# Math 249B. Cartan's connectedness theorem

## 1. Introduction

Let $G$ be a connected semisimple group over $\mathbf{R}$. The group $G(\mathbf{R})$ is often disconnected for its analytic topology (in contrast with the situation over $\mathbf{C}$). For example, if $G = \mathrm{PGL}_{2m}$ then there is a natural continuous surjection

$$\det : G(\mathbf{R}) \to \mathbf{R}^\times/(\mathbf{R}^\times)^2 = \{\pm 1\}$$

induced by the determinant on $\mathrm{GL}_{2m}(\mathbf{R})$. Likewise, if $(V, q)$ is an indefinite non-degenerate quadratic space over $\mathbf{R}$ then $\mathrm{SO}(q)(\mathbf{R})$ is generally disconnected (with 2 or 4 connected components).

It is a general theorem of Whitney that if $X$ is an $\mathbf{R}$-scheme of finite type then $X(\mathbf{R})$ has only finitely many connected components. (For a proof, see Appendix A of Milnor's book *Singular points of complex hypersurfaces*, which rests on the Lemma in §1 of "The Lefschetz theorem on hyperplane sections" by Andreotti and Frankel in Annals **69** (1959).) That is overkill for our purposes, and gives very limited information. We shall directly prove that $\pi_0(G(\mathbf{R}))$ is a 2-torsion finite abelian group controlled by the maximal split $\mathbf{R}$-tori in $G$. This will emerge from our proof of:

**Theorem 1.1** (E. Cartan)**.** *If $G$ is simply connected then $G(\mathbf{R})$ is connected.*

The original approach of Cartan used Riemannian geometry. Briefly, since $G(\mathbf{C})$ is a connected and topologically simply connected Lie group with $G(\mathbf{R})$ the fixed points of the involution given by complex conjugation, the problem is reduced to showing that any involution of a connected and simply connected Lie group has connected locus of fixed points, a problem Cartan solved by geometric methods.

We will take a different approach, due to Borel and Tits (in §4 of their 1972 paper in IHES 41 that is a supplement to their big 1965 paper on reductive groups in IHES 27). This deduces the general case from the anisotropic and split cases by using the relationship between absolute and relative roots. Our presentation of their technique explains some points in more detail and streamlines other aspects (to keep the exposition self-contained relative to this course) by using our prior work with relative root systems.

*Example* 1.2. As a warm-up, let's discuss the anisotropic and split cases. If $G$ is anisotropic then $G(\mathbf{R})$ is compact, as can be proved by adapting Prasad's method from the handout on compactness and anisotropicity over local fields. This is explained more fully as Theorem D.2.4 in the Luminy SGA3 article on reductive group schemes, where it is also shown that $G(\mathbf{R})$ is connected in such cases without any need to assume $G$ is simply connected (in the sense of algebraic groups).

The split case is more interesting. For a split maximal $\mathbf{R}$-torus $T$, the "simply connected" hypothesis (which amounts to a coroot basis $\Delta^\vee$ of $\Phi(G,T)^\vee$ being a $\mathbf{Z}$-basis of $\mathrm{X}_*(T)$) ensures that $G(\mathbf{R})$ is generated by its subgroups $U_a(\mathbf{R})$ for $a \in \Phi(G,T)$ (see Proposition 2.5 in the handout on geometric Bruhat decomposition); for $\mathrm{SL}_2$ this is very classical (over any field). But each $U_a(\mathbf{R}) = \mathbf{R}$ is connected and passes through the identity, so any word in finitely many elements of such subgroups lies in the connected component of the identity (consider the associated "word map" using entire $U_a(\mathbf{R})$'s). Hence, $G(\mathbf{R})$ is connected!

The strategy for the general case consists of three steps, using a maximal split $\mathbf{R}$-torus $S$ and the associated relative root system ${}_{\mathbf{R}}\Phi := \Phi(G,S)$:

(I) Show that $S(\mathbf{R}) \to \pi_0(G(\mathbf{R}))$ is surjective, so it suffices to prove that $S(\mathbf{R}) \subset G(\mathbf{R})^0$. (Surjectivity does not use the "simply connected" hypothesis, and gives precise control over $\#\pi_0(G(\mathbf{R}))$ for general connected semisimple $G$ by applying Cartan's theorem to the simply connected central cover; see Remark 4.4.)

(II) Prove that for a basis $\Delta$ of the root system ${}_{\mathbf{R}}\Phi^{\mathrm{nm}}$ of non-multipliable relative roots, the associated set $\Delta^\vee$ of relative coroots is a basis of $\mathrm{X}_*(S)$. In particular, $S$ is a direct product of copies of $\mathrm{GL}_1$ embedded by the coroots in $\Delta^\vee$, so to prove $S(\mathbf{R}) \subset G(\mathbf{R})^0$ it suffices to check the result after replacing $G$ with the rank-1 connected semisimple subgroup $\mathscr{D}(Z_G(S_a))$ containing $a^\vee(\mathrm{GL}_1)$ for each $a \in {}_{\mathbf{R}}\Phi^{\mathrm{nm}}$. (This derived group is *simply connected*, as for derived groups of torus centralizers in simply connected groups in general; see Corollary 9.5.11 of the course notes.)

(III) For a connected semisimple group with relative rank 1 over a field $k$, construct $\mathrm{SL}_2$ as a $k$-subgroup containing a given maximal split $k$-torus. This reduces the task from (II) to the connectedness of $\mathrm{SL}_2(\mathbf{R})$ that is a special case of Example 1.2 (or more concretely, $\mathrm{SL}_2(\mathbf{R})$ is generated by the $\mathbf{R}$-points of the standard root groups).

Step (II) rests on some input from Bourbaki that we will review when needed, and step (III) involves some clever group-theoretic considerations (due to Borel and Tits).

