Math 249B. Geometric Bruhat decomposition

1. Introduction

Let (G,T) be a split connected reductive group over a field k, and $\Phi = \Phi(G,T)$. Fix a positive system of roots $\Phi^+ \subset \Phi$, and let B be the unique Borel k-subgroup of G containing T such that $\Phi(B,T) = \Phi^+$. Let $W = W(G,T)(k) = N_{G(k)}(T)/T(k)$, and for each $w \in W$ let $n_w \in N_{G(k)}(T)$ be a representative of W. For each $a \in \Phi$ we let $r_a \in W$ be the associated involution, and we let Δ denote the base of Φ^+.

Under the action of $B \times B$ on G by $(b,b').g = bgb'^{-1}$, the orbit BgB through any $g \in G(k)$ is a locally closed subvariety (as for orbits of linear algebraic groups in general). The double coset $C(w) = Bn_wB$ (a *Bruhat cell*) depends only on w, not n_w, and in class we proved that $C(w)$ decomposes as a direct product scheme under multiplication: for the closed subset

$$\Phi'_w = \Phi^+ \bigcap w(-\Phi^+) \subset \Phi^+,$$

the multiplication map

$$U_{\Phi'_w} n_w \times B \to C(w)$$

is an isomorphism of k-schemes. Equivalently, the left translate $n_w^{-1}C(w)$ is identified with the closed subscheme $U_{w^{-1}(\Phi'_w)} \times B$ in the open cell $\Omega = U_{-\Phi^+} \times B \subset G$. (Thus, $C(w)$ is open in G if and only if $\Phi'_w = \Phi^+$, which is to say that $w(\Phi^+) = -\Phi^+$. There is exactly one such w_0, and $C(w_0)$ is the unique open Bruhat cell.)

Remark 1.1. The k-scheme $U_{\Phi'_w}$ is a direct product (under multiplication, in any order) of the root groups U_a for $a \in \Phi'_w$, so it is an affine space of dimension $\#\Phi'_w$. This cardinality has a simple combinatorial interpretation: it is the *length* $\ell(w)$ relative to the generating set $\{r_a\}_{a\in\Delta}$ of W. (That is, this is the minimal length of a word in the r_a's whose product is equal to w.) In particular, $\ell(w) \le \#\Phi^+$ with equality if and only if $C(w)$ is the unique open Bruhat cell. See Corollary 2 to Proposition 17 in §1.6 of Chapter VI of Bourbaki for a proof.

By Corollary 3 to Proposition 17 in §1.6 of Chapter VI of Bourbaki there is a unique longest element $w_0 \in W$ relative to $\{r_a\}_{a\in\Delta}$, it is given by the product $\prod_{a\in\Phi^+} r_a$ taken in *any* order, and $w_0(\Phi^+) = -\Phi^+$. In particular, $w_0^2 = 1$ since $w_0^2(\Phi^+) = \Phi^+$. We call w_0 the *long Weyl elements.*

The purpose of this handout is to prove that the subvarieties $C(w)$ are a pairwise disjoint covering of G; we call this the *geometric Bruhat decomposition* since it concerns covering the entire scheme G (and since each $C(w)$ is locally closed, this is equivalent to a covering statement at the level of $\overline{k}$-points). In the setting of possibly non-split connected reductive groups there will be an analogous covering result (called the *Bruhat decomposition*) that is only valid at the level of k-points, recovering the geometric version over algebraically closed fields.

Remark 1.2. The Zariski closure of each $C(w)$ is a union of Bruhat cells $C(w')$ for some subset of elements $w' \in W$. Indeed, to prove this it suffices to work over $\overline{k}$ (since the formation of the Zariski closure of a locally closed subscheme commutes with extension of the ground field), and this closure is visibly stable under left and right multiplication by B. Hence, the closure of $C(w)(\overline{k})$ inside $G(\overline{k})$ is a union of $B(\overline{k})$ double cosets, so it is indeed a union of Bruhat cells.

