Math 249B. Group cohomology and group extensions

1. Motivation

Let G be a group, and let M be a G-module. Consider exact sequences of groups

$$1 \to M \to E \to G \to 1$$

in which the left action by $G = E/M$ on M induced by E-conjugation on the commutative normal subgroup M is the given G-module structure on M. An *isomorphism* between two such extensions (with the same G and M) is defined to be a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \simeq\downarrow f & & \| & & \\ 1 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

in which $f : E' \to E$ is a group isomorphism restricting to the identity on M and inducing the identity on the common quotient G. Of course, if f is merely assumed to be a group homomorphism respecting the extension structures in this way then it is automatically an isomorphism (by a simple diagram chase).

In this handout, we will see that $\mathrm{H}^2(G, M)$ is naturally identified with the set of isomorphism classes of such extensions of G by M. But we emphasize that just as this particular group cohomology depends very much on the G-module structure on M, it will be essential that we have fixed the G-action induced on M from the extension structures that we consider. For example, if M has *trivial* G-action then $\mathrm{H}^2(G, M)$ is the set of isomorphism classes of central extensions of G by M (i.e., exact sequences as above for which M is in the center of E), but if we modify the G-module structure on M to be nontrivial then $\mathrm{H}^2(G, M)$ completely changes in general and likewise the class of extensions of G by M that we are considering completely changes too. So don't forget that the G-action on M in the exact sequences which we consider has been specified in advance!

2. Interpretation of H^2

To describe the possible exact sequences as above (inducing a given G-action on M!), let us first describe E as a set: we choose a set-theoretic section $s : G \to E$ to the given quotient map $\pi : E \twoheadrightarrow G$, so the M-cosets of E have a unique representative $s(g)$ for varying $g \in G$. We do *not* assume $s(1) = 1$. As a set, we have a disjoint union decomposition

$$E = \coprod_{g \in G} M \cdot s(g) = M \times G,$$

where $M \cdot s(g) = \pi^{-1}(g)$. To describe the group structure on this disjoint union, we note that the subgroup structure on M has been specified (with $1 \in M$ as the identity for the group law on E) but $s(1)$ may not equal the identity of E, so we cannot expect the element $(0, 1) \in M \times G$ to correspond to the identity of the group law.

As far as the composition law on E is concerned, what needs to be defined is $s(g_1)s(g_2)$ for $g_1, g_2 \in G$, since the way that $s(g)$ acts on M by conjugation within E (i.e., $s(g) \cdot m \cdot s(g)^{-1}$ for $g \in G$ and $m \in M$) has been specified in advance. Since π is to be a group homomorphism, we must have $s(g_1)s(g_2) \in \pi^{-1}(g_1 g_2)$, which is to say $s(g_1)s(g_2) = c_{g_1,g_2} s(g_1 g_2)$ for a unique $c_{g_1,g_2} \in M$.

(Note in particular that $s(1) = c_{1,1}$.) Thus, there is a function $c : G \times G \to M$ such that the group law on the set $E = M \times G$ is defined by the rule

$$(m, g)(m', g') = (m + g.m' + c(g, g'), gg').$$

The condition that this be associative says

$$g.c(g', g'') - c(gg', g'') + c(g, g'g'') - c(g, g') = 0, \tag{2.1}$$

and this relation forces $c(1, g) = c(1, 1)$ and $g.c(1, 1) = c(g, 1)$ for all $g \in G$ (by specializing $g = g' = 1$ and $g' = g'' = 1$). The condition in (2.1) is exactly the 2-cocycle condition: it says $c \in Z^2(G, M)$. Also, if we change s to another section $s' : G \to E$, which is to say we replace s with $s' = f \cdot s$ for a function $f : G \to M$ then c is replaced with $c' = c + \mathrm{d}f$, or in other words $c' - c \in B^2(G, M)$. Hence, the class $[c] \in \mathrm{H}^2(G, M)$ is independent of s and so depends only on the isomorphism class of the given extension structure E of G by M.