## 2. Control of $\pi_0(G(\mathbf{R}))$ by $S(\mathbf{R})$

Let $G$ be a connected reductive $\mathbf{R}$-group, and $S$ a maximal split $\mathbf{R}$-torus. We shall show that $S(\mathbf{R})$ meets every connected component of $G(\mathbf{R})$, which is to say $S(\mathbf{R})G(\mathbf{R})^0 = G(\mathbf{R})$. If $r = \dim S$ then $S(\mathbf{R}) = (\{\pm 1\} \times \mathbf{R}_{>0})^r$, so it would follow that $\pi_0(G(\mathbf{R}))$ is 2-torsion abelian with size at most $2^r$. (A description of the size of $\pi_0(G(\mathbf{R}))$ is given in Remark 4.4.) To prove $S(\mathbf{R}) \to \pi_0(G(\mathbf{R}))$ is surjective, we need:

**Lemma 2.1.** *Let $P \subset G$ be a minimal parabolic $\mathbf{R}$-subgroup containing $S$. Then the quotient space $G(\mathbf{R})/P(\mathbf{R}) = (G/P)(\mathbf{R})$ is connected.*

The equality $G(\mathbf{R})/P(\mathbf{R}) = (G/P)(\mathbf{R})$ is a special case of a general equality we have proved over any field (with any parabolic subgroup over the ground field), and crucially it is a *topological equality* since passage to $\mathbf{R}$-points carries smooth morphisms (such as $G \to G/P$) to submersions due to the Submersion Theorem.

*Proof.* We have a dynamic description for $P$, so $P = P_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(S)$. Thus, for $U = U_G(-\lambda)$ we have a Zariski-dense open immersion $U \subset G/P$. As a variety $U$ is an affine space. Hence, $U(\mathbf{R})$ is connected. Thus, it suffices to show that $U(\mathbf{R})$ is dense in $(G/P)(\mathbf{R})$.

Rather generally, if $X$ is a smooth $\mathbf{R}$-scheme and $Z \subset X$ is a nowhere-dense locally closed subset then we claim that $Z(\mathbf{R}) \subset X(\mathbf{R})$ has measure zero and hence empty interior. (Applying this to $X = G/P$ and $Z = (G/P) - U$ would then give the desired density on $\mathbf{R}$-points.) By stratifying $Z$ we may assume it is smooth, yet its dimension is everywhere strictly smaller than that of $X$. The map of manifolds $Z(\mathbf{R}) \to X(\mathbf{R})$ is therefore nowhere a submersion, so by Sard's Theorem its image has measure 0. ■

If $\Gamma$ is a locally connected topological group with $\Gamma^0$ denoting is its (necessarily open) identity component and if $H$ is a subgroup such that $\Gamma/H$ is connected then $H$ meets every connected component of $\Gamma$ (so $\pi_0(H) \to \pi_0(\Gamma)$ is surjective). Indeed, since $\Gamma/\Gamma^0$ is discrete (by openness of $\Gamma^0$), the quotient space $H\backslash\Gamma/\Gamma^0$ is trivial. It follows that $H \to \Gamma/\Gamma^0$ is surjective.

Setting $\Gamma = G(\mathbf{R})$ and $H = P(\mathbf{R})$, it follows that $\pi_0(P(\mathbf{R})) \to \pi_0(G(\mathbf{R}))$ is surjective. The Levi decomposition $P = Z_G(S) \rtimes U$ with $U(\mathbf{R})$ connected then implies that $\pi_0(Z_G(S)(\mathbf{R})) \to \pi_0(G(\mathbf{R}))$ is surjective. But the connected reductive $\mathbf{R}$-group $Z_G(S)/S$ is anisotropic, so its group $(Z_G(S)/S)(\mathbf{R})$ of $\mathbf{R}$-points is compact and *connected*. The submersive homomorphism of Lie groups $Z_G(S)(\mathbf{R}) \to (Z_G(S)/S)(\mathbf{R})$ has image that is open, hence closed, so by connectedness of the target it is surjective. Thus, $Z_G(S)(\mathbf{R})/S(\mathbf{R}) = (Z_G(S)/S)(\mathbf{R})$, so this quotient modulo $S(\mathbf{R})$ is connected; hence, $S(\mathbf{R})$ meets every connected component of $Z_G(S)(\mathbf{R})$. This says $S(\mathbf{R}) \to \pi_0(Z_G(S)(\mathbf{R}))$ is surjective, so we obtain:

**Proposition 2.2.** *For a connected reductive* $\mathbf{R}$*-group* $G$ *with maximal split* $\mathbf{R}$*-torus* $S$*, the natural map* $S(\mathbf{R}) \to \pi_0(G(\mathbf{R}))$ *is surjective.*

Now assume $G$ is *semisimple*. Consider a basis $\Delta$ for ${}_{\mathbf{R}}\Phi^{\mathrm{nm}}$, so the associated relative coroots provide an isogeny

$$\prod_{a\in\Delta} \mathrm{GL}_1 \to S$$

defined by $(y_a) \mapsto \prod_{a\in\Delta} a^\vee(y_a)$. This isogeny is an isomorphism if and only if $\Delta^\vee$ is a $\mathbf{Z}$-basis of $\mathrm{X}_*(S)$, which is to say that the root datum

$$(\mathrm{X}(S), {}_{\mathbf{R}}\Phi^{\mathrm{nm}}, \mathrm{X}_*(S), ({}_{\mathbf{R}}\Phi^{\mathrm{nm}})^\vee = ({}_{\mathbf{R}}\Phi^\vee)^{\mathrm{nd}})$$

is "simply connected" in the sense that $({}_{\mathbf{R}}\Phi^{\mathrm{nm}})^\vee$ generates $\mathrm{X}_*(S)$ over $\mathbf{Z}$. We will establish this property when $G$ is simply connected, and that will permit us to reduce the proof of Cartan's theorem to the case of $\mathbf{R}$-rank equal to 1 (which needs some work too!).

