Math 249B. Reductive centralizer

1. Motivation

Let G be a connected reductive group over a field k. Let M be a closed k-subgroup scheme of G of multiplicative type. (The case of interest to us is $M = \ker(a)$ for a nontrivial character $a : S \to \mathbf{G}_{\mathrm{m}}$ on a k-split torus S; in positive characteristic this scheme-theoretic kernel might not be smooth.) We want to study smoothness and reductivity properties of the *scheme-theoretic centralizer* $Z_G(M)$, but first we need to discuss what $Z_G(M)$ means if M is not smooth (as this will be very important later on, to have a robust theory in positive characteristic).

By definition, if it exists, $Z_G(M)$ is the closed subgroup scheme of G representing the functor whose points valued in a k-algebra R consist of those $g \in G(R)$ such that g-conjugation on G_R restricts to the identity on M_R. We have established its existence in some special cases: if $g \in T(k)$ and M is the Zariski-closure of $g^{\mathbf{Z}}$ in T then M is a smooth (possibly disconnected) closed k-subgroup of T and $Z_G(M) = Z_G(g)$ with Lie algebra $\mathfrak{g}^{g=1} = \mathfrak{g}^M$. More generally, if M is smooth then $Z_G(M)$ was constructed by a Galois-theoretic method in Exercise 3 of HW3 of the previous course, where we saw that its Lie algebra is $\mathfrak{g}^M$.

In fact, $Z_G(M)$ exists and has Lie algebra $\mathfrak{g}^M$ *without* smoothness hypotheses on M nor reductivity (or even smoothness!) hypotheses on G. More broadly, for any closed subgroup scheme H of an affine k-group scheme G of finite type, the scheme-theoretic centralizer $Z_G(H)$ (defined to represent the evident functor on k-algebras) always exists as a closed k-subgroup scheme of G with Lie algebra $\mathfrak{g}^H$. To see this, all we need from the group structure of G is its role in defining the action of the k-group H on G with the identity point as a fixed point for the action. Hence, to clarify the method, we prove a more general result:

Proposition 1.1. *Let Y be an affine k-scheme of finite type and H an affine k-group scheme of finite type acting on Y. There exists a closed subscheme $Y^H \subset Y$ representing the functor of H-fixed points, and its tangent space at any $y \in Y^H(k) = Y(k)^H$ is $\mathrm{T}_y(Y)^H$.*

This is proved in Proposition A.8.10(1),(2) of [CGP] with no affineness hypotheses on Y (and even over rings, under some hypotheses on $k[H]$). But that proof simplifies *a lot* for affine Y with base ring a field, as we show below.

Proof. Let $\{e_j\}$ be a k-basis of $k[H]$, so if R is a k-algebra then $\{e_j\}$ is an R-basis of the coordinate ring of H_R. For any k-algebra R and f in the coordinate ring $R[H] \otimes_R R[Y]$ of $(H \times Y)_R = H_R \times Y_R$, we can uniquely write $f = \sum e_j \otimes c_j(f)$ for $c_j(f) \in R[Y]$.

Let $I \subset k[H \times Y]$ be the ideal of the pullback of the diagonal of Y under the action map $\alpha : H \times Y \to Y \times Y$ defined by $\alpha(h, y) = (h.y, y)$. For any k-algebra R and $y \in Y(R)$, we have that y is fixed by the H_R-action on Y_R precisely when the map $i_y : H_R \to H_R \times Y_R$ defined by $h \mapsto (h, y)$ lands inside $\alpha^{-1}(\Delta_{Y/k})$, which is to say that the pullback ideal $i_y^*(I_R)$ inside $R[H] = \bigoplus Re_j$ vanishes. Since I_R is generated by I as an R-module, this vanishing condition is equivalent to the vanishing of $y^*(c_j(f)) \in R$ for every $f \in I$, which is to say that $y : \mathrm{Spec}(R) \to Y$ factors through the common zero scheme of the elements $c_j(f) \in k[Y]$. Hence, this latter zero scheme in Y represents Y^H.

For $y \in Y^H(k) = Y(k)^H$, to prove that $\mathrm{T}_y(Y^H) = \mathrm{T}_y(Y)^H$ inside $\mathrm{T}_y(Y)$ an argument is really needed, as the tangent space of Y^H involves studying an $H_{k[\varepsilon]}$-action whereas $\mathrm{T}_y(Y)^H$ involves an H-action. The problem is to show that a vector $v \in \mathrm{T}_y(Y)$ is H-fixed if and only if when viewed inside $Y(k[\varepsilon])$ it is fixed by the $H_{k[\varepsilon]}$-action.