But which $C(w')$ lie in the closure of $C(w)$? This has a nice combinatorial answer: if we fix a "reduced decomposition" $w = r_{a_1}\cdots r_{a_{\ell(w)}}$ (i.e., a minimal length expression for w as a product of the r_a's, allowing some repetition in the sequence $\{a_j\}$) then $C(w') \subset \overline{C(w)}$ if and only if w' is obtained as a subproduct of $\prod r_{a_j}$ by removing some terms: $w' = r_{a_{i_1}}\cdots r_{a_{i_n}}$ with

$1 \leq i_1 < \cdots < i_n \leq \ell(w)$. We will not use this important result; see 8.5.4–8.5.5 in Springer's book for a proof.

2. Bruhat decomposition

We begin by proving the disjointness of the Bruhat cells. This will ultimately reduce to a general result concerning torus actions on unipotent groups.

Proposition 2.1. *If $w \neq w'$ then $C(w) \bigcap C(w') = \emptyset$.*

Proof. Since any two $B(\overline{k})$ double cosets in $G(\overline{k})$ are either equal or disjoint, we just have to rule out the possibility that $n_w \in C(w')(k) = U_{\Phi'_{w'}}(k) n_{w'} B(k)$. That is, for $w, w' \in W$ we shall prove that an equality

$$n_w = u n_{w'} b$$

with $u \in U_{\Phi'_w}(k)$ and $b \in B(k)$ forces $w' = w$.

Let $U = U_{\Phi^+}$, and define the closed set of roots $\Phi^+_w = \Phi^+ \bigcap w^{-1}(\Phi^+)$ inside Φ^+, so $\Phi^+ = \Phi'_w \coprod \Phi^+_w$ and $U = U_{\Phi^+_w} \times U_{\Phi'_w}$ under multiplication. Similarly define $\Phi^+_{w'}$. For u as above, we claim that

$$U_{\Phi^+_w} = u U_{\Phi^+_{w'}} u^{-1}. \tag{1}$$

To establish this equality, we first prove the formula

$$U_{\Phi^+_w} = U \bigcap n_w B n_w^{-1}$$

for any $w \in W$. A point u of U lies in $n_w B n_w^{-1}$ if and only if $n_w^{-1} u n_w \in B$. But if we write $u = u' u^+$ under the decomposition $U = U_{\Phi'_w} \times U_{\Phi^+_w}$ (via multiplication inside G) then

$$n_w^{-1} u n_w = (n_w^{-1} u' n_w)(n_w^{-1} u^+ n_w)$$

with $n_w^{-1} u' n_w \in n_w^{-1} U_{\Phi'_w} n_w = U_{w^{-1}(\Phi'_w)} \subset U_{-\Phi^+}$ and likewise $n_w^{-1} u n_w \in U_{\Phi^+}$. Hence, the direct product structure of the open cell $\Omega = U_{-\Phi^+} \times B$ implies that $n_w^{-1} u n_w \in B$ if and only if $u' = 1$, which is to say $u \in U_{\Phi^+_w}$.

But we can compute $U \bigcap n_w B n_w^{-1}$ in another way: since $n_w = u n_{w'} b$, clearly $n_w B n_w^{-1} = u n_{w'} n_{w'}^{-1} u^{-1}$, so

$$U \bigcap n_w B n_w^{-1} = u(U \bigcap n_{w'} B n_{w'}^{-1}) u^{-1} = u U_{\Phi^+_{w'}} u^{-1}.$$

Thus, we have proved (1).

Now comes the key point:

Lemma 2.2. *Let U be a unipotent smooth connected k-group equipped with an action by a split k-torus T such that all T-weights on* $\mathrm{Lie}(U)$ *are non-trivial, pairwise linearly independent in* $\mathrm{X}(T)_{\mathbf{Q}}$*, and have a 1-dimensional weight space.*

If $V, V' \subset U$ are T-stable smooth connected k-subgroups that are $U(k)$-conjugate then they are equal inside U.

Proof. We may and do assume $k = \overline{k}$, and we argue by induction on $\dim U$ (the case $\dim U = 0$ being trivial). For each $a \in \Phi(U, T)$, let $T_a = (\ker a)^0_{\mathrm{red}}$. The centralizer $U_a = Z_U(T_a)$ of the T_a-action on U is smooth and connected, with $\mathrm{Lie}(U_a) = \mathfrak{u}_a$ since $\Phi(U,T) \bigcap \mathbf{Q}a = \{a\}$ by the hypotheses. Thus, U_a is 1-dimensional, so $U_a \simeq \mathbf{G}_{\mathrm{a}}$. The 1-dimensionality implies by dynamic considerations (!) that U_a is the *unique* nontrivial smooth connected T-stable k-subgroup of U with T-weight a on its Lie algebra. The k-subgroups U_a for $a \in \Phi(U, T)$ generate U since their Lie algebras span $\mathrm{Lie}(U)$. These conclusions also apply to any T-stable smooth connected k-subgroup of U in place of U.