## 3. A result on relative root systems

We call a semisimple root datum $(X, \Phi, X^\vee, \Phi^\vee)$ *simply connected* when $\mathbf{Z}\cdot\Phi^\vee = X^\vee$, and *adjoint type* when $\mathbf{Z}\cdot\Phi = X$. For a connected semisimple group $G$ over a field $k$, since the ranks of the absolute and relative root systems can be very different, it is not at all apparent whether the properties of being simply connected or adjoint type for the absolute root datum should be inherited by the relative root datum. Hence, it may be surprising that this works well in the simply connected case:

**Theorem 3.1.** *Let* $G$ *be a connected semisimple group over a field* $k$*, with maximal split* $k$*-torus* $S$*. If* $G$ *is simply connected then the relative root datum formed by the non-multipliable relative roots is simply connected; i.e.,* $(\Phi(G,S)^{\mathrm{nm}})^\vee$ *spans* $\mathrm{X}_*(S)$*.*

*Remark* 3.2. If a root $a$ is multipliable then $2a$ is not multipliable and $(2a)^\vee = a^\vee/2$, so $a^\vee = 2(2a)^\vee$. Hence, the $\mathbf{Z}$-span of $(\Phi(G,S)^{\mathrm{nm}})^\vee$ coincides with that of $\Phi(G,S)^\vee$.

To prove Theorem 3.1, we first reduce to the absolutely simple case. Recall that $G = \mathrm{R}_{k'/k}(G')$ for a finite étale $k$-algebra $k'$ and a smooth affine $k'$-group $G'$ whose fiber $G'_i$ over each factor field $k'_i$ of $k'$ is connected semisimple, absolutely simple, and simply connected. For the split $k$-torus images $S_i \subset \mathrm{R}_{k'_i/k}(G'_i)$ we have $S \subset \prod S_i$, so by maximality of $S$ this is an equality and each $S_i$ is maximal. Hence, we can pass to factors and assume $k'$ is a field.

As we saw in §6 of the handout on the relative Bruhat decomposition, the $k$-torus $S$ is the maximal split $k$-subtorus of $\mathrm{R}_{k'/k}(S')$ for a unique maximal split $k'$-torus $S' \subset G'$. Moreover, as we saw in the handout on relative root systems, naturally $\mathrm{X}(S') \simeq \mathrm{X}(S)$ identifying $\Phi(G', S')$ with $\Phi(G, S)$. The construction of reflections associated to relative roots shows that if $a' \in \Phi(G', S')$ restricts to $a \in \Phi(G, S)$ then $r_a$ is induced by $\mathrm{R}_{k'/k}(r_{a'})$, so in this way we see that $r_a$ on $\mathrm{X}(S)$ coincides with $r_{a'}$ on $\mathrm{X}(S')$. It follows that the associated coroots $a^\vee$ and $a'^\vee$ which are built to compute these respective reflections must coincide. Hence, it suffices to treat $G'$ instead of $G$, so we may now assume $G$ is *absolutely simple* over $k$.

The absolute root system $\Phi := \Phi(G_{k_s}, T_{k_s})$ for a maximal $k$-torus $T \supset S$ in $G$ is now *irreducible*. Our study of relative root systems yields from the irreducibility of $\Phi$ that ${}_{\mathbf{R}}\Phi := \Phi(G, S)$ is also irreducible (but possibly non-reduced!), with a basis ${}_{\mathbf{R}}\Delta$ given by the set of nontrivial restrictions to $S$ of a choice of basis $\Delta$ of $\Phi$. This opens the door to applying two general properties of irreducible root systems, as follows.

To state the first of these, we recall from §1.8 in Chapter VI of Bourbaki that for any *irreducible* root system equipped with a specified root basis, there is always a unique positive root each of whose coefficients relative to the chosen basis is at least as large as the corresponding coefficient occurring in all other roots (in particular, all coefficients are positive for this distinguished root); we call it the *highest root*. The following lemma concerning the highest root is the unique Corollary in §2.3 of Chapter VI of Bourbaki:

**Lemma 3.3.** *Let $(V, \Phi)$ be an irreducible and reduced root system, and let $\Delta$ be a root basis. Let $\{\varpi_a\}_{a \in \Delta} \subset P(\Phi^\vee)$ be the dual basis to $\Delta$ (the "fundamental weights" for the dual root system $\Phi^\vee$). Let $b = \sum_{a \in \Delta} m_a a$ be the highest root for $\Phi$ relative to $\Delta$.*

*A set of representatives for the nonzero elements of the fundamental group $P(\Phi^\vee)/Q(\Phi^\vee)$ of the dual root system is given by the dual weights $\varpi_a$ associated to those $a \in \Delta$ whose coefficient $m_a$ in the highest root $b$ of $\Phi$ is equal to 1.*

This lemma could be proved by case-checking of the tables at the end of Bourbaki, but Bourbaki gives a case-free proof. Here is a companion result in the non-reduced case.

**Lemma 3.4.** *For a non-reduced irreducible root system $(V, \Phi)$ and the irreducible root system $(V, \Phi^{\mathrm{nm}})$ of non-multipliable roots,*

$$P(\Phi^{\mathrm{nm}}) = P(\Phi) = Q(\Phi).$$

*Proof.* The first equality in Lemma 3.4 is dual to the equality $Q((\Phi^\vee)^{\mathrm{nd}}) = Q(\Phi^\vee)$ that says $Q(\Phi^\vee)$ is spanned over $\mathbf{Z}$ by elements of a coroot basis, which in turn holds since elements of a root coroot basis are certainly non-divisible.

To prove that $P(\Phi) = Q(\Phi)$, we first note that the inclusion of root lattices $Q(\Phi^{\mathrm{nm}}) \subset Q(\Phi)$ has index 2 (because a root basis $\Delta$ for $\mathrm{BC}_n$ consists of one multipliable root $a$ and $n-1$ additional roots that are neither multipliable nor divisible, so the root system of non-multipliable roots has a root basis consisting of $2a$ along with the same $n-1$ additional roots). Since $P(\Phi) = P(\Phi^{\mathrm{nm}})$, to establish that the inclusion $Q(\Phi) \subset P(\Phi)$ has index 1 it suffices to show that $Q(\Phi^{\mathrm{nm}})$ has index 2 inside $P(\Phi^{\mathrm{nm}})$; i.e., the root system $\Phi^{\mathrm{nm}}$ of type

$\mathrm{C}_n$ $(n \geq 1)$ has fundamental group of order 2. But this is clear from the $\mathrm{C}_n$-table at the end of Bourbaki for $n \geq 3$. For $n \leq 2$, we check type-A and type-B for $\mathrm{A}_1$ and $\mathrm{B}_2$. ■

We are finally ready to prove Theorem 3.1, completing step II of the proof of Cartan's connectedness theorem.

