Math 248B. Homework 1

1. (i) Prove that an absolutely convergent power series on a polydisc in $\mathbf{R}^n$ is uniformly convergent on compacts and C^∞, with partial derivatives computed termwise and again absolutely convergent.

(ii) Prove the equivalence among the several proposed definitions of holomorphicity for a C^∞-map $U \to V$ between open sets in $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively.

(iii) As an example, rigorously prove that the map $(\mathbf{C} \times (\mathbf{C} - \mathbf{R})) - \Lambda \to \mathbf{C}$ defined by $(w, \tau) \mapsto \wp_{\Lambda_\tau}(w)$ (with $\Lambda_\tau = \mathbf{Z}\tau \oplus \mathbf{Z}$, making Λ closed in $\mathbf{C} \times (\mathbf{C} - \mathbf{R})$) is holomorphic.

2. Let M be a complex manifold. A *local system* over M is a covering space $\pi : L \to M$, and it is *trivial* if it is split as a covering space.

(i) Prove that L admits a unique complex-analytic structure making π a local holomorphic isomorphism, and prove that if $\{U_i\}$ is an open cover of M by connected domains such that each $L_i = \pi^{-1}(U_i) \to U_i$ is split then to give a M-group structure to L is equivalent to imposing a group structure on the set $G_i = L_i(U_i)$ of holomorphic sections over U_i such that for any $m \in U_i \cap U_j$ the resulting identification $G_i \simeq L_m \simeq G_j$ is a group isomorphism.

(ii) Let $\Delta^* = \{q \in \mathbf{C} \,|\, 0 < |q| < 1\}$ be the punctured unit disc. For $n > 1$, consider the map $f : \mathbf{C}^\times \times \Delta^* \to \mathbf{C}^\times \times \Delta^*$ defined by $(w, q) \mapsto (w^n/q, q)$. Prove that $f^{-1}(1 \times \Delta^*)$ inside $\mathbf{C}^\times \times \Delta^*$ is a non-split local system over Δ^* via pr_2, with fibers of size n.

3. Read the handout "Vector bundles and relative lattices" (which uses Exercise 2 above). Make sure you understand the concrete example in §1 of that handout before reading the rest.

4. This exercise assumes that you have already done Exercise 3.

(i) Prove rigorously that

$$\mathscr{E} := \{([x, y, z], \tau) \in \mathbf{CP}^2 \times (\mathbf{C} - \mathbf{R}) \,|\, y^2 z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3\}$$

is a complex submanifold of $\mathbf{CP}^2 \times (\mathbf{C} - \mathbf{R})$ (i.e., verify the Jacobian criterion, and don't ignore the situation near $\{z = 0\} \times (\mathbf{C} - \mathbf{R})$). Composing with the projection to $\mathbf{C} - \mathbf{R}$ and using the holomorphic section $\infty : m \mapsto ([0, 1, 0], m)$, compute analytic fibers to justify that $(\mathscr{E}, \infty)$ is an elliptic curve over $\mathbf{C} - \mathbf{R}$.

(ii) Let $V = \mathbf{C} \times (\mathbf{C} - \mathbf{R})$ and define $\Lambda = \mathbf{Z}^2 \times (\mathbf{C} - \mathbf{R}) \hookrightarrow V$ via $((m, n), \tau) \mapsto (m\tau + n, \tau)$; this quotient was discussed in class, and (over each connected component of $\mathbf{C} - \mathbf{R}$) is the "$g = 1$" case of the construction in §1 of the handout you read for Exercise 3. Consider $E = V/\Lambda$ as defined in that handout. Composing the holomorphic zero-section of $V = \mathbf{C} \times (\mathbf{C} - \mathbf{R}) \to \mathbf{C} - \mathbf{R}$ with the holomorphic map $V \to V/\Lambda$ defines a holomorphic section $e : \mathbf{C} - \mathbf{R} \to E$. Rigorously compute analytic fibers (i.e., construct a holomorphic isomorphism $V_\tau/\Lambda_\tau \simeq E_\tau$) to prove that (E, e) is an elliptic curve over $\mathbf{C} - \mathbf{R}$.

(iii) For each $\tau \in \mathbf{C} - \mathbf{R}$, explain how the fibers $(\mathscr{E}_\tau, \infty(\tau))$ and $(E_\tau, e(\tau))$ are isomorphic elliptic curves via classical Weierstrass theory. We want to make precise the sense in which this identification varies holomorphically in τ. Using Exercise 1(iii), construct a *holomorphic* map $E \to \mathscr{E}$ over $\mathbf{C} - \mathbf{R}$ respecting identity sections and inducing the classical Weierstrass isomorphism on fibers over each $\tau \in \mathbf{C} - \mathbf{R}$. (Be rigorous in your justification of holomorphicity of this map $E \to \mathscr{E}$ between 2-dimensional complex manifolds.) Use Lemma 3.1 in the handout "Vector bundles and relative lattices" to deduce that this map is a holomorphic isomorphism!