Math 248B. Homework 9

1. This exercise gives a rigorous proof that for $N \geq 3$ and $\zeta = e^{2\pi i/N} \in \mathbf{C}$, the generic fiber E of the universal elliptic curve over the connected affine curve $Y = Y_\zeta(N)$ over $\mathbf{C}$ does *not* descend to the subfield $\mathbf{C}(j) \subseteq \mathbf{C}(Y)$ (so the construction in the Diamond/Shurman book is incorrect).

Let $G = \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$, $\overline{G} = G/\langle -1\rangle = \mathrm{Aut}(\mathbf{C}(Y)/\mathbf{C}(j))$, and denote the natural map $G \to \overline{G}$ by $g \mapsto \overline{g}$. Suppose E descends to an elliptic curve over $\mathbf{C}(j)$, and let ϕ be the generic fiber of the universal level structure. For each $g \in G$ there is a unique $\mathbf{C}(Y)$-isomorphism $\theta_g : (E, \phi \circ g^t) \simeq \overline{g}^*(E, \phi)$, and $\overline{h}^*(\theta_g) \circ \theta_h = \theta_{gh}$. The $\mathbf{C}(j)$-descent of E specifies $\mathbf{C}(Y)$-isomorphisms $\varphi_{\overline{g}} : E \simeq \overline{g}^*(E)$ satisfying $\overline{h}^*(\varphi_{\overline{g}}) \circ \varphi_{\overline{h}} = \varphi_{\overline{g}\cdot\overline{h}}$.

(i) Prove that $\theta_g = \varepsilon(g)\varphi_{\overline{g}}$ for some $\varepsilon(g) = \pm 1$, and deduce that $\varepsilon(gh) = \varepsilon(g)\varepsilon(h)$.

(ii) Prove that $\theta_{-g} = -\theta_g$, and deduce that $\varepsilon(-1) = -1$.

(iii) Prove that there is no homomorphism $G \to \langle -1\rangle$ satisfying $-1 \mapsto -1$, and conclude that no $\mathbf{C}(j)$-descent of E exists. Can you interpret this "quadratic obstruction" in terms of topological monodromy?

2. Let A be a ring equipped with an action by a finite group G.

(i) Prove that $A^G \to A$ is integral, and that the map $\pi : X = \operatorname{Spec} A \to \operatorname{Spec} A^G = Y$ is topologically the quotient by the G-action, with $\mathscr{O}_Y \simeq \pi_*(\mathscr{O}_X)^G$.

(ii) Prove that π is initial among G-invariant maps from $\operatorname{Spec} A$ to an arbitrary locally ringed space (and in particular to any scheme, not necessarily affine).

(iii) Assume A is an R-algebra and that G acts R-linearly on A. For any R-algebra R', construct a natural R'-algebra map $R' \otimes_R A^G \to (R' \otimes_R A)^G$ (the *base change morphism* for G-quotients), and prove it is an isomorphism when R' is R-flat. Give an example for which it is *not* an isomorphism.

(iv) For a subgroup $G \subset \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ and $\mathbf{Z}[1/N]$-algebra R, explain why the coarse moduli scheme $Y_{G,N,R}$ on the category of R-schemes exists and construct a natural map $Y_{G,N,R} \to (Y_{G,N}) \otimes_{\mathbf{Z}[1/N]} R$ bijective on geometric points. Prove that $Y_{1,1,R} \to \operatorname{Spec} R[j]$ is an isomorphism for all R. (Hint: Invert $j(j-1728)$.)

3. This exercise constructs elliptic curves with no global Weierstrass model. Let F be imaginary quadratic, and fix an elliptic curve E_η with $F = \mathrm{End}^0(E_\eta)$ over a number field K. Use the tangential action to embed F into K. Increase K so E_η has everywhere good reduction, with Néron model E an elliptic curve over $\mathscr{O}_K$.

(i) Find a suitable K-isogenous elliptic curve so that the endomorphism ring is $\mathscr{O}_F$.

(ii) For each finite projective $\mathscr{O}_F$-module M, prove that the functor $S \rightsquigarrow M \otimes_{\mathscr{O}_F} E(S)$ is represented by an abelian scheme over $\mathscr{O}_K$ of relative dimension $\dim M_F$, denoted $M \otimes E$. (Hint: M is a direct summand of a finite free $\mathscr{O}_F$-module.) In particular, $\mathfrak{a} \otimes E$ is an elliptic curve over $\mathscr{O}_K$ for any fractional $\mathscr{O}_F$-ideal $\mathfrak{a}$.

(iii) Prove that the natural map $\mathfrak{a} \otimes_{\mathscr{O}_F} \omega_{E/\mathscr{O}_K} \to \omega_{(\mathfrak{a}\otimes E)/\mathscr{O}_K}$ is an isomorphism.

(iv) Deduce that if $h_F \nmid [K : F]$ then some $\mathfrak{a} \otimes E$ does *not* admit a global Weierstrass model over $\mathscr{O}_K$. Construct such (F, E, K).

4. This exercise uses p-divisible groups to prove that if $N \geq 1$ and $p \nmid N$ then $Y_1(N,p)$ is normal and $\mathbf{Z}[1/N]$-flat (hence of pure relative dimension 1), and regular when $N \geq 4$ and $p \nmid N$.

(i) Reduce to the case $N \geq 4$. For $N \geq 4$, prove that $\pi : Y_1(N,p) \to Y_1(N)$ is finite étale over $\mathbf{Z}[1/Np]$ and that all generic points of $Y_1(N,p)$ are in $Y_1(N,p)_{\mathbf{Q}}$. Deduce that $Y_1(N,p)$ has dimension 2 at all closed points of $Y_1(N,p)_{\mathbf{F}_p}$, and that it suffices to prove regularity at all such points when $N \geq 4$.

(ii) Fix $N \geq 4$. Choose a geometric closed point $y \in Y_1(N)$ corresponding to (E_0, ϕ_0) over $k = \overline{\mathbf{F}}_p$. Explain why $(\mathscr{O}^{\mathrm{sh}}_{Y_1(N),y})^\wedge$ is naturally a $W(k)$-algebra isomorphic to $W(k)[\![t]\!]$, and a universal deformation ring for E_0 on the category of complete local noetherian $W(k)$-algebras.

(iii) By Serre–Tate, the infinitesimal deformation theories of E_0 and $\Gamma_0 = E_0[p^\infty]$ coincide. Let $R(\Gamma_0)$ be the deformation ring and Γ the universal deformation of Γ_0. For $y' \in Y_1(N,p)$ over y, prove the finite $R(\Gamma_0)$-algebra $(\mathscr{O}^{\mathrm{sh}}_{Y_1(N,p),y'})^\wedge$ classifies order-p finite flat subgroup schemes of $\Gamma[p]$ (see HW8, Exercise 3(iv)).

(iv) Over an algebraically closed field of characteristic $p > 0$, up to isomorphism there is a unique 1-dimensional connected p-divisible group with each height $h \geq 1$. Deduce via (iii) that it suffices to prove the 2-dimensional $(\mathscr{O}^{\mathrm{sh}}_{Y_1(N,p),y'})^\wedge$ is regular for *one* supersingular closed point y' in characteristic p. (Hint: The regular locus in $Y_1(N,p)$ is open, due to the excellence of finite type $\mathbf{Z}$-schemes.) Now read the "Katz–Mazur" handout to see the deformation-theoretic proof of such regularity.