*Proof.* We have reduced to the case that $G$ is absolutely irreducible, and we may and do assume $S \neq 1$. Let $T \subset G$ be a maximal $k$-torus containing $S$, so the reduced absolute root system $\Phi = \Phi(G_{k_s}, T_{k_s})$ is irreducible (as $G$ is absolutely simple) and $\mathrm{X}(T_{k_s}) = P(\Phi)$ (i.e., the coroots span $\mathrm{X}_*(T_{k_s})$) by the hypothesis that $G$ is simply connected.

Let $V = \mathrm{X}(T_{k_s})_{\mathbf{Q}}$ and $V_0 = \mathrm{X}(S)_{\mathbf{Q}} = \mathrm{X}(S_{k_s})_{\mathbf{Q}}$. The restriction map $\rho : \mathrm{X}(T_{k_s}) \to \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ is surjective since $S$ is a $k$-subtorus of $T$, so in terms of the map $r = \rho_{\mathbf{Q}} : V \twoheadrightarrow V_0$ it suffices to show that $r(P(\Phi)) = P({}_k\Phi)$ inside $V_0$, where ${}_k\Phi := \Phi(G, S)$. Certainly $r(P(\Phi)) = \rho(\mathrm{X}(T_{k_s})) = \mathrm{X}(S) \subset P({}_k\Phi)$, and we need to prove the reverse containment.

Under the quotient map $r : V \twoheadrightarrow V_0$, for any lattice $L \subset V$ the image $r(L) \subset V_0$ is a lattice and its $\mathbf{Z}$-dual $r(L)^* \subset V_0^*$ coincides with $L^* \cap V_0^*$ (as we verify immediately from the definitions). Taking $L = P(\Phi)$, our problem is equivalent to proving

(1) $$Q(\Phi^\vee) \cap V_0^* \overset{?}{\subset} Q({}_k\Phi^\vee)$$

and we know the reverse containment holds.

Since $Q(\Phi)$ and $P(\Phi^\vee)$ are dual lattices, clearly

$$Q(\Phi^\vee) \cap V_0^* \subset P(\Phi^\vee) \cap V_0^* = r(Q(\Phi))^* = Q({}_k\Phi)^* = P({}_k\Phi^\vee),$$

(the equality $r(Q(\Phi)) = Q({}_k\Phi)$ is immediate from the following facts: $Q(\Phi)$ is the $\mathbf{Z}$-span of $\Delta$, $r(\Delta)$ lies between ${}_k\Delta$ and ${}_k\Delta \cup \{0\}$ for a basis ${}_k\Delta$ of ${}_k\Phi$ (as we showed in our study of relative root system), and $Q({}_k\Phi)$ is the $\mathbf{Z}$-span of ${}_k\Delta$). Combining this with the known reverse of (1), we get a natural map of fundamental groups

(2) $$P({}_k\Phi^\vee)/Q({}_k\Phi^\vee) \to P(\Phi^\vee)/Q(\Phi^\vee)$$

induced by the inclusion of $V_0^*$ into $V^*$, and the desired containment (1) expresses exactly that this map is injective. Hence, we need to prove such injectivity.

Recall that ${}_k\Phi$ (which is non-empty, since $S \neq 1$ and $G$ is semisimple) is the set of nonzero elements in $r(\Phi)$. Our study of relative root systems gave that ${}_k\Phi$ is irreducible since $\Phi$ is irreducible. If the irreducible root system ${}_k\Phi$ is non-reduced then so is ${}_k\Phi^\vee$, so its fundamental group is trivial by Lemma 3.4 and hence (2) is clearly injective. We therefore may and do assume that the irreducible root system ${}_k\Phi$ is *reduced*. Lemma 3.3 therefore applies to $\Phi$ and ${}_k\Phi$, giving us a handle on both fundamental groups in (2) and motivating the following considerations with highest roots.

Let $b = \sum_{a \in \Delta} m_a a$ be the highest root in $\Phi$, so $m_a \geq 1$ for all $a$. For every $c \in \Phi$ we have $c = \sum_{a \in \Delta} \nu_a a$ with $\nu_a \leq m_a$ for all $a$. Thus, restricting to $S$ and dropping those $a$ killed by such restriction gives

$$r(c) = \sum_{a' \in {}_k\Delta} \nu_{a'} a'$$

where

$$\nu_{a'} = \sum_{a\in\Delta, r(a)=a'} \nu_a.$$

For each $c$ we have $c \le b$ coefficient-wise along $\Delta$, so $r(c) \le r(b)$ coefficient-wise along ${}_k\Delta$. This says that $r(b)$ is the highest root $\sum_{a'\in {}_k\Delta} m_{a'}a'$ for ${}_k\Phi$ with respect to ${}_k\Delta$. Hence, the respective coefficients $\{m_a\}$ and $\{m'_{a'}\}$ for the highest roots $b \in \Phi$ and $r(b) \in {}_k\Phi$ satisfy the relations

$$m'_{a'} = \sum_{a\in\Delta, r(a)=a'} m_a \tag{3}$$

for all $a' \in {}_k\Delta$.

Now consider a nonzero $\xi \in P({}_k\Phi^\vee)/Q({}_k\Phi^\vee)$. By Lemma 3.3, there exists a unique $a' \in {}_k\Delta$ such that $m'_{a'} = 1$ and the dual weight $\varpi_{a'} \in P({}_k\Phi^\vee)$ represents $\xi$. We have to show that $\varpi_{a'}$ viewed as an element of the dual weight lattice $P(\Phi^\vee)$ (using the inclusion of $V_0^*$ into $V^*$) does not lie inside coroot lattice $Q(\Phi^\vee)$. But (3) with $m'_{a'} = 1$ implies that there is a unique $a \in \Delta$ satisfying $r(a) = a'$ and moreover that $m_a = 1$. We shall prove that $\varpi_{a'} = \varpi_a$ via the inclusion of $V_0^*$ into $V^*$, so then (2) carries $\xi$ to the class of $\varpi_a$ which in turn is nonzero by Lemma 3.3 (as $m_a = 1$).