The unipotent U is *nilpotent* (see the handout "Nilpotence of unipotent groups"), so $(Z_U)^0_{\mathrm{red}}$ is nontrivial and T-stable. The above reasoning can be applied to $(Z_U)^0_{\mathrm{red}}$ in place of U, so $(Z_U)^0_{\mathrm{red}}$ contains U_a for each T-weight a on $\mathrm{Lie}((Z_U)^0_{\mathrm{red}})$. Fix such a weight a, and consider the central quotient U/U_a. The images of V and V' in this quotient coincide (by dimension induction), so $V \cdot U_a = V' \cdot U_a$. If the *central* $U_a \subset U$ is contained in one of V or V' then it is contained in both (as V and V' are $U(k)$-conjugate), in which case $V = V \cdot U_a = V' \cdot U_a = V'$ as desired. Hence, we can assume that U_a is not contained in V nor in V'.

The T-weight a cannot occur on $\mathrm{Lie}(V)$ or $\mathrm{Lie}(V')$ (as otherwise the construction of U_a could be carried out inside V or V', a contradiction), so the intersections $V \bigcap U_a$ and $V' \bigcap U_a$ have vanishing Lie algebra and thus are étale. But $U_a = \mathbf{G}_{\mathrm{a}}$ on which T acts as $t.x = a(t)x$ for the nontrivial character a of T, so visibly U_a has no nontrivial T-stable finite étale k-subgroup. In other words, the surjective homomorphisms $V \times U_a \to V \cdot U_a$ and $V' \times U_a \to V' \cdot U_a$ are isomorphisms. Passing to Lie algebras and comparing T-weights, we see that $\Phi(V,T) = \Phi(V',T)$ inside $\Phi(U,T)$. But the unipotent smooth connected V is generated by the groups U_a for $a \in \Phi(V,T)$, and similarly for V', so $V = V'$ inside U as desired. ∎

It follows from the lemma that $U_{\Phi^+_w} = U_{\Phi^+_{w'}}$ inside $U = U_{\Phi^+}$, so $\Phi^+_w = \Phi^+_{w'}$ inside Φ^+. Passing to complements in Φ^+, we also have $\Phi'_w = \Phi'_{w'}$. Thus, $\Phi^+_w \coprod -\Phi'_w = \Phi^+_{w'} \coprod -\Phi'_{w'}$. Denoting this set as Ψ, we have

$$w(\Psi) = w(\Phi^+_w) \coprod w(-\Phi'_w) = (\Phi^+ \bigcap w(\Phi^+)) \coprod (\Phi^+ \bigcap w(-\Phi^+)) = \Phi^+$$

and similarly $w'(\Psi) = \Phi^+$. Thus, $w'w^{-1}(\Phi^+) = \Phi^+$, so by simple transitivity of the W-action on the set of positive systems of roots in Φ^+ we have $w' = w$. ∎

Proposition 2.3 (Geometric Bruhat decomposition). *The locally closed subvarieties $\{C(w)\}_{w \in W}$ cover G. In particular, $G(k)$ is covered by the disjoint subsets $C(w)(k) = U_{\Phi'_w}(k) n_w B(k)$ and the natural map $W(G,T)(k) \to B(k)\backslash G(k)/B(k)$ is bijective.*

Proof. We first treat the case when the split connected reductive G has semisimple-rank 1 (i.e., $\mathscr{D}(G)$ is k-isomorphic to SL_2 or PGL_2). Let $w \in W$ be the unique nontrivial element, so $C(w) = U n_w B$ for $U = U_{\Phi^+} = \mathbf{G}_{\mathrm{a}}$. Thus, $C(w)/B$ is an affine line that is open in $G/B \simeq \mathbf{P}^1_k$, so its complement in G/B is a single k-rational point. The extra point $C(1)/B$ accounts for this, so $C(w) \cup C(1) = G$.

