Math 249B. Local residue pairing

Let K be a *local* function field with finite residue field k of characteristic $p > 0$, so the reduction map $\mathscr{O}_K \twoheadrightarrow k$ of $\mathbf{F}_p$-algebras has a unique ring-theoretic section (by Hensel's Lemma). That is, $\mathscr{O}_K$ has a unique structure of k-algebra respecting the identification of k with the residue field, and in this way K is naturally an extension of k. Topologically, $K \simeq k(\!(x)\!)$ over k. In class we discussed that there is a canonical k-linear "local residue map"

$$\mathrm{Res} : \widehat{\Omega}^1_{K/k} \to k$$

which is uniquely characterized by the formula

$$\mathrm{Res}(\sum_{n \gg -\infty} a_n x^n \mathrm{d}x) = a_{-1}$$

for any choice of uniformizer x of $\mathscr{O}_K$. The hard part, which we will not discuss in this course, is why such a mapping is independent of the choice of x.

In class we considered a bi-additive pairing $(\cdot, \cdot) : K \times K^\times \to \mathbf{F}_p$ defined by

$$(f, t) = \mathrm{Tr}_{k/\mathbf{F}_p}(\mathrm{Res}(f \cdot \mathrm{d}t/t).$$

We also considered a counterpart in the global setting by adding up local contributions. In the local case we checked that when $f \in \mathscr{O}_K \cap \wp(K)$ then $(f, \cdot) : K^\times \to \mathbf{F}_p$ vanishes. The aim of this handout is to prove the same vanishing for any $f \in \wp(K)$. Curiously, this is not actually proved in Artin-Tate. Rather, they only check the analogous vanishing in the global case by using that the global ideles are killed under the global residue pairing (thanks to the residue theorem for algebraic curves), and this in turn is proved by a clever use of weak approximation to reduce to the local vanishing against elements of $\mathscr{O}_K \cap \wp(K)$ which has already been proved. It seems more natural to prove the local vanishing against all of $\wp(K)$, as we shall now do.

Say $f = g^p - g$ with $g \in K$, and choose $t \in K^\times$. We wish to prove that $(f, t) = 0$ in $\mathbf{F}_p$. Fix an isomorphism $K \simeq k(\!(x)\!)$ of topological fields over k. It is easy to check by direct inspection of the definition that the biadditive local residue pairing $K \times K^\times \to \mathbf{F}_p$ is continuous, so by additivity in the first variable we may assume $g = ax^n$ for some $a \in k^\times$ and $n \in \mathbf{Z}$. That is, we may assume $f = a^p x^{np} - ax^n$.

We can write $t = x^m \cdot cu$ with $m \in \mathbf{Z}$, $c \in k^\times$, and $u \in 1 + \mathfrak{m}$. Biadditivity in the second variable reduces us to separately treating the cases $t = x$, $t = c \in k^\times$, and $t \in 1 + \mathfrak{m}$. The case $t \in k^\times$ is trivial since then $\mathrm{d}t = 0$. The case $t = x$ says

$$\mathrm{Tr}_{k/\mathbf{F}_p}(\mathrm{Res}((a^p x^{np} - ax^n) \cdot \mathrm{d}x/x)) = 0,$$

or in other words that the constant term of the Laurent polynomial $a^p x^{np} - ax^n$ is killed by $\mathrm{Tr}_{k/\mathbf{F}_p}$. If $n \neq 0$ this is clear since the constant term itself vanishes, whereas if $n = 0$ then the constant term is $a^p - a$, which is clearly killed by $\mathrm{Tr}_{k/\mathbf{F}_p}$. Thus, we may now assume $t \in 1 + \mathfrak{m}$. By a successive approximation argument, it is easy to show that every element $t \in 1 + \mathfrak{m}$ is expressible as an infinite product

$$t = \prod (1 + c_m x^m)$$

with $c_m \in k$. Hence, by continuity and biadditivity we are reduced to the case $t = 1 + cx^m$ for some $c \in k$ and $m \geq 1$. In this case we have

$$f \cdot \frac{\mathrm{d}t}{t} = (a^p x^{np} - ax^n) \cdot \frac{mcx^{m-1}\mathrm{d}x}{1 + cx^m}.$$

If $n \geq 0$ then this has no pole and so the residue vanishes. Hence, we can assume $n < 0$. That is, $n = -N$ with $N \geq 1$.

Expanding out $1/(1+cx^m)$ as a geometric series, we get

$$f \cdot \frac{\mathrm{d}t}{t} = m(a^p x^{-pN} - ax^{-N})(cx^m - c^2x^{2m} + c^3x^{3m} - \dots) \cdot \frac{\mathrm{d}x}{x},$$

and this vanishes if $p|m$, so we may assume $p \nmid m$. Hence, it suffices to prove that the constant term of the Laurent series

$$(ax^{-pN} - ax^{-N}) \cdot \sum_{j \geq 0} (-1)^{j+1} c^j x^{jm} \in k((x))$$

has vanishing trace down to $\mathbf{F}_p$ when $p \nmid m$. If N is not divisible by m then likewise Np is also not divisible by m (since $p \nmid m$), so the constant term vanishes in such cases. Hence, we may assume $N = j_0 m$ for some positive integer j_0. In this case $pN = (j_0 p)m$, so the constant term is easily seen to be $(-1)^{j_0} a c^{j_0} + (-1)^{j_0 p+1} a^p c^{j_0 p}$. Since $(-1)^{j_0 p+1} = -(-1)^{j_0}$ in k, we are finally reduced to checking that $ac^{j_0} - (ac^{j_0})^p \in k$ is killed by $\mathrm{Tr}_{k/\mathbf{F}_p}$, which is obvious.