To establish the equality of $\varpi_{a'}$ and $\varpi_a$ using the inclusion $V_0^* \hookrightarrow V^*$ dual to the natural quotient map $r : V \twoheadrightarrow V_0$, we note that $r$ is identified with the natural map

$$\mathbf{Q}^{\Delta} \twoheadrightarrow \mathbf{Q}^{{}_k\Delta}$$

killing the factors corresponding to $\Delta \cap r^{-1}(0)$ and sending the factor indexed by any other element of $\Delta$ onto the factor indexed by its $S$-restriction in ${}_k\Delta$ (via the identity map on $\mathbf{Q}$-coefficients). In particular, the fact that $a \in \Delta$ is the unique element carried onto $a'$ by $r$ implies that the member $\varpi_{a'}$ in the dual basis to ${}_k\Delta$ is carried under $V_0^* \hookrightarrow V^*$ to the member $\varpi_a$ in the dual basis to $\Delta$. ∎

## 4. A split subgroup

We have reduced the proof of Cartan's Theorem to when $G$ is absolutely simple (and simply connected). For a maximal split $\mathbf{R}$-torus $S \subset G$, it suffices to prove $S(\mathbf{R}) \subset G(\mathbf{R})^0$ (by Proposition 2.2). The case $S = 1$ is trivial, so we may assume $S \neq 1$.

The cocharacter lattice $\mathrm{X}_*(S)$ has a basis consisting of non-divisible coroots $\{a_i^\vee\}$ by Theorem 3.1, so the isomorphism $\prod_i \mathrm{GL}_1 \simeq S$ via $(y_i) \mapsto \prod a_i^\vee(y_i)$ gives $\prod a_i^\vee(\mathbf{R}^\times) = S(\mathbf{R})$ with non-multipliable $\{a_i\} \subset \Phi(G,S)$. It therefore suffices to show that $a^\vee(\mathbf{R}^\times) \subset G(\mathbf{R})^0$ for each $a \in \Phi(G,S)$.

Fix a choice of $a \in \Phi(G,S)$, and let $S_a = (\ker a)^0$. The connected semisimple group $G'_a := \mathscr{D}(Z_G(S_a))$ is simply connected with $\mathbf{R}$-rank 1. The coroot $a^\vee$ is valued in $G'_a$, and via the identification $\Phi(Z_G(S_a), S) = \Phi(G'_a, a^\vee(\mathrm{GL}_1))$ defined by restriction to the isogeny complement $a^\vee(\mathrm{GL}_1)$ inside $S$ to the central $S_a \subset Z_G(S_a)$, the relative root $a|_{a^\vee(\mathrm{GL}_1)} \in \Phi(G'_a, a^\vee(\mathrm{GL}_1))$ has the same associated coroot $a^\vee$ (why?). Since it is enough to prove that $a^\vee(\mathbf{R}^\times) \subset (G'_a)(\mathbf{R})^0$, we may rename $G'_a$ as $G$ to reduce to the case that $\dim S = 1$.

We shall build a copy of $\mathrm{SL}_2$ as an $\mathbf{R}$-subgroup of $G$ containing $S$. Then we would have $S(\mathbf{R}) \subset \mathrm{SL}_2(\mathbf{R}) \subset G(\mathbf{R})^0$, completing the proof. The rank-1 root system $\Phi(G,S)$ is either $\mathrm{A}_1$

or $\mathrm{BC}_1$. Let $a$ be a non-multipliable relative root. Since $G$ is simply connected, $a^\vee : \mathrm{GL}_1 \to S$ is an isomorphism by Theorem 3.1 and the relative root group $U_a$ is a vector group over $\mathbf{R}$ on whose Lie algebra $S$ acts through the character $a$. Relative to the unique linear structure on $U_a$, let $E \subset U_a$ be a line.

We claim there exists a unique split connected semisimple $\mathbf{R}$-subgroup $L \subset G$ containing $S$ and $E$ with $S$ a maximal $\mathbf{R}$-torus in $L$. Granting this, $L$ must be $\mathrm{SL}_2$ or $\mathrm{PGL}_2$, and $\Phi(L, S)$ of type $\mathrm{A}_1$ contains the character $a$ for which there is an *isomorphism* $a^\vee : \mathrm{GL}_1 \simeq S$ satisfying $\langle a, a^\vee \rangle = 2$. It then follows that $a \in 2\mathrm{X}(S)$, so necessarily $L = \mathrm{SL}_2$ and we would be done. The construction of $L$ is a special case of a general result over any field:

**Proposition 4.1.** *Let $G$ be a connected semisimple group over a field $k$ with a maximal split $k$-torus $S$ of dimension 1. For any non-multipliable $a \in {}_k\Phi := \Phi(G, S)$ and line $E$ inside the vector group $U_a$, there exists a unique split connected semisimple $k$-subgroup $L \subset G$ containing $S$ and $E$ with $S$ maximal in $L$.*

Before we prove this result, we make a few remarks. Note that the $S$-action on $U_a$ has $a$ as its only weight on the Lie algebra since $a$ is non-multipliable, so $U_a$ is a vector group and admits a unique $S$-equivariant linear structure (the "$S$-equivariant" condition can be dropped when $\mathrm{char}(k) = 0$). This defines the notion of "line" inside $U_a$.

Also, this proposition has a generalization allowing any $\dim S > 0$: for a basis $\Delta$ of the reduced root system of *non-multipliable* roots in $\Phi(G, S)$ and a choice of line $E_a \subset U_a$ for each $a \in \Delta$, there exists a unique split connected semisimple $k$-subgroup $L \subset G$ containing $S$ as a maximal $k$-torus and containing each $E_a$; the root datum of $(L, S)$ coincides with the reduced root datum

$$(\mathrm{X}(S), \Phi(G, S)^{\mathrm{nm}}, \mathrm{X}_*(S), (\Phi(G, S)^{\mathrm{nm}})^\vee = (\Phi(G, S)^\vee)^{\mathrm{nd}}).$$

The proof of this generalization is given by Borel and Tits in §7 of their IHES 27 paper on reductive groups (see 7.2). A more conceptual and much simpler version of their argument given in [CGP, Theorem C.2.30] (where $k$ is assumed to be infinite, but from which the finite case can be deduced by the same method as used in the argument below); we specialize the latter to the case of $k$-rank equal to 1.