From the construction of Y^H, we have $v \in \mathrm{T}_y(Y^H)$ if and only if the composite k-algebra map

$$k[Y \times Y] \stackrel{\alpha^*}{\to} k[H \times Y] \stackrel{\mathrm{id}_H \otimes v^*}{\longrightarrow} k[H][\varepsilon]$$

factors through $\Delta^*_{Y/k} : k[Y \times Y] \twoheadrightarrow k[Y]$ (i.e., the restrictions to the two tensor factors of $k[Y \times Y] = k[Y] \otimes_k k[Y]$ coincide). On the other hand, v is H-invariant in $\mathrm{T}_y(Y)$ if and only if for every k-algebra R and $h \in H(R)$ the composite map of R-algebras

$$R[Y] \stackrel{h^*}{\simeq} R[Y] \stackrel{v_R^*}{\to} R[\varepsilon]$$

(using the effect of the h-action on Y_R) coincides with v_R. Taking the universal case $R = k[H]$ and $h = \mathrm{id}_H$, this is the condition that the composite map of $k[H]$-algebras

$$k[H \times Y] \simeq k[H \times Y] \stackrel{\mathrm{id}_H \otimes v^*}{\to} k[H][\varepsilon]$$

(where the first step corresponds to $(h, y) \mapsto (h, h.y)$) coincides with $\mathrm{id}_H \otimes v_H^*$. But this latter equality of $k[H]$-algebra maps can be checked on the second tensor factor of $k[H \times Y] = k[H] \otimes k[Y]$, where it becomes exactly the equality of maps $k[Y] \rightrightarrows k[H][\varepsilon]$ that encodes when $v \in \mathrm{T}_y(Y^H)$. ■

As a refinement, for any smooth affine k-scheme Y equipped with an action by a k-subgroup M of multiplicative type, the k-scheme Y^M is always *smooth* (even when M is not smooth!). Indeed, to check this we may assume $k = \overline{k}$, so M is a "split" group of multiplicative type, and then we can verify the infinitesimal smoothness criterion for Y^M by using the complete reducibility of k-linear representations of split multiplicative-type k-group schemes. This calculation is exactly Exercise 3 of HW8 of the previous course (applied to the M-action on H via conjugation), which was stated only for actions of tori because at that time we didn't construct centralizers for non-smooth subgroup schemes (as that generality wasn't needed in the previous course, where the Galois-theoretic construction of schematic centralizers against *smooth* subgroups was sufficient).

Now let's return to the original setup with a connected reductive k-group G and a closed k-subgroup scheme $M \subset G$ of multiplicative type, so $Z_G(M)$ is smooth. One source of M as above are k-subgroup schemes of k-tori in G. But there are other examples not arising in that way:

Example 1.2. For $n \geq 3$, let q_n denote the standard "split" quadratic form ($x_1x_2 + \cdots + x_{n-1}x_n$ for n even, and $x_0^2 + q_{n-1}(x_1, \ldots, x_{n-1})$ for n odd). Let G be the split connected semisimple group $\mathrm{SO}_n = \mathrm{SO}(q_n) \subset \mathrm{SL}_n$. Consider the k-subgroup

$$M' = \{(\zeta_1, \ldots, \zeta_n) \in \mu_2^n \mid \prod \zeta_j = 1\} \simeq \mu_2^{n-1}$$

inside G. The maximal tori of $G_{\overline{k}}$ have dimension $\lfloor n/2 \rfloor$, and so have 2-torsion equal to $\mu_2^{\lfloor n/2 \rfloor}$. Since $\lfloor n/2 \rfloor < n - 1$ for $n \geq 3$, M' is not contained in any k-torus of G.

Remark 1.3. The special case $\mathrm{char}(k) = p > 0$ with $M = \mu_p$ makes an appearance in the classical theory of reductive groups in the sense that for a nonzero element X in the line $\mathrm{Lie}(M) \subset \mathrm{Lie}(G)$, Proposition A.8.10(3) in [CGP] shows that the smooth closed k-subgroup $Z_G(M)$ equals the group denoted $Z_G(X)$ in the classical theory (see 9.1 in Borel's textbook on linear algebraic groups).

It is an important fact in the classical theory that the smooth connected k-group $Z_G(M)^0$ is *reductive* when M is smooth with cyclic étale component group or when $M = \mu_p$ with $\mathrm{char}(k) = p > 0$. The former case immediately reduces to $Z_G(g)$ for $g \in T(k)$, and the latter case can be expressed in the form of $Z_G(X)$ as explained above. In Borel's textbook, the reductivity of $Z_G(M)^0$ for such M is proved in 13.19.

The goal of this handout is to generalize a classical reductivity result in our scheme-theoretic framework: $Z_G(M)^0$ is reductive for *any* multiplicative type k-subgroup M of a k-torus inside G. In the special case that M is smooth and connected, hence a torus, this is a ubiquitous fact in the theory of connected reductive groups that we have used all the time. We remove connectedness and especially smoothness hypotheses on M.

