Math 248B. Line bundles and sections to elliptic curves

1. Motivation

Let (E, e) be an elliptic curve over a complex manifold M, and let $\mathscr{L}$ be an invertible sheaf on E with degree 1 on all fibers: for all $m \in M$, $\deg_{E_m}(\mathscr{L}_m) = 1$. Also assume that $\mathscr{L}$ is equipped with an e-rigidification $i : \mathscr{O}_M \simeq e^*(\mathscr{L})$.

We seek to construct a closed submanifold $Z \hookrightarrow E$ mapping isomorphically onto M (i.e., defining a section to $E \to M$) such that $\delta_e(\mathscr{I}_Z^{-1}) \simeq \mathscr{L}$ respecting the e-rigidifications on each side. As we saw in class, this will suffice to complete the construction of an M-group structure on (E, e), and the absence of nontrivial automorphisms of e-rigidified line bundles and the known uniqueness of Z if it exists (which we deduced from the classical theory on fibers) makes our problem *local* over M.

2. The hard analytic input

In the classical case, one uses Riemann-Roch to show that $\mathscr{L} \simeq \mathscr{O}(z) = \mathscr{I}_z^{-1}$ for a unique point $z \in E$ (viewed as a divisor). In our relative setting, we have that $\mathrm{H}^0(E_m, \mathscr{L}_m)$ is 1-dimensional for all $m \in M$ and we seek $Z \subset E$ as above (defining a section to $E \to M$) such that $\mathscr{L} \simeq \delta_e(\mathscr{I}_Z^{\vee})$ respecting e-rigidifications. To replace the role of Riemann-Roch in the classical case, we use the following deep (but fundamental) result whose proof rests on hard analytic tools and higher sheaf cohomology:

Theorem 2.1. *Let $f : X \to M$ be a proper submersion, and $\mathscr{F}$ a vector bundle on X such that $\mathrm{H}^0(X_m, \mathscr{F}_m)$ has dimension d independent of $m \in M$, where $\mathscr{F}_m$ is the pullback of $\mathscr{F}$ to a vector bundle on the fiber X_m. Then $f_*(\mathscr{F})$ is a locally free $\mathscr{O}_M$-module of rank d and the natural specialization map*

$$f_*(\mathscr{F})_m / \mathfrak{m}_m f_*(\mathscr{F})_m \to \mathrm{H}^0(X_m, \mathscr{F}_m)$$

induced by

$$f_*(\mathscr{F})_m = \varinjlim (f_*(\mathscr{F}))(U) = \varinjlim \mathscr{F}(f^{-1}(U)) \to \mathrm{H}^0(X_m, \mathscr{F}_m)$$

(with U varying through opens around m in M) is an isomorphism for all $m \in M$.

Example 2.2. To get a feeling for the theorem, here is a re-interpretation of the isomorphism property of the specialization map. It says that any global section s of $\mathscr{F}_m$ over X_m lifts to an element of $\mathrm{H}^0(U, f_*(\mathscr{F})) = \mathscr{F}(f^{-1}(U))$ for some open U in M around m (with U possibly depending on s), and this lift has m-stalk that is unique modulo $\mathfrak{m}_m$.

Proof. The original proof due to Kodaira and Spencer used elliptic PDE methods (Sobolev estimates, etc.), and implicitly used higher sheaf cohomology. The role of higher cohomology is more explicit in later proofs that came about from Grauert's deep work relativizing the finite-dimensionality theorems of Cartan–Serre for the cohomology of coherent sheaves on compact complex manifolds. For this latter approach, see Chapter 10 of the remarkable book "Coherent Analytic Sheaves", and more specifically 10.5/5 for the above result. (Alternatively, in the series of books by Gunning and Rossi on several complex variables, volume 3 presents an account of the cohomological theory of coherent analytic sheaves, including the above result.) ■

Remark 2.3. The analogous theorem in the (noetherian) scheme case is due to Grauert, and is proved in §12 of Chapter III of Hartshorne's "Algebraic Geometry" textbook for projective f (and in Mumford's book on abelian varieties for general proper f) as an application of Grothendieck's general work on base change theorems for cohomology (which are the main results in §12 of Chapter III of Hartshorne). These latter base change theorems are also discussed near the end of Vakil's notes. (Hartshorne requires his morphism f to be projective only because he proved that f_* preserves coherence just for projective f. The preservation of coherence in the general proper case is a result of Grothendieck, deduced from the projective case via Chow's Lemma. Vakil's notes discuss the general proper case.)