*Proof.* In the $\mathrm{BC}_1$-case, $\ker a = \mu_2$ and this centralizes $U_{\pm a}$, so it centralizes any possibility for $L$. Hence, in such cases we can replace $G$ with $\mathscr{D}(Z_G(\ker a)^0)$ to reduce to the $\mathrm{A}_1$-case. (This step might lose contact with a "simply connected" hypothesis were one imposed on $G$, since $\ker a$ is finite rather than a torus, so it is important that there is no such assumption concerning $G$ in Proposition 4.1.) Hence, ${}_k\Delta = \{a\}$ is a basis of ${}_k\Phi$, so $U_a = U_G(a^\vee)$ and $U_{-a} = U_G(-a^\vee)$. Note also that $Z_G(S) = Z_G(a^\vee)$, so the multiplication map

$$U_{-a} \times Z_G(S) \times U_a \to G$$

is an open immersion by the dynamic method; denote its image as $\Omega$.

Let $N = N_G(S)$ and $Z = Z_G(S)$, so $N(k)/Z(k) = (N/Z)(k)$ has order 2 since $G$ has $k$-rank 1. Pick a nonzero $u \in E(k)$ and any $n \in N(k) - Z(k)$. Note that $n$-conjugation swaps the two minimal parabolic $k$-subgroups $P_{\pm a} = S \ltimes U_{\pm a}$ containing $S$. The Bruhat decomposition of $G$ relative to $S$ and $P_{-a}$ is

$$G(k) = P_{-a}(k) \coprod P_{-a}(k) n P_{-a}(k) = P_{-a}(k) \coprod U_{-a}(k)(N(k) - Z(k))U_{-a}(k)$$

since $N - Z = nZ$. We have $u \notin P_{-a}(k)$ since $U_a \cap P_{-a} = 1$, so there exist $u', u'' \in U_{-a}(k)$ such that $u'uu'' \in N(k) - Z(k)$. Such $u'$ and $u''$ satisfy the following additional properties (inspired by a result of Tits in the split reductive case):

**Lemma 4.2.** *There exist unique $u', u'' \in U_{-a}(k)$ such that $m(u) := u'uu'' \in N(k)$. Also:*

(i) *$m(u) \notin Z(k)$ and $u', u'' \neq 1$,*

(ii) *if $K/k$ is an extension field and $z \in Z(K)$ satisfies $zuz^{-1} \in U_a(k)$ then $zu'z^{-1} \in U_{-a}(k)$ and $m(zuz^{-1}) = zm(u)z^{-1}$,*

(iii) *$u'' = u' = m(u)^{-1}um(u) \neq 1$, and $m(u)^2 \in S(k)$.*

Before proving this lemma, we explain the motivation for it in a special case:

*Example* 4.3. In the special case of $\mathrm{SL}_2$ or $\mathrm{PGL}_2$, $u \mapsto u'$ recovers the map of varieties $U_a - \{0\} \simeq U_{-a} - \{0\}$ given by

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -1/x & 1 \end{pmatrix}$$

for $x \in k^\times$, as the reader is urged to check. This special case motivates the properties in (i), (ii), and (iii) that the reader should also directly verify for $\mathrm{SL}_2$ and $\mathrm{PGL}_2$ (fun exercise).

*Proof.* We have seen that there exist $u', u'' \in U_{-a}(k)$ such that $u'uu'' \in N(k)$. For any such $u', u''$, the product $n' = u'uu''$ cannot lie in $Z(k)$, since otherwise we would have $u = u'^{-1}n'u''^{-1} \in U_{-a}(k)Z(k)U_{-a}(k) = P_{-a}(k)$, contradicting that $U_a \cap P_{-a} = 1$. Since $N(k) - Z(k) = n'Z(k)$, uniqueness for $u'$ and $u''$ amounts to checking that if $v', v'' \in U_{-a}(k)$ and $v'n'v'' = n'z$ for some $z \in Z(k)$ then $v' = v'' = 1$ (so $z = 1$). But

$$v'(n'v''n'^{-1}) = n'zn'^{-1} \in Z(k)$$

yet $v' \in U_{-a}(k)$ and $n'v''n'^{-1} \in U_a(k)$, so this forces $v' = 1$ and $n'v''n'^{-1} = 1$ (so $v'' = 1$) since the multiplication map

$$U_{-a} \times Z \times U_a \to G$$

is an open immersion (hence injective on $k$-points).

To prove $u', u'' \neq 1$, by passing to inversion if necessary we just need to get a contradiction if there exists $u' \in U_{-a}(k)$ such that $u'u =: n \in N(k)$. But $u' = nu^{-1} = (nu^{-1}n^{-1})n \in U_{-a}(k)n$ yet $u' \in U_{-a}(k)$, so we would have $n \in U_{-a}(k)$ and hence $u = u'^{-1}n \in U_{-a}(k)$, impossible since $u \in U_a(k) - \{0\}$ and $U_a \cap U_{-a} = 1$. We have proved (i).