In general, for every $a \in \Phi(G,T)$ and the associated codimension-1 torus $T_a = (\ker a)^0_{\mathrm{red}}$, $G_a := Z_G(T_a)$ is split connected reductive with semisimple-rank 1 and $W(G_a,T)(k) = \{1, r_a\}$. A Borel subgroup of G_a containing T is $B_a = T \cdot U_a \subset B$, so for $n_a \in N_{G_a(k)}(T)$ representing r_a we have

$$G_a = B_a \coprod U_a n_a B_a. \tag{2}$$

For each $a \in \Delta$, we claim that

$$G_a B n_w B \subset B n_w B \cup B n_a n_w B, \tag{3}$$

or equivalently $G_a C(w) \subset C(w) \cup C(r_a w)$. Once this is proved, it follows that $\bigcup_{w \in W} C(w)(\overline{k})$ is stable under left multiplication by the subgroups $G_a(\overline{k})$ for $a \in \Delta$. But $G(\overline{k})$ is generated by the subgroups $G_a(\overline{k})$ for $a \in \Delta$ since $T, U_{\pm a} \subseteq G_a$ and the elements $n_a \in \langle U_a, U_{-a} \rangle$ generate W (and $W \cdot \Delta = \Phi$, so every root group is in the W-orbit of some U_a with $a \in \Delta$). Thus, $\bigcup_{w \in W} C(w)(\overline{k})$ is stable under left multiplication by $G(\overline{k})$ and hence it coincides with $G(\overline{k})$. This implies the Bruhat decomposition (conditional on (3)), as each $C(w)$ is locally closed in G.

To prove (3), we fix $a \in \Delta$, so $\Psi = \Phi^+ - \{a\}$ is a closed set of roots contained in Φ^+. The associated unipotent smooth connected k-subgroup $U_\Psi \subset U_{\Phi^+} = U$ satisfies

$$U = U_\Psi \times U_a = U_a \times U_\Psi$$

under multiplication. Thus,

$$G_a B n_w B = G_a U T n_w B = G_a U_a U_\Psi n_w B = G_a U_\Psi n_w B.$$

Lemma 2.4. *For each $a \in \Delta$ and $\Psi = \Phi^+ - \{a\}$, $G_a U_\Psi = U_\Psi G_a$.*

Proof. Since G_a is generated by T, U_a, U_{-a}, it suffices to prove that U_Ψ is normalized by each of T, U_a, and U_{-a}. Normalization by T is obvious, and the cases of U_a and U_{-a} are equivalent upon replacing the positive system of roots $\Phi^+ = \Psi \cup \{a\}$ in Φ with $w_a(\Phi^+) = \Psi \cup \{-a\}$. Thus, we focus on U_a-conjugation.

Fix isomorphisms $u_c : \mathbf{G}_{\rm a} \simeq U_c$ for $c \in \Phi$, so for $b \neq \pm a$ in Φ we have

$$u_a(x) u_b(y) u_a(x)^{-1} u_b(y)^{-1} \in U_{(\langle a \rangle + \langle b \rangle) \cap \Phi^+}.$$

Thus, $u_a(x) u_b(y) u_a(x)^{-1} \in U_{(\langle a \rangle + \langle b \rangle) \cap \Phi^+} \cdot U_b \subset U_\Psi$ since U_Ψ is directly spanned in any order by the root groups U_c for $c \in \Psi$. This gives that $u_a(x)$ conjugates U_b into U_Ψ for all $b \in \Psi$, so U_a normalizes U_Ψ. ∎

The preceding lemma implies (via (2)) that

$$G_a B n_w B = U_\Psi G_a n_w B = U_\Psi (B_a \cup U_a n_a B_a) n_w B \subset B n_w B \cup U n_a B_a n_w B = B n_w B \cup U n_a U_a n_w B$$

since $B_a n_w = U_a T n_w = U_a n_w T$ with $T \subset B$. But $U n_a U_a = U U_{-a} n_a$, so

$$G_a B n_w B \subset B n_w B \cup B U_{-a} n_a n_w B.$$

We claim that

$$G_a \subset U_a n_w B n_w^{-1} \cup U_a n_a n_w B n_w^{-1}, \tag{4}$$

from which it would follow that $U_{-a} n_a n_w \subset U_a n_w B \cup U_a n_a n_w B$, so