Remark 1.4. It is natural to wonder if the reductivity of $Z_G(M)^0$ requires the assumption that M occurs inside a k-torus T of G (that we have seen in Example 1.2 need not always hold for multiplicative type subgroups of split connected semisimple groups) That is, if M is *any* closed k-subgroup scheme of multiplicative type inside G then is the smooth connected k-subgroup $Z_G(M)^0$ reductive? The answer is affirmative, but our technique of proof (which uses the structure of root groups relative to $\Phi(G_{k_s}, T_{k_s})$) for a maximal k-torus $T \supset M$ is not applicable without the crutch of such a T.

Rather generally, consider any finite type affine k-group scheme H such that the representation theory of $H_{\overline{k}}$ is completely reducible. For any action by H on a connected reductive k-group G, the schematic centralizer G^H is smooth with reductive identity component. This result lies *much* deeper than the case "$H \subseteq T$ acting through conjugation" treated below, and a proof is given in Proposition A.8.12 in [CGP]. The proof rests on a remarkable necessary and sufficient reductivity criterion for smooth connected k-subgroups G' of G independently due to Borel and Richardson: G' is reductive if and only if G/G' is affine. (Borel's proof rests on the general apparatus of étale cohomology, and Richardson's proof rests on the work of Haboush and Mumford in geometric invariant theory).

2. Reductivity

To prove the reductivity of $Z_G(M)^0$ when M is contained in a k-torus $T \subset G$ (that we may and do assume is maximal), we may and do assume $k = \overline{k}$. Suppose to the contrary that $U = \mathscr{R}_u(Z_G(M)^0)$ is nontrivial, so $\mathrm{Lie}(U)$ is a nonzero representation space for T through its adjoint action on the smooth connected group $Z_G(M)^0$. This representation space cannot support the trivial weight, since $\mathfrak{g}^T = \mathrm{Lie}(T)$ by reductivity of G and $\mathrm{Lie}(T) \cap \mathrm{Lie}(U) = \mathrm{Lie}(T \cap U) = 0$ (as $T \cap U$ is a multiplicative type subgroup scheme of the unipotent U, so it has to be trivial since $\mathbf{G}_\mathrm{a}$ contains no nontrivial multiplicative type closed subgroup scheme). Thus, for some $a \in \Phi(G, T)$ the 1-dimensional weight space $\mathfrak{g}_a$ occurs inside $\mathrm{Lie}(U)$.

Let $H = Z_G(T_a \cdot M)^0$ where $T_a = (\ker a)^0_{\mathrm{red}}$, so H is smooth and connected inside $Z_G(M)^0$. In particular, $U \cap H$ is a normal subgroup scheme of H. Note that since T normalizes U (by working inside $Z_G(M)^0$ in which U is normal), the schematic centralizer U^{T_a} is smooth. But $U \cap H = U^{T_a}$ and this has Lie algebra $\mathrm{Lie}(U)^{T_a} \supseteq \mathfrak{g}_a \neq 0$, so $(U \cap H)^0$ is a nontrivial smooth connected unipotent subgroup of H that is normal. In other words, by replacing M with $T_a \cap M$ we may assume that $T_a \subseteq M$ without losing the hypothesis that H is not reductive.

But $H \subset Z_G(T_a)$ and $Z_G(T_a)$ is an almost direct product of the torus T_a and the rank-1 connected semisimple group $H' := \mathscr{D}(Z_G(T_a)) = \langle U_a, U_{-a} \rangle$ that is either SL_2 or PGL_2 and meets T in the diagonal torus D. Since $T_a \subseteq M$, by writing $T = T_a \cdot D$ we have $M = T_a \cdot \mu$ for $\mu = D \cap M$. Thus, $Z_G(M)^0 = T_a \cdot Z_{H'}(\mu)^0$ as an almost direct product of smooth connected k-groups, so the failure of reductivity for $Z_G(M)^0$ forces the failure for $Z_{H'}(\mu)^0$.

To get a contradiction, we're now reduced to checking for H' equal to either SL_2 or PGL_2 and any closed k-subgroup scheme μ of the diagonal $D = \mathbf{G}_{\mathrm{m}}$ that $Z_{H'}(\mu))^0$ is reductive. The cases $\mu = 1, D$ are trivial, so we can assume $\mu = \mu_n$ for some $n > 1$. Since $\mathrm{Lie}(Z_{H'}(\mu)^0) = \mathrm{Lie}(Z_{H'}(\mu)) = \mathrm{Lie}(H')^\mu$, if $H' = \mathrm{PGL}_2$ then $\mathrm{Lie}(H')^\mu = \mathrm{Lie}(D)$. Hence, in such cases the inclusion $D \subset Z_{H'}(\mu)^0$ between smooth connected groups is an equality on Lie algebras, so it is an equality of k-groups. Suppose instead that $H' = \mathrm{SL}_2$. If $\mu = \mu_2$ then $Z_{H'}(\mu) = H'$ and we are done, so we may assume $\mu = \mu_n$ with $n > 2$. Thus, squaring on μ_n is nontrivial, so it is easy to check that $\mathrm{Lie}(H')^\mu = \mathrm{Lie}(D)$, and hence once again $D = Z_{H'}(\mu)^0$ by Lie algebra considerations.