In the setting of (ii), applying the preceding to $zuz^{-1} \in U_a(k) - \{0\}$ provides unique $u'_z, u''_z \in U_{-a}(k)$ such that $m(zuz^{-1}) := u'_z(zuz^{-1})u''_z \in N(k)$. From the definition of $m(\cdot)$ on $U_a(k) - \{0\}$ we then have

$$z^{-1}m(zuz^{-1})z = (z^{-1}u'_z z \cdot u'^{-1})m(u)(u''^{-1}z^{-1}u''_z z),$$

so multiplying on the left by $m(u)^{-1}$ and on the right by $(u''^{-1}z^{-1}u''_z z)^{-1}$ gives

$$(m(u)^{-1} \cdot z^{-1}m(zuz^{-1})z)(u''^{-1} \cdot z^{-1}u''_z z)^{-1} = m(u)^{-1}(z^{-1}u'_z z \cdot u'^{-1})m(u).$$

Since the left side lies in $Z(K) \ltimes U_{-a}(K) = P_{-a}(K)$ and the right side lies in $U_a(K)$, both sides are trivial (as $P_{-a} \cap U_a = 1$) Analyzing this on each side then yields

$$zu'z^{-1} = u'_z \in U_{-a}(k), zu''z^{-1} = u''_z \in U_{-a}(k), m(zuz^{-1}) = zm(u)z^{-1},$$

establishing (ii).

To prove (iii), pick a geometric point $s \in S(\overline{k})$ satisfying $a(s) = -1$ so that $s$-conjugation on $(U_{\pm a})(\overline{k})$ is negation (i.e., inversion). Thus,

$$sm(u)s^{-1} = u'^{-1}u^{-1}u''^{-1}$$

inside $G(\overline{k})$, so applying inversion gives that

$$u''uu' = sm(u)^{-1}s^{-1} \in G(k) \cap N(\overline{k}) = N(k).$$

The uniqueness of $u'$ and $u''$ therefore forces $u' = u''$ and that $sm(u)^{-1}s^{-1} = m(u)$, so

$$m(u)^2 = (m(u)sm(u)^{-1})s^{-1} \in G(k) \cap S(\overline{k}) = S(k).$$

To prove the elements $u'$ and $m(u)^{-1}um(u) \in U_{-a}(k)$ coincide (where $m(u)^{-1}um(u) \in U_{-a}(k)$ because $u \in U_a(k)$ and conjugation on $S$ by $m(u) \in N(k) - Z(k)$ is inversion), the general equality for (iii) that we have just established (i.e. $u' = u''$ for $u \in U_a(k) - \{0\}$) reduces us to finding $v \in U_a(k) - \{0\}$ such that

$$u'v(m(u)^{-1}um(u)) \in N(k)$$

(as then $u' = v' = v'' = m(u)^{-1}um(u)$). The choice $v := m(u)^{-1}u'm(u) \neq 1$ works (directly from the definition of $m(u)$) since this $v$ lies in $U_a(k)$ due to $m(u)$-conjugation carrying $U_{-a}(k)$ into $U_a(k)$. ■

Now we prove the existence and uniqueness of $L$. Pick $u \in E(k) - \{1\}$. By Lemma 4.2, there is a unique $v \in U_{-a}(k)$ such that $n := m(u) = vuv$ normalizes $S$, and $n$-conjugation on $S$ is inversion. Lemma 4.2 also gives that $n^2 \in S(k)$ and $v = n^{-1}un$. The 1-dimensional smooth connected $k$-subgroup $E_- := n^{-1}En$ of $U_{-a}$ is stable under the conjugation action of $S$, so it is a 1-dimensional $k$-linear subgroup. Clearly $v \in E_-(k)$.

We claim that the $k$-subgroup $E_-$ does not depend on the choice of $u$. Since $S$ acts on $E \simeq \mathbf{G}_{\mathrm{a}}$ through $a \neq 1$, if $u' \in E(k) - \{0\}$ is another choice and $E'_-$ is the analogue of $E_-$ resting on $u'$ in place of $u$ then extracting a square root in $\overline{k}^{\times}$ of the scaling factor relating the nonzero points $u'$ and $u$ in the $k$-line $E(k)$ gives that $u' = sus^{-1}$ for some $s \in S(\overline{k})$. Hence, Lemma 4.2(ii) yields that $n' := m(u') = sns^{-1}$, so

$$(E'_-)_{\overline{k}} = n'^{-1}E_{\overline{k}}n' = sn^{-1}s^{-1}E_{\overline{k}}sns^{-1} = n^{-1}(nsn^{-1}s^{-1})E_{\overline{k}}(nsn^{-1}s^{-1})^{-1}n.$$

Since $nsn^{-1}s^{-1} \in S(\overline{k})$ and $E$ is normalized by $S$, the right side is $n^{-1}E_{\overline{k}}n_a = (E_-)_{\overline{k}}$. Thus, the $k$-subgroups $E'_-$ and $E_-$ coincide over $\overline{k}$, so they are equal over $k$.

Now it is well-posed to define $L$ to be the smooth connected $k$-subgroup of $G$ generated by $S$, $E$, and $E_-$. In view of the construction of $E_-$ and Example 4.3 and the uniqueness in Lemma 4.2, this is the *only possibility* that can work. We will prove that $L$ is a $k$-split connected semisimple $k$-subgroup of $G$ with maximal $k$-torus $S$ such that $\Phi(L, S) = \{\pm a\}$. First we shall treat the case of infinite $k$, and then at the end we settle finite $k$ by using a well-chosen infinite-degree algebraic extension (arranged to preserve the relative rank).

Assume $k$ is infinite. The key point is then to give a concrete subgroup of $L(k)$ that is Zariski-dense in $L$. Since $n = vuv \in L(k)$, the subset

$$\Gamma = E(k)\{1, n\}S(k)E(k)$$

is contained in $L(k)$. We will now prove that $\Gamma$ is a subgroup of $L(k)$ that is Zariski-dense in $L$ (Zariski-density is where we use that $k$ is infinite).

It is clear that $\Gamma$ contains 1 and is stable under inversion in $G(k)$ (as $n$ normalizes $S$ and $n^2 \in S(k)$), so we just have to show that $\Gamma$ is stable under multiplication. Since it is stable under left and right multiplication by $E(k)$ and $S(k)$, and $n^2 \in S(k)$, to prove that $\Gamma$ is a subgroup it suffices to show that $nE(k)n^{-1}$ is contained in $\Gamma$.