In fact, the reason that the result in the algebraic case is attributed to Grauert is because he first proved a generalization of Theorem 2.1 using coherent analytic sheaves in place of vector bundles (and replacing the submersion condition with a flatness hypothesis on the coherent sheaf) and his arguments for the analytic case carry over virtually without change to the algebraic case once Grothendieck's results are in place.

Historically, analytic results such as Theorem 2.1 were an important source of motivation for the work of Grothendieck on higher cohomology in the scheme case. Neither the scheme case nor the analytic case logically imply the other; they are merely analogous, and can be proved using similar ideas once adequate general foundations in coherent sheaf theory are available in each category (schemes and complex-analytic spaces).

3. Application to elliptic curves

Applying Theorem 2.1 to $E \to M$ and the vector bundle $\mathscr{L}$ on E (taking $d = 1$) gives that $f_*(\mathscr{L})$ is an invertible $\mathscr{O}_M$-module and the natural map

$$f_*(\mathscr{L})_m \to \mathrm{H}^0(E_m, \mathscr{L}_m)$$

is surjective for all $m \in M$. Now shrink M so that the invertible $f_*(\mathscr{L})$ is free on a global section s over M. Composing with the map

$$\Gamma(E, \mathscr{L}) = \Gamma(M, f_*(\mathscr{L})) \to f_*(\mathscr{L})_m$$

yields the natural restriction map $\Gamma(E, \mathscr{L}) \to \Gamma(E_m, \mathscr{L}_m)$. Thus, by viewing $s \in \Gamma(M, f_*(\mathscr{L})) = \Gamma(E, \mathscr{L})$ as a global section of $\mathscr{L}$ over E, it follows that s restricts to a *nonzero* element s_m in the line $\mathscr{L}_m(X_m)$ for all $m \in M$. The $\mathscr{O}_X$-linear map $s : \mathscr{O}_X \to \mathscr{L}$ carrying $a \in \mathscr{O}_X(U)$ to $a \cdot s|_U$ in $\mathscr{L}(U)$ for all open U in X therefore has *nonzero* restriction $s_m : \mathscr{O}_{X_m} \to \mathscr{L}_m$ over each fiber X_m.

Dualizing, we get a map $s^\vee : \mathscr{L}^\vee \to \mathscr{O}_X$ whose pullback to each X_m is the dual map $s_m^\vee$ that is *nonzero*. But a nonzero map between line bundles on a *connected* complex manifold is injective (exercise, using analytic continuation and connectedness), so each $s_m^\vee$ is injective and hence $s^\vee$ is injective. Thus, $s^\vee$ identifies $\mathscr{L}^\vee$ with an ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$. Moreover, since $\mathscr{I} \simeq \mathscr{L}^\vee$ as $\mathscr{O}_X$-modules, $\mathscr{I}$ is invertible and its pullback $\mathscr{I}|_{X_m}$ to a line bundle on each X_m is isomorphic to $\mathscr{L}_m^\vee$ which has degree -1.

Now comes the geometric construction. Locally on E the invertible ideal sheaf $\mathscr{I}$ has a generator that is a holomorphic function. This holomorphic function on an open set in E is unique up to a unit multiple, so the zero loci of these local holomorphic functions agree on overlaps and hence glue to define a closed subset $Z \subset E$. By Exercise 1(iii) in HW2, it follows from the fibral degree -1 condition that Z is a submanifold of E and $Z \to M$ is an isomorphism. This defines a holomorphic section to $E \to M$, and by construction $\mathscr{I}_Z = \mathscr{I} \simeq \mathscr{L}^\vee$.

Dualizing gives $\mathscr{I}_Z^\vee \simeq \mathscr{L}$ over E, so pulling back along e gives $e^*(\mathscr{I}_Z^\vee) \simeq e^*(\mathscr{L}) \simeq \mathscr{O}_M$ and hence $\delta_e(\mathscr{I}_Z^\vee) \simeq \mathscr{I}_Z^\vee \simeq \mathscr{L}$. This latter composite isomorphism might not carry the canonical e-rigidification on $\delta_e(\mathscr{I}_Z^\vee)$ over to the given e-rigidification i on $\mathscr{L}$. But as we saw in class, we can scale the $\mathscr{O}_E$-linear isomorphism $\delta_e(\mathscr{I}_Z^\vee) \simeq \mathscr{L}$ by the pullback of a unique global unit on M so that it respects the e-rigidifications. Hence, we are done.