Math 249B. The $*$-action

1. Motivation

Let G be a connected semisimple (not just reductive) group over a field k, and let S be a maximal split k-torus and P a minimal parabolic k-subgroup of G containing S. Let T be a maximal k-torus of P containing S (so T is also a maximal k-torus of G, since P is parabolic in G). Define the notation

$$ {}_k\Phi = \Phi(G,S),\ \ {}_k\Phi^+ = \Phi(P,S),\ \ \Phi = \Phi(G_{k_s}, T_{k_s}). $$

Choose a Borel k_s-subgroup $B \subset P_{k_s}$ containing T_{k_s} (so $B = P_{k_s}$ if G is quasi-split over k). This amounts to choosing a positive system of roots $\Phi^+ = \Phi(B, T_{k_s})$ for Φ contained inside the parabolic set of roots $\Phi(P_{k_s}, T_{k_s})$ in Φ. We define Δ to be the basis of Φ^+ (so its elements correspond to the nodes of the Dynkin diagram obtained from (G_{k_s}, T_{k_s}, B)), and Δ_0 denotes the set of $a \in \Delta$ for which the restriction $a|_{S_{k_s}} \in \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ is trivial. Let ${}_k\Delta \subset \mathrm{X}(S) - \{0\} = \mathrm{X}(S_{k_s}) - \{0\}$ denote the restriction of $\Delta - \Delta_0$ along the inclusion $S_{k_s} \hookrightarrow T_{k_s}$, so restriction defines a map

$$ \Delta \to {}_k\Delta \cup \{0\}. $$

In class, we defined an action of $\Gamma = \mathrm{Gal}(k_s/k)$ on the set Δ, called the "$*$-action", as follows. There is an evident *left* action of Γ on Φ defined through base change of characters $T_{k_s} \to \mathrm{GL}_1$. For each $\gamma \in \Gamma$, $\gamma(\Phi^+)$ is a positive system of roots for Φ, so there is a unique $w_\gamma \in W(G_{k_s}, T_{k_s})$ such that $w_\gamma(\gamma(\Phi^+)) = \Phi^+$. Considering minimal elements of these positive systems of roots, we see that $w_\gamma(\gamma(\Delta)) = \Delta$. We saw in class that

$$ w_{\gamma'\gamma} = w_{\gamma'}\gamma'(w_\gamma), $$

so $\Gamma \times \Delta \to \Delta$ defined by

$$ (\gamma, a) \mapsto \gamma * a := w_\gamma(\gamma(a)) $$

is an action of Γ on the set Δ. This is visibly continuous, since the action factors through $\mathrm{Gal}(K/k)$ for a finite Galois extension K/k inside k_s that splits T and over which representatives in $N_G(T)(k_s)$ for the elements of the finite group $W = W(G_{k_s}, T_{k_s})$ are defined.

Example 1.1. As was noted in class, if G is quasi-split (i.e., P is a Borel k-subgroup of G) then $w_\gamma = 1$ for all γ. Thus, in the quasi-split case the $*$-action is induced by the natural Γ-action on Φ. The converse is true too: if the $*$-action is induced by the Γ-action on Φ then G is quasi-split.

Indeed, since any nontrivial element of $W(G_{k_s}, T_{k_s})$ moves Φ^+ to another positive system of roots, any two of which are disjoint from each other, in such a situation necessarily $w_\gamma = 1$ for all γ. Thus, $\Phi^+ = w_\gamma(\gamma(\Phi^+)) = \gamma(\Phi^+)$, which is to say that $\Phi(B, T_{k_s})$ is Γ-stable inside Φ. But any parabolic k_s-subgroup of G_{k_s} containing T_{k_s} (such as a Borel k_s-subgroup) is uniquely determined by its associated parabolic set of roots, so B is Γ-stable inside G_{k_s} and hence descends to a Borel k-subgroup of G.

The $*$-action on the set Δ respects a lot of structure, such as the data encoded in the Dynkin diagram (directed edges and edge multiplicities), and a bit more. To see this, note that by definition, the Γ-action on Φ is induced by the natural Γ-action on $\mathrm{X}(T_{k_s})$. This latter action and its left action on the $\mathbf{Z}$-dual $\mathrm{X}_*(T_{k_s})$ (related through inversion on Γ!) permute the sets of absolute roots and coroots, respecting the evaluation pairing between them. The same holds for the action of the absolute Weyl group $W(G_{k_s}, T_{k_s})$ (whose *left* action on $\mathrm{X}(T_{k_s})$ is defined by $w.a : t \mapsto a(n_w^{-1} t n_w)$ for $n_w \in N_G(T)(k_s)$ representing w; note the placement of n^{-1} to keep this a left action).