Conversely, given c satisfying (2.1) and defining a composition law on $E = M \times G$ as indicated above, one checks that $(-c(1, 1), 1)$ is a 2-sided identity element for this composition law and that the maps $M \to E$ defined by $m \mapsto (m - c(1, 1), 1)$ and $E \to G$ defined by $(m, g) \mapsto g$ are compatible with the composition laws. Finally, one checks that $(-g^{-1}.m - c(g^{-1}, g) - c(1, 1), g^{-1})$ is a 2-sided inverse to (m, g) (using that $g.c(g^{-1}, g) - c(1, g) + c(g, 1) - c(g, g^{-1}) = 0$ with $c(1, g) = c(1, 1)$ and $c(g, 1) = g.c(1, 1)$ for all $g \in G$). Thus, we have constructed a group extension of G by M inducing the given G-module structure on M via conjugation on the extension structure. Moreover, if we replace c with any 2-cocycle c' representing the same cohomology class then the new extension structure thereby constructed is isomorphic to the one constructed from c. (Explicitly, if $c' = c + \mathrm{d}f$ and E' denotes the group extension structure on $M \times G$ defined via c' then the asserted isomorphism $E' \simeq E$ as group extensions is $(m, g) \mapsto (m + f(g), g)$.)

The preceding considerations provide a natural bijection between the set $\mathrm{H}^2(G, M)$ and the set of isomorphism classes of group extensions of G by M inducing the given G-module structure on M via conjugation on the group extension. Note that in $\mathrm{H}^2(G, M)$ there is a distinguished element, the origin, and this is represented by the 2-cocycle $c = 0$. Hence, the corresponding group extension is easily seen to be the split one, namely $E = M \times G$ as a product group with its evident extension structure.

2.1. **Automorphisms and H^1.** Having interpreted degree-2 group cohomology in terms of isomorphism classes of group extensions, we now interpret degree-1 group cohomology in terms of automorphisms of a fixed such group extension. To this end, consider an automorphism of a group extension

$$1 \to M \to E \xrightarrow{\pi} G \to 1,$$

which is to say an automorphism $f : E \simeq E$ respecting the extension structure. We write $\mathrm{Aut}(E)$ to denote the set of such automorphisms (the extension structure on E being understood from context). One trivial example of such an automorphism is conjugation $\gamma_m : E \simeq E$ by some $m \in M$. We say that two automorphisms $f_1, f_2 : E \simeq E$ of this extension structure are *equivalent* if $f_1 = \gamma_m \circ f_2$ for some $m \in M$; we then write $f_1 \sim f_2$, and $[f]$ will denote the equivalence class of f in $\mathrm{Aut}(E)$.

There is a natural action of $\mathrm{H}^1(G, M)$ on $\mathrm{Aut}(E)/\sim$ as follows. If $\xi \in \mathrm{H}^1(G, M)$ and $c : G \to M$ is a 1-cocycle representing the cohomology class ξ, then for any $f \in \mathrm{Aut}(E)$ it is easy to check that $c.f : x \mapsto c(\pi(x)) \cdot f(x)$ is another such automorphism of E as a group extension. (It is a group automorphism of E since it is clearly a group homomorphism from E to E that induces the identity automorphisms on the subgroup M and on the quotient $G = E/M$.). Moreover, if we

change the choice of representative c for ξ then this automorphism of E changes up to equivalence as just defined. Hence, we get a well-defined pairing

$$\mathrm{H}^1(G, M) \times (\mathrm{Aut}(E)/\sim) \to \mathrm{Aut}(E)/\sim$$

via $([c], [f]) \mapsto [c.f]$. One checks that this really is an action of the group $\mathrm{H}^1(G, M)$ on the set $\mathrm{Aut}(E)/\sim$.

Rather interestingly, the verification that the equivalence class of $c.f$ only depends on the equivalence class of f (and the cohomology class of c) shows that this action by $\mathrm{H}^1(G, M)$ is simply transitive on $\mathrm{Aut}(E)/\sim$. In particular, if $\mathrm{H}^1(G, M) = 0$ then *all* automorphisms of E as a group extension are necessarily of the trivial type arising from conjugation by an element of M. This special case plays an essential role in the Artin–Tate construction of Weil groups (especially for number fields, in which case group cohomology provides the only known method of construction), where the relevant instance of the vanishing property $\mathrm{H}^1(G, M) = 0$ is the multiplicative Hilbert Theorem 90 in the local case and the variant for idele class groups in the global case.