$$B U_{-a} n_a n_w B \subset B n_w B \cup B n_a n_w B$$

and hence $G_a B n_w B \subset B n_w B \cup B n_a n_w B$ as desired for (3). To prove (4) it is harmless to replace the Borel subgroup $n_w B n_w^{-1}$ of G containing T with its intersection $G_a \bigcap (n_w B n_w^{-1})$. This intersection is a Borel subgroup of $G_a = Z_G(T_a)$ containing T, so it is equal to one of the groups $B_{\pm a} = T \cdot U_{\pm a}$. Hence, (4) is equivalent to

$$G_a \stackrel{?}{=} U_a B_{\pm a} \cup U_a n_a B_{\pm a}$$

for each of the roots $\pm a$. For the case of B_a this is the Bruhat decomposition (2) of the semisimple-rank 1 group G_a relative to (B_a, T), and for the case of B_{-a} this is the left n_a-translate of the Bruhat decomposition of G_a relative to the pair (B_{-a}, T). ∎

Here is an important application of the Bruhat decomposition.

Proposition 2.5 (Chevalley)**.** *Let (G, T) be a split connected semisimple group over a field k, and assume that G is simply connected. Then $G(k)$ is generated by the subgroups $U_c(k)$ for $c \in \Phi(G, T)$.*

Proof. Let B be a Borel k-subgroup containing T, $\Phi^+ = \Phi(B,T)$ a positive system of roots in $\Phi = \Phi(G,T)$, and Δ the base of Φ^+. By the Bruhat decomposition, $G(k)$ is generated by the subgroups $U_c(k)$, $T(k)$, and $n_a \in N_{G(k)}(T)$ for representatives n_a of the generating set $\{r_a\}_{a\in\Delta}$ of $W = W(G,T)(k)$.

Since G is *simply connected*, we have an isomorphism $\mathbf{G}_m^{\Delta^\vee} \simeq T$ via $(\lambda_a)_{a\in\Delta} \mapsto \prod_{a\in\Delta} a^\vee(\lambda_a)$. This direct product structure implies that even on rational points, $T(k)$ is generated by its subgroups $a^\vee(k^\times)$ for $a \in \Delta$. Hence, $G(k)$ is generated by: the subgroups $\{U_c(k)\}_{c\in\Phi}$, the subgroups $a^\vee(k^\times)$ for $a \in \Delta$, and representative elements n_a for $a \in \Delta$. But $\Phi = W \cdot \Delta$, so in this generating list we can limit the root groups considered to just those associated to roots in Δ. But $\langle U_a, U_{-a}\rangle \simeq \mathrm{SL}_2$ in which $U_{\pm a}$ go over to the standard unipotent k-subgroups $U^\pm$, $a^\vee(k^\times)$ goes over to the subgroup of diagonal elements, and n_a goes over to the standard Weyl element. It is classical that $\mathrm{SL}_2(k)$ is generated by $U^\pm(k)$, so we are done. ■

The "simply connected" hypothesis in Proposition 2.5 cannot be dropped. To prove this, consider a split connected semisimple k-group G. Let $f : \widetilde{G} \to G$ be the simply connected central cover and $\widetilde{T}$ the split maximal k-torus preimage of T in $\widetilde{G}$. We have seen in class (Example 5.2.6) that the bijection $\Phi(\widetilde{G}, \widetilde{T}) = \Phi(G,T)$ induced by $\mathrm{X}(f) : \mathrm{X}(T)_{\mathbf{Q}} \simeq \mathrm{X}(\widetilde{T})_{\mathbf{Q}}$ yields isomorphisms $U_{c'} \simeq U_c$ for corresponding roots $c \in \Phi(G,T)$ and $c' \in \Phi(\widetilde{G},\widetilde{T})$. Proposition 2.5 implies that the subgroups $U_c(k) \subset G(k)$ generate the image of $\widetilde{G}(k) \to G(k)$, a normal subgroup for which the cokernel is a generally non-trivial commutative group. (For example, $\mathrm{SL}_n(k) \to \mathrm{PGL}_n(k)$, has cokernel $k^\times/(k^\times)^n$, and in general the cokernel is a subgroup of $\mathrm{H}^1(k, \ker f)$.) More explicitly, the commutator morphism $\widetilde{G} \times \widetilde{G} \to \widetilde{G}$ factors uniquely through a morphism $G \times G \to \widetilde{G}$ since G is a central quotient of $\widetilde{G}$, and this induced morphism is a lift of the commutator morphism of G. Hence, the commutator subgroup of $G(k)$ is contained in the image of $\widetilde{G}(k) \to G(k)$.