Thus, the $*$-action of any $\gamma \in \Gamma$ also respects these structures, and hence acts through not only an automorphism of the based root system (i.e., the root system equipped with a choice of positive system of roots, or equivalently a choice of basis) – which is to say an automorphism of the Dynkin diagram – but even an automorphism of the *based root datum* for (G_{k_s}, T_{k_s}) (i.e., the *root datum* equipped with a choice of positive system or roots, or equivalently a choice of basis).

We say that a subset of Δ is $*$-*stable* if it is stable for the above action of Γ on Δ. In this handout, our main aim is to prove two properties of this action:

Theorem 1.2. *The restriction map $\Delta \to {}_k\Delta \cup \{0\}$ has Γ-stable fibers, and for a subset $\Delta' \subset \Delta - \Delta_0$ the parabolic set $\Phi^+ \cup [\Delta_0 \coprod \Delta'] \subset \Phi$ is Γ-stable inside Φ if and only if the subset $\Delta' \subset \Delta - \Delta_0$ is $*$-stable.*

In the final section of this handout, we explain a more conceptual perspective on the $*$-action that links it up with Galois cohomological considerations to be studied later.

2. Proof of Theorem 1.2

The key point is to show:

Lemma 2.1. *For any $\gamma \in \Gamma$, $w_\gamma \in N_{Z_G(S)}(T)(k_s)/T(k_s)$ inside $N_G(T)(k_s)/T(k_s)$.*

Proof. Let $U = \mathscr{R}_{u,k}(P)$, so $P = Z_G(S) \ltimes U$ due to the minimality of P. Thus, Borel k_s-subgroups of P necessarily contain U and so the set of these corresponds bijectively to the set of Borel k_s-subgroups of $P/U = Z_G(S)$ via "image" and "preimage". In particular, the set of Borel k_s-subgroups of P_{k_s} containing T_{k_s} is in bijective correspondence with the set of Borel k_s-subgroups of $Z_G(S)_{k_s}$ containing T_{k_s}. The group $W(Z_G(S)_{k_s}, T_{k_s})$ acts (simply) transitively on the set of Borel k_s-subgroups of $Z_G(S)_{k_s}$ containing T_{k_s}. Thus, for the purpose of choosing w_γ we can find a choice inside $N_{Z_G(S)}(T)(k_s)$. ■

Since $\mathrm{Lie}(Z_G(S)) = \mathrm{Lie}(G)^S$, an element of Φ occurs in $\mathrm{Lie}(Z_G(S))_{k_s}$ if and only if S_{k_s} is killed by that absolute root. In other words, the elements of Δ whose 1-dimensional weight space in $\mathrm{Lie}(G)_{k_s}$ occurs inside $\mathrm{Lie}(Z_G(S))_{k_s}$ are exactly the elements of Δ_0. Writing $P = Z_G(S) \ltimes U$, so $U_{k_s} \subset B$, the T_{k_s}-weights occurring on $\mathrm{Lie}(U_{k_s})$ lie inside $\Phi(B_{k_s}, T_{k_s}) = \Phi^+$. Hence, any element of $\Phi(P_{k_s}, T_{k_s})$ nontrivial on S_{k_s} lies in Φ^+ and thus its negative cannot lie in $\Phi(P_{k_s}, T_{k_s})$, so the roots in $\Phi(P_{k_s}, T_{k_s})$ whose negative also lies in there are precisely the elements of $\Phi(Z_G(S)_{k_s}, T_{k_s}) = [\Delta_0]$.

We conclude that Δ_0 is the basis of the positive system of roots for $(Z_G(S)_{k_s}, T_{k_s})$ associated to the Borel k_s-subgroup of $Z_G(S)_{k_s} = P_{k_s}/U_{k_s}$ whose preimage in P_{k_s} is B. Hence, the Weyl group $W(Z_G(S)_{k_s}, T_{k_s})$ is generated by the reflections r_a for $a \in \Delta_0$. In view of the Lemma above, we conclude that $w_\gamma = r_{a_1} \cdots r_{a_m}$ for a sequence $a_1, \dots, a_m \in \Delta_0$. For $a \in \Delta_0$ and $x \in \mathrm{X}(T_{k_s})$,

$$r_a(x) = x - \langle x, a^\vee \rangle a \in x + \mathbf{Z}\Delta_0.$$

Since moreover $r_a(\Delta_0) \in \mathbf{Z}\Delta_0$, it follows that $r_a(x+\mathbf{Z}\Delta_0) = x+\mathbf{Z}\Delta_0$. Thus, $w_\gamma(x) \in x+\mathbf{Z}\Delta_0$ for any such x, so $\gamma * a = w_\gamma(\gamma(a)) \in \gamma(a) + \mathbf{Z}\Delta_0$. Restricting to S_{k_s} kills Δ_0, so using the triviality of the natural Γ-action on $\mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ implies that

$$(\gamma * a)|_{S_{k_s}} = \gamma(a)|_{S_{k_s}} = a|_{S_{k_s}}.$$

This proves that the map $\Delta \to {}_k\Delta \cup \{0\}$ is invariant under the $*$-action on Δ.

