Math 248B. Homework 8

1. Let $f : X \to S$ be a flat map of schemes. Let $j : Y \hookrightarrow X$ be a closed subscheme that is S-flat (e.g., a section when f is separated, with $Y = S$). Let $\mathscr{I}_Y \subseteq \mathscr{O}_X$ denote the associated quasi-coherent ideal on X.

(i) Prove that $\mathscr{I}_Y$ is S-flat; i.e., its x-stalk is $\mathscr{O}_{S,f(x)}$-flat for all $x \in X$. (Hint: If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over a ring A with M and M'' both A-flat, prove that M' is A-flat.)

(ii) Let S' be an S-scheme, let $f' : X' \to S'$ and $j' : Y' \to S'$ the base changes of f and j over S', and let $\pi : X' \to X$ be the projection. Using (i), prove that the π-pullback $\pi^*(\mathscr{I}_Y) \to \mathscr{O}_{X'}$ of the $\mathscr{O}_X$-linear map $\mathscr{I}_Y \hookrightarrow \mathscr{O}_X$ is an isomorphism onto $\mathscr{I}_{Y'}$ (in particular, it is injective). In this sense, the formation of $\mathscr{I}_Y$ commutes with any base change on S; this is essential in the construction of Hilbert schemes.

2. Let S be a locally noetherian scheme and $f : X \to Y$ a map between flat and locally finite type S-schemes. (The noetherian hypotheses can be removed for what follows, but let's not get into that here.)

(i) For $s \in S$, prove that if $f_s : X_s \to Y_s$ is flat at $x \in X_s$ then f is flat at x. (Hint: artfully use Theorem 22.5 in Matsumura's *Commutative Ring Theory* book, by locally viewing X in an affine space over Y.) In particular, f is flat if and only if every f_s is flat (the "fibral flatness criterion"). Show also that it suffices to check flatness on geometric fibers over S.

(ii) Working with $\Omega^1_{X/Y}$, prove that if f_s is smooth (resp. étale) for all s then the same holds for f, and likewise using geometric fibers over S ("fibral smoothness/étaleness criterion").

(iii) Prove that if f is finite type and f_s is an isomorphism for all $s \in S$ then f is quasi-finite flat with fibral-degree 1.

(iv) Recall *Zariski's Main Theorem*: a quasi-finite separated map between noetherian schemes factors as an open immersion followed by a finite morphism. Use this to prove the following fantastic lemma of Deligne and Rapoport: such a map that is flat with constant fibral degree is *finite*. (Hint: use "proper + quasi-finite = finite" and the valuative criterion for properness to reduce to the case when the base is a discrete valuation ring. Then apply ZMT and consider closure of the generic fiber over the Dedekind base.)

Deduce that in (iii), if f is also separated then f is an isomorphism ("fibral isomorphism criterion")!

3. Let $f : X \to S$ be a finite locally free map of schemes, of constant rank n. For every $1 \le r \le n$, the functor $\mathbf{Hilb}^r_{X/S}$ on the category of S-schemes is defined by assigning to any S-scheme S' the set of closed subschemes $Z \hookrightarrow X_{S'}$ that are finite locally free over S' of constant rank r. (For locally noetherian S', it is equivalent to say that the map $Z \to S'$ is flat with fibers of rank r.)

(i) By considering the ideal sheaf of Z in $\mathscr{O}_{X_{S'}}$, explain why $\mathbf{Hilb}^r_{X/S}$ is a subfunctor of the Grassmannian of rank-$(n-r)$ subbundles of the vector bundle $f_*\mathscr{O}_X$ over S.

(ii) By shrinking S so that the $\mathscr{O}_S$-module $f_*\mathscr{O}_X$ admits a global frame, and working locally over the Grassmannian in (i) (covered by affine opens corresponding to r-tuples in the chosen frame that project isomorphically onto a basis of the universal rank-r quotient bundle), express in "linear algebra" terms the condition that a rank-$(n-r)$ subbundle of $f_*\mathscr{O}_{X_{S'}}$ arises from an ideal sheaf on $X_{S'}$. Deduce that $\mathbf{Hilb}^r_{X/S}$ is represented by a closed subscheme $\mathrm{Hilb}^r_{X/S}$ of the Grassmannian.

(iii) If X is also an S-group and $d|n$, prove the functor $\mathbf{Ord}^d_{X/S}$ taking an S-scheme S' to the set of closed S'-subgroups $H \subseteq X_{S'}$ finite locally free of rank d over S' is represented by a closed subscheme of $\mathrm{Hilb}^d_{X/S}$.

(iv) Let $C \to S$ be a smooth separated S-group of finite type with all $\dim C_s = 1$ (e.g., elliptic curve, or smooth locus of generalized elliptic curve), and let $G \subseteq C$ be a finite locally free closed S-subgroup of rank n. Prove that the proper map $\mathrm{Ord}^d_{G/S} \to S$ is *finite* for any $d|n$. (Hint: prove finiteness of *geometric* fibers.)

4. (i) Let $f : X \to S$ be a separated étale map, and $e : S \to X$ a section. Prove that e is étale, hence an open and closed immersion. That is, prove $X = S \coprod X'$ where e corresponds to S.

(ii) Let $X \to S$ be finite étale of constant degree d. Using (i), construct a finite étale surjection $S' \to S$ such that $X_{S'}$ is a disjoint union of d copies of S'. (Hint: $X \times_S X \to X$ has the diagonal section!) Find connected S' if S is connected. How does this recover the construction of splitting fields in Galois theory?

(iii) If X is a finite étale S-group and G is a finite group so that $X(\overline{s}) \simeq G$ for all geometric points $\overline{s}$ of S, construct an étale surjection $S' \to S$ so that $X_{S'} \simeq G \times S'$ as S'-groups. (Hint: reduce to connected S; e.g., local S by proving $\mathrm{Hom}_{\mathscr{O}_{S,s}}(Z_{\mathscr{O}_{S,s}}, Y_{\mathscr{O}_{S,s}}) = \varinjlim_{s \in U} \mathrm{Hom}_U(Z_U, Y_U)$ for finite locally free $Z, Y \rightrightarrows S$.)