Math 248B. Strong approximation in algebraic groups

1. Setup and SL_2

For any global field k and finite set S of places of k, let $\mathbf{A}_k^S$ denote the factor ring of $\mathbf{A}_k$ obtained by removing $k_S := \prod_{v\in S} k_v$, so $\mathbf{A}_k = \mathbf{A}_k^S \times k_S$ as topological rings. A smooth affine k-group G satisfies *strong approximation* with respect to a given non-empty S if the image of $G(k)$ in $G(\mathbf{A}_k^S)$ is dense. (Note that since $\mathbf{A}_k^S$ is a locally compact Hausdorff topological ring, $G(\mathbf{A}_k^S)$ is a locally compact Hausdorff topological group; see §2 and (3.51) through Theorem 3.6 in the notes on adelic topologies on my webpage.) The discreteness of k in $\mathbf{A}_k$ implies the discreteness of $G(k)$ in $G(\mathbf{A}_k)$, so the possibility of denseness of $G(k)$ in $G(\mathbf{A}_k^S)$ is quite striking. However, this possibility is already seen in number theory for the special case $G = \mathbf{G}_{\mathrm{a}}$, where it is the classical strong approximation theorem for the adele ring (i.e., k is dense in $\mathbf{A}_k^S$ for any non-empty finite S). For many problems in number theory which involve an interplay between local and global considerations, the strong approximation property (for quite general S) is extremely important when it is available.

In this handout, we explain how to easily prove the strong approximation property for SL_2 with respect to *any* non-empty finite S (and any global field), and for readers familiar with the theory of algebraic groups we explain why this special case implies the same for many other interesting linear algebraic groups over global fields. (The ultimate result is that for k-simple connected semisimple $G \neq 1$ and a non-empty finite S, strong approximation holds for G with respect to S if and only if G is simply connected and $G(k_v)$ is non-compact for some $v \in S$. That is the *strong approximation theorem* for linear algebraic groups, and it lies quite deep.)

Proposition 1.1. *For any k and non-empty finite S, $\mathrm{SL}_2(k)$ is dense in $\mathrm{SL}_2(\mathbf{A}_k^S)$. That is, SL_2 over k satisfies the strong approximation property with respect to S.*

Proof. The closure Z of $\mathrm{SL}_2(k)$ in $\mathrm{SL}_2(\mathbf{A}_k^S)$ is a subgroup. It suffices to prove that Z contains $\mathrm{SL}_2(k_v)$ (embedded canonically as the v-factor) for every $v \notin S$. Indeed, if such containment holds then Z contains all finite direct factors $\prod_{v\in S'} \mathrm{SL}_2(k_v)$ for finite S' disjoint from S, as well as $\mathrm{SL}_2(\mathscr{O}_v)$ for all $v \notin (S \cup S')$, so by closedness of Z and stability under multiplication it would follow that Z contains every open subgroup

$$\prod_{v\in S'} \mathrm{SL}_2(k_v) \times \prod_{v\notin S\cup S'} \mathrm{SL}_2(\mathscr{O}_v) = \mathrm{SL}_2(k_{S'} \otimes \prod_{v\notin S\cup S'} \mathscr{O}_v).$$

But these exhaust $\mathrm{SL}_2(\mathbf{A}_k^S)$, so $Z = \mathrm{SL}_2(\mathbf{A}_k^S)$ as desired.

To verify that $\mathrm{SL}_2(k_v) \subset Z$, we recall the classical fact (see Lemma 8.1 in Chapter XIII of 3rd edition of Lang's "Algebra") that for any field F the group $\mathrm{SL}_2(F)$ is generated by the F-points of the unipotent F-subgroups $U^+ = \left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)$ and $U^- = \left(\begin{smallmatrix}1 & 0\\ * & 1\end{smallmatrix}\right)$ in the F-group SL_2 (i.e., $U^+(F)$ and $U^-(F)$ generate $\mathrm{SL}_2(F)$). Thus, it suffices to prove that Z contains $U^\pm(k_v)$. Since Z contains the closure of $U^\pm(k)$, and $U^\pm$ is a closed k-subgroup of the k-group SL_2, we may replace SL_2 with the k-groups $U^\pm \simeq \mathbf{G}_{\mathrm{a}}$ to reduce to the analogous problem with $\mathbf{G}_{\mathrm{a}}$ in place of SL_2. But this is exactly the classical strong approximation property for the adele ring of k! ■

2. Generalization

For readers familiar with the basics of the structure theory of connected semisimple groups, we now deduce strong approximation over any k relative to any non-empty finite S for a connected semisimple k-group G that is *simply connected* and k-split (i.e., contains a split maximal k-torus). This includes cases such as SL_n $(n \geq 2)$ and Sp_{2g} $(g \geq 1)$.

Fix a choice of split maximal torus T in G, and a positive system of roots in $\Phi(G,T)$. Let B be the corresponding Borel k-subgroup. This specifies a collection of simple reflections $\{r_\alpha\}$ that generate the Weyl group $W = W(G,T)(k) = N_G(T)(k)/T(k)$; the indexing set $\{\alpha\}$ is the set of simple positive roots. For each positive (not necessarily simple) root $\alpha \in \Phi(G,T)^+$, let G_α be the type-A_1 subgroup generated by the opposite root groups $U_{\pm\alpha}$. This is k-split (with maximal k-torus $T_\alpha = G_\alpha \cap T$) and more specifically is k-isomorphic to SL_2 or PGL_2. By the simple connectedness of G, every G_α is actually k-isomorphic to SL_2

and the simple positive coroots $\alpha^\vee : \mathbf{G}_m \to T$ form a basis of the cocharacter group of T. In view of the Bruhat decomposition

$$G(k) = \coprod_{w \in W} B(k)wB(k),$$

and the fact that W is generated by the simple reflections r_α that in turn represent the nontrivial element in $W(G_\alpha, T_\alpha)$, it follows that $G(k)$ is generated by the subgroups $G_\alpha(k)$ as α varies through $\Phi(G,T)^+$ (not just the simple roots). Thus, exactly as we reduced the strong approximation property for SL_2 to the case of $\mathbf{G}_\mathrm{a}$ in the earlier proof, now the case of a general k-split simply connected G is reduced to the case of the k-groups $G_\alpha \simeq \mathrm{SL}_2$.