Math 248B. Homework 5

1. Fix $N \geq 1$ and a choice of $i = \sqrt{-1} \in \mathbf{C}$. Consider $r \mapsto \langle r \rangle := \left(\begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as a section to $\det : \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \to (\mathbf{Z}/N\mathbf{Z})^\times$, identify $(\mathbf{Z}/N\mathbf{Z})^\times$ with $\mu_N^\times$ via $r \mapsto \zeta_r = e^{2\pi i r/N}$, and for any $\gamma \in \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ let $\gamma^\sim$ denote an arbitrary preimage in $\mathrm{SL}_2(\mathbf{Z})$ (which exists: "Chinese Remainder Theorem" for SL_2).

(i) Prove that under the identification of $\coprod_{(\mathbf{Z}/N\mathbf{Z})^\times} \Gamma(N)\backslash\mathfrak{h}_i$ as a coarse moduli space for the functor of isomorphism classes of full level-N structures (where (τ, r) corresponds to $(\mathbf{C}/\Lambda_\tau, \{r\tau/N, 1/N\})$), the natural left action of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ is $g.(\tau, r) = ([(\langle r \det g\rangle^{-1} g \langle r \rangle)^\sim](\tau), (\det g) r)$.

(ii) Prove that the map $Y(N) \to \mathrm{GL}_2(\mathbf{Z})\backslash((\mathbf{C} - \mathbf{R}) \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}))$ defined by $(\tau, r) \mapsto (\tau, \langle r \rangle^{-1})$ is a holomorphic isomorphism that is $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-*equivariant* when using the natural left action on $Y(N)$ and the natural left action on the target via $g.(m_0, g_0) = (m_0, g_0 g^{-1})$.

2. (i) Prove that every compact open subgroup of $\mathrm{GL}_n(\mathbf{Q}_p)$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbf{Z}_p)$ (hint: consider $\mathbf{Z}_p$-lattices in $\mathbf{Q}_p^n$) and that $\mathrm{GL}_n(\mathbf{Z}_p)$ is maximal as a compact subgroup. Deduce the analogue for $\mathrm{GL}_n(\mathbf{A}_f)$ using $\mathrm{GL}_n(\prod_p \mathbf{Z}_p)$, and show that all compact subrings of $\mathbf{A}_f$ lie in $\prod_p \mathbf{Z}_p$.

(ii) For $n \geq 2$, use (i) and the conjugation action of $\mathrm{GL}_n(\mathbf{Q}_p)$ on $\mathrm{SL}_n(\mathbf{Q}_p)$ to prove that every compact subgroup of $\mathrm{SL}_n(\mathbf{Q}_p)$ lies in a maximal compact open subgroup, of which there are n conjugacy classes, all conjugate in $\mathrm{GL}_n(\mathbf{Q}_p)$. (Hint: $\mathrm{GL}_n(\mathbf{Q}_p)$ acts on $\mathrm{SL}_n(\mathbf{Q}_p)$ through its quotient $\mathrm{PGL}_n(\mathbf{Q}_p)$.) How about for PGL_n? (Hint: $\mathrm{SL}_n(\mathbf{Q}_p) \to \mathrm{PGL}_n(\mathbf{Q}_p)$ is proper with finite fibers.)

(iii) For a finite set S of places of $\mathbf{Q}$ containing ∞, define $\mathbf{A}^S$ to be the factor ring of $\mathbf{A}$ obtained by removing $\prod_{v \in S} \mathbf{Q}_v$. The *strong approximation theorem* says that for any connected semisimple $\mathbf{Q}$-group G that is simply connected (e.g., SL_n, Sp_{2g}), $G(\mathbf{Q})$ has dense image in $G(\mathbf{A}^S)$ for any S when $G(\mathbf{Q}_v)$ is non-compact for some $v \in S$ (e.g., $S = \{\infty\}$). An analogue is valid over any global field.

For a finite type affine $\mathbf{Z}$-scheme $\mathscr{X}$ with generic fiber X, prove that if $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_f) = X(\mathbf{A}^\infty)$ then $\mathscr{X}(\mathbf{Z}) \to \mathscr{X}(\mathbf{Z}/N\mathbf{Z})$ is surjective for all $N \geq 1$. (Hint: equivalently, $\mathscr{X}(\mathbf{Z})$ has dense image in $\mathscr{X}(\widehat{\mathbf{Z}})$!) In this sense, strong approximation vastly generalizes the Chinese Remainder Theorem. Deduce that PGL_n *violates* strong approximation over $\mathbf{Q}$. (Can you generalize to any G that isn't simply connected?)