It remains to show for $\Delta' \subset \Delta - \Delta_0$ that $\Phi^+ \cup [\Delta_0 \cup \Delta']$ is Γ-stable inside Φ if and only if Δ' is $*$-stable inside Δ. (Keep in mind that $\Phi(P_{k_s}, T_{k_s}) = \Phi^+ \cup [\Delta_0]$.) It suffices to show that under the natural Γ-action on Φ,

$$\gamma(\Phi^+ \cup [\Delta_0 \cup \Delta']) = \Phi^+ \cup [\Delta_0 \cup \gamma * \Delta']$$

for all $\gamma \in \Gamma$, where $\gamma * \Delta'$ denotes the image of Δ' under the $*$-action of $\gamma \in \Gamma$.

Clearly $\gamma(\Phi(P_{k_s}, T_{k_s})) = \Phi(P_{k_s}, T_{k_s})$, and $\Phi^+ \cup [\Delta_0 \cup \Delta'] = \Phi(P_{k_s}, T_{k_s}) \cup [\Delta_0 \cup \Delta']$, so

$$\gamma(\Phi^+ \cup [\Delta_0 \cup \Delta']) = \Phi(P_{k_s}, T_{k_s}) \cup [\gamma(\Delta_0 \cup \Delta')] = \Phi^+ \cup [\Delta_0] \cup [\gamma(\Delta_0) \cup \gamma(\Delta')].$$

We want this to coincide with $\Phi^+ \cup [\Delta_0 \cup \gamma * \Delta']$. We have $\gamma * \Delta' = w_\gamma(\gamma(\Delta')) \subset \gamma(\Delta') + \mathbf{Z}\Delta_0$ since w_γ arises from reflections in roots from the basis Δ_0 of $\Phi(Z_G(S)_{k_s}, T_{k_s})$, so $[\Delta_0 \cup \gamma * \Delta'] = [\Delta_0 \cup \gamma(\Delta')]$. It follows that

$$\Phi^+ \cup [\Delta_0 \cup \gamma * \Delta'] \subset \gamma(\Phi^+ \cup [\Delta_0 \cup \Delta']).$$

These have the same size (recall that $\gamma * \Delta_0 = \Delta_0$), so it is an equality since the $*$-action is through automorphisms of the root system.

Remark 2.2. There is another viewpoint one can take: a continuous Γ-action on a finite set is a finite étale k-scheme, so the $*$-action gives rise to a finite étale k-scheme whose set of k_s-points is identified with the set of nodes of the Dynkin diagram of (G_{k_s}, T_{k_s}, B). Note that the $*$-action preserves the structure of the diagram (directed edges and edge multiplicities), and this structure can be encoded in terms of (i) specifying a subset of $\Delta \times \Delta$ away from the diagonal (directed edges that are not loops) and (ii) a map from that subset to $\{1, 2, 3\}$ (edge multiplicity).

To summarize, the $*$-action defines a finite étale k-scheme $\mathrm{Dyn}(G)$ and a finite étale closed subscheme $\mathrm{DirEdge}(G) \subset \mathrm{Dyn}(G) \times \mathrm{Dyn}(G)$ disjoint from the diagonal along with a map from $\mathrm{DirEdge}(G)$ to the constant k-scheme $\{1, 2, 3\}$ (and an identification of this structure on k_s-points with the Dynkin diagram). Actually, the $*$-action is a bit finer, since it respects information related to the root datum and not just the root system (which is all that is "known" through the diagram).

In SGA3, Exp. XXIV, §3, the notion of the finite étale *scheme of Dynkin diagrams* is defined for semisimple group schemes over a general (non-empty) base scheme S. This is a finite étale S-scheme D equipped with a finite étale closed subscheme of $D \times D$ disjoint from the diagonal and a map from that closed subscheme to the constant scheme $\{1, 2, 3\}_S$ (satisfying some axioms which ensure it arises from an actual Dynkin diagram on geometric fibers). Working over the field k and applying this to G, we recover $\mathrm{Dyn}(G)$ with its additional structure built above via the $*$-action.