3. The long Weyl element

As we noted in Remark 1.1, there is a unique longest element $w_0 \in W$ relative to $\{r_a\}_{a\in\Delta}$, and it is characterized by the property that $w_0(\Phi^+) = -\Phi^+$. It is therefore natural to wonder if perhaps $w_0 = -1$. This holds if and only if $-1 \in W$ inside $\mathrm{GL}(\mathrm{X}(T))$, since if $-1 \in W$ then it satisfies the property uniquely characterizing $w_0 \in W$ through its effect on Φ^+. But does -1 belong to W?

Inspection of the Bourbaki Plates shows that $-1 \in W$ precisely for the root systems A_1, B_n ($n \geq 2$), C_n ($n \geq 3$), D_{2m} ($m \geq 2$), E_7, E_8, F_4, and G_2. In the other cases (i.e., A_n for $n \geq 2$, D_{2m+1} for $m \geq 2$, and E_6), the Bourbaki Plates show that $w_0 = -\iota$ where ι on $\mathbf{Q}^\Delta$ arises from the unique diagram involution of Φ.

Remark 3.1. Don't confuse $w_0 = \prod_{a\in\Phi^+} r_a$ (independent of the order of multiplication) with the *Coxeter element* $\prod_{a\in\Delta} r_a$ that does depend on the order of multiplication but has conjugacy class independent of all choices (see Bourbaki Chapter IV, §1.11). Whereas w_0 has order 2, the order of the Coxeter element is the *Coxeter number* h that is generally larger than 2.

We finish by describing the long Weyl element and the Coxeter element (up to conjugacy) for the root system A_n with $n \geq 1$. Recall that the weight lattice is $X = \mathbf{Z}^{n+1}/\mathrm{diag}$. Letting $\varepsilon_1, \ldots, \varepsilon_{n+1}$ be the standard basis of $\mathbf{Z}^{n+1}$, a root basis Δ is given by $a_i = \varepsilon_i - \varepsilon_{i+1}$ modulo the diagonal, for $1 \leq i \leq n$.

For $1 \leq i \leq n$, the effect of the reflection r_{a_i} is given by negation of a_i, $a_{i+1} \mapsto a_{i+1} + a_i$ if $i < n$, $a_{i-1} \mapsto a_{i-1} + a_i$ if $i > 1$, and no effect on any other a_j. This is induced by the linear automorphism of $\mathbf{Z}^{n+1}$ (preserving the diagonal!) given by swapping ε_i and ε_{i+1} and leaving all other ε_j unaffected. In terms of the indexed set of residue classes of ε_j's in the weight lattice, it follows that the subgroup

$S_{n+1} \subset \mathrm{GL}(\mathbf{Z}^{n+1})$ arising as permutations of the standard basis (preserving the diagonal) maps onto $W \subset \mathrm{GL}(X)$. The resulting map $S_{n+1} \twoheadrightarrow W$ is an isomorphism since the finite kernel consists of unipotent elements.

Under this description of W, the long Weyl element w_0 is induced by the swapping of ε_i and ε_{n+2-i} for all $1 \leq i \leq n+1$; i.e., it swaps a_i and $-a_{n+1-i}$ for all $1 \leq i \leq n$. This shows quite explicitly that $w_0 \neq 1$ when $n \geq 2$. Likewise, the Coxeter element (up to conjugacy) is induced by the "left shift" $\varepsilon_i \mapsto \varepsilon_{i-1}$ using indexing modulo $n+1$ (so $\varepsilon_1 \mapsto \varepsilon_{n+1}$). In terms of Δ, this is induced by $a_i \mapsto a_{i-1}$ for $1 < i \leq n$ and $a_1 \mapsto -(a_1 + \cdots + a_n)$. Clearly this latter operation is not an involution when $n > 2$.