(iv) For an affine $\mathbf{Z}$-group scheme G of finite type and compact open subgroup K of $G(\mathbf{A}_f)$, prove that $K \cap G(\mathbf{Q})$ is commensurable with $\ker(G(\mathbf{Z}) \to G(\mathbf{Z}/N\mathbf{Z}))$ for some $N \geq 1$. Thus, compact open subgroups of $G(\mathbf{A}_f)$ replace congruence subgroups when considering $\mathbf{Q}$-groups *without* a specified $\mathbf{Z}$-structure.

3. Let $f : X \to M$ be a genus-g curve, and $\mathrm{T}_{X/M}$ the relative tangent bundle $(\Omega^1_{X/M})^\vee$.

(i) Prove that $\mathrm{R}^p f_*((\Omega^1_{X/M})^{\otimes q})$ is a vector bundle (of what rank?) whose formation commutes with any base change on M, vanishing for $p \geq 2$. Also prove that the natural map $\mathscr{F} \otimes_{\underline{\mathbf{C}}} \mathrm{R}^j f_*(\mathbf{C}) \to \mathrm{R}^j f_*(f^{-1}\mathscr{F})$ is an isomorphism for any $\underline{\mathbf{C}}$-module $\mathscr{F}$ on M. (Hint: consider stalks, and use that cohomology commutes with direct limits on a compact Hausdorff space.) Prove $\mathrm{R}^2 f_*(\underline{\mathbf{C}}) = \underline{\mathbf{C}}$ canonically (no i-dependence!).

(ii) Prove that the complex of abelian sheaves $0 \to f^{-1}\mathscr{O}_M \to \mathscr{O}_X \to \Omega^1_{X/M} \to 0$ is exact, and deduce that the connecting map $\mathrm{R}^1 f_*(\Omega^1_{X/M}) \to \mathrm{R}^2 f_*(f^{-1}\mathscr{O}_M) = \mathscr{O}_M \otimes_{\underline{\mathbf{C}}} \mathrm{R}^2 f_*(\mathbf{C}) = \mathscr{O}_M$ is an isomorphism. Negating this (for reasons you'll see in (iv)), prove $\mathrm{R}^1 f_*(\mathrm{T}_{X/M})$ is naturally $\mathscr{O}_M$-dual to $f_*((\Omega^1_{X/M})^{\otimes 2})$.

(iii) Dualizing $0 \to f^*\Omega^1_M \to \Omega^1_X \to \Omega^1_{X/M} \to 0$, the connecting map $\mathrm{KS}_{X/M} : \mathrm{T}(M) = f_* f^*(\mathrm{T}(M)) \to \mathrm{R}^1 f_*(\mathrm{T}_{X/M})$ is called the *Kodaira-Spencer* map. If you have experience with deformation theory and complex-analytic spaces, describe how the image of a tangent vector $\vec{v}$ under the m-fiber map $\mathrm{T}_m(M) \to \mathrm{H}^1(X_m, \mathrm{T}(X_m))$ classifies the way $X \to M$ yields a "first-order deformation" of X_m in the direction $\vec{v}$. (Thus, this measures the variation of complex structure on the fibers of f; e.g., it vanishes when $X = M \times X_0$.)

(iv) Dualizing the Kodaira-Spencer map in (iii) and using (ii), obtain a map $f_*((\Omega^1_{X/M})^{\otimes 2}) \to \Omega^1_M$. Prove that the natural map $f_*(\Omega^1_{X/M})^{\otimes 2} \to f_*((\Omega^1_{X/M})^{\otimes 2})$ is an isomorphism if and only if $g = 1$; in particular, for elliptic curves $E \to M$ the dual of the Kodaira-Spencer map takes the form $\mathrm{KS}'_{E/M} : \omega_{E/M}^{\otimes 2} \to \Omega^1_M$. (This is often called the "Kodaira-Spencer map"!) Prove that $\mathrm{KS}'_{\mathscr{E}/(\mathbf{C}-\mathbf{R})} : (2\pi i_\tau \mathrm{d}z)^{\otimes 2} \mapsto 2\pi i_\tau \mathrm{d}\tau$, and deduce that for the Tate curve $\mathbf{E} \to \Delta^*$ from HW3 Exercise 2 we have $(\mathrm{d}w/w)^{\otimes 2} \mapsto \mathrm{d}q/q$. Note that these are *isomorphisms*; if you did (iii), prove this isomorphism property via nontriviality of first-order deformations.