composite homeomorphism $C \simeq U \simeq C'$ is a holomorphic isomorphism. This problem is local on U, in the sense that it suffices to check it over a collection of connected open sets that cover U. Pick a point $u \in U$ and let $v \in C$ and $v' \in C'$ be the corresponding points in $\pi^{-1}(u)$. Let $m \in M$ be the common image point, so $\ell := v' - v \in L_m$ inside of V_m. Since $L \to M$ is a local system, by working locally on M around m we can arrange that there exists a holomorphic section $s \in L(M)$ such that $s(m) = \ell$ in the fiber L_m. Now $j : L \to V$ is an M-group map, so translation on the group $V(M)$ by $j \circ s \in V(M)$ preserves the subgroup $L(M)$, inducing translation by s. This translation by $j \circ s$ is a *holomorphic* automorphism of V over M that preserves L and carries v to v' on V_m. In particular, this automorphism restricts to a holomorphic U-automorphism of $\pi^{-1}(U)$ carrying v to v' and thus carrying the open connected component C through v onto the connected component C' through v' via a holomorphic isomorphism respecting the homeomorphic projections $C, C' \rightrightarrows U$. It follows that the composite homeomorphism of interest $C \simeq C'$ between open sets in V is precisely the holomorphic U-automorphism induced by translation by $j \circ s$ on the M-group V, so we are done! ∎

Corollary 3.3. *There is a unique M-group structure on V/L that makes $\pi : V \to V/L$ an M-group homomorphism, and its formation is functorial in (V, L) and commutes with base change on M. Moreover, for $\ker \pi := V \times_{V/L,0} M$, the natural map of M-groups $L \to \ker \pi$ is an isomorphism.*

Proof. Since π is surjective, so $V \times_M V \to (V/L) \times_M (V/L)$ is also surjective, such an M-group structure is certainly unique if it exists, and it inherits the functoriality and compatibility with base change from comparison with the M-group law on V. To prove existence, we first note that the composite map

$$V \oplus V = V \times_M V \xrightarrow{+} V \to V/L$$

is invariant under the equivalence relation that defines $(V \oplus V)/(L \oplus L)$, so it uniquely factors through an M-map $(V \oplus V)/(L \oplus L) \to V/L$. But the source of this map is naturally identified with $(V/L) \times_M (V/L)$ (by the preceding Theorem), so we have constructed an M-map $\mu : (V/L) \times_M (V/L) \to V/L$. By construction, on fibers over $m \in M$ this is the usual group law on V_m/L_m. We likewise construct the candidate $i : V/L \simeq V/L$ over M for inversion, and comparison with V via the surjective π shows that μ, i, and the composite "zero section" $M \xrightarrow{0} V \to V/L$ satisfy the M-group axioms.

Finally, to prove that the M-group map $L \to \ker \pi$ is an isomorphism, it suffices to check that it is an isomorphism on fibers over m. This reduces us to the case when M is a point, in which case the assertion is clear. ∎

We now wish to explain the sense in which the M-group V/L behaves like a quotient sheaf over M. To this end, for any $X \to M$, let $[X]$ denote the corresponding sheaf of sets on M assigning to any open set $U \subset M$ the set $X(U) = \mathrm{Hom}_M(U, X) = \mathrm{Hom}_U(U, X_U)$ of sections of X_U over U. When X is a commutative

M-group, then $[X]$ has a structure of abelian sheaf on M. By the preceding corollary, the natural diagram of abelian sheaves on M

$$0 \to [L] \to [V] \to [V/L]$$

is left exact, so the quotient sheaf $[V]/[L]$ over M is naturally a subsheaf of $[V/L]$.

Proposition 3.4. *The inclusion of sheaves $[V]/[L] \to [V/L]$ over M is an equality.*

Proof. For any open U in M and $x \in (V/L)(U) = \mathrm{Hom}_M(U, V/L)$, we need to find an open cover $\{U_i\}$ of U such that each $x|_{U_i} \in (V/L)(U_i)$ lifts to $V(U_i) = \mathrm{Hom}_M(U_i, V)$. Equivalently (check!), this says that the projection $V \times_{V/L,x} U \to U$ admits holomorphic sections locally over U. But $V \to V/L$ is a covering map that is a local analytic isomorphism, so these properties are inherited by the base change $V \times_{V/L,x} U \to U$. Consequently, the existence of local holomorphic sections is clear. ■