By the transitivity of the conjugation action of $S(\overline{k})$ on $E(\overline{k}) - \{0\}$, every nontrivial $v \in E(k)$ has the form $v = sus^{-1}$ for some $s \in S(\overline{k})$. For such an $s$, $a(s) \in k^{\times}$ since $v, u \in E(k)$. Thus, the conjugation action over $\overline{k}$ of $s$ on $E$ and $E_{-}$, and so also of $nsn^{-1}$ on $E$ and $E_{-}$, is actually defined over $k$. Now

$$nvn^{-1} = nsus^{-1}n^{-1} = nsn^{-1} \cdot nun^{-1} \cdot (nsn^{-1})^{-1},$$

so it suffices to prove that $nun^{-1} \in E(k)nE(k)$. Indeed, the conjugation action of $nsn^{-1}$ keeps $E(k)$ stable (since it is a $k$-rational action), and the conjugate

$$c_s := nsn^{-1} \cdot n \cdot (nsn^{-1})^{-1}$$

of $n$ under $nsn^{-1}$ is equal to the product $n \cdot sns^{-1}n^{-1}$ that lies in $nS(k)$ (because, by Lemma 4.2(ii), $sns^{-1} = m(sus^{-1}) = m(u) \in G(k)$, forcing $sns^{-1}n^{-1} \in S(k)$).

The formula $n = u'uu'$ with $u' = n^{-1}un$ yields

$$n = nu'n^{-1} \cdot nun^{-1} \cdot nu'n^{-1} = u \cdot nun^{-1} \cdot u,$$

so $nun^{-1} = u^{-1}nu^{-1} \in E(k)nE(k) \subseteq \Gamma$. This proves that $\Gamma$ is a subgroup. Since $S(k)$, $E(k)$, and $n^{-1}E(k)n = E_{-}(k)$ are Zariski-dense in $S$, $E$, and $E_{-}$ respectively (as $k$ is infinite!), we conclude that $\Gamma$ is Zariski-dense in $L$.

The multiplication map

$$U_{-a} \times Z \times U_a \longrightarrow G$$

is an open immersion by the dynamic method with $\lambda = a^{\vee}$ (recall we arranged $\Phi(G, S) = \{\pm a\}$). Let $\Omega$ be the left $n$-translate of this open subscheme; i.e., $\Omega = U_a nZU_a$. Since $P_a := Z \ltimes U_a$ is a minimal pseudo-parabolic $k$-subgroup of $G$ containing $S$, the relative Bruhat decomposition gives that $\Omega(k) \cap P_a(k)$ is empty. In particular, the set $E(k)S(k)E(k)$ ($\subseteq P_a(k)$) is disjoint from $\Omega(k)$, so $\Gamma \cap \Omega(k) = E(k)nS(k)E(k)$.

The formation of closures is of local nature in any topological space, so by the Zariski-density of $\Gamma$ in $L$ we conclude that the subset $\Gamma \cap \Omega(k) = E(k)nS(k)E(k)$ is Zariski-dense in $L \cap \Omega$. The Zariski-closure of $E(k)nS(k)E(k)$ in $\Omega$ is clearly $EnSE$ (since $k$ is infinite), so the open subscheme $L \cap \Omega$ of $L$ is equal to $EnSE$. In particular, $\dim L = 2 + \dim S = 3$, so the locally closed immersion $E_{-} \times S \times E \to L$ via multiplication is an open immersion. But $n \in L(k)$ and $E_{-} = n^{-1}En$, so

$$L \cap U_{-a}ZU_a = L \cap n^{-1}\Omega = n^{-1}(L \cap \Omega) = n^{-1}EnSE = E_{-}SE.$$

Since $U_{-a}ZU_a$ is a direct product (as a scheme), we conclude that $L \cap Z = S$ (hence $S$ is a split maximal $k$-torus of $L$), $L \cap U_a = E$, and $L \cap U_{-a} = E_{-}$.

The derived group of any solvable smooth connected affine group $H$ is unipotent, so in any such $H$ the normalizer of a maximal torus is equal to its centralizer. Since the element $n \in L(k)$ normalizes $S$ but does not centralize $S$, it follows that $L$ is not solvable. Thus, the connected solvable codimension-1 subgroups $B := S \ltimes E$ and $B_{-} := S \ltimes E_{-}$ of $L$ are Borel

$k$-subgroups of $L$. Since $B \cap B'_- = S$ is a torus, we conclude that $L$ is reductive. The root system of $L$ with respect to $S$ is clearly $\{\pm a\}$. Hence, $\mathscr{D}(L)$ is $\mathrm{SL}_2$ or $\mathrm{PGL}_2$. But we saw that $\dim L = 3$, so $L = \mathscr{D}(L)$ is semisimple too. This settles the case of infinite $k$.

Now assume $k$ is finite. We know that the construction $L$ is the only possibility that can actually work, and we must prove that it does work (i.e., $L$ is a $k$-split connected semisimple $k$-group with maximal $k$-torus $S$ such that $\Phi(L, S) = \{\pm a\}$). Suppose we could find an infinite-degree algebraic extension $k'/k$ such that $S_{k'}$ is maximal split in $G_{k'}$. Then $L_{k'}$ works by the settled case of infinite ground fields, so $L$ is connected semisimple in which the split $k$-torus $S$ is maximal since $S_{k'}$ is maximal in $L_{k'}$ by hypothesis on $k'/k$. Thus, $L$ is split and $\Phi(L, S) = \Phi(L_{k'}, S_{k'}) = \{\pm a\}$, so we would be done.

To find the desired $k'/k$, consider the centralizer $Z_G(S)$. This is a maximal $k$-torus of $G$ since $G$ is quasi-split (as $k$ is finite), so if $k'/k$ is *any* extension and we pick a maximal split $k'$-torus $S' \subset G_{k'}$ containing $S_{k'}$ then $S' \subset Z_{G_{k'}}(S_{k'}) = Z_G(S)_{k'}$. Hence, for our purposes it is sufficient that the anisotropic $k$-torus $Z_G(S)/S$ remains anisotropic over $k'$. That is, it is enough that $k'/k$ is linearly disjoint over $k$ from the finite Galois splitting field $K/k$ of the $k$-torus $Z_G(S)$. Hence, for a prime $\ell$ not dividing $[K : k]$, we may take $k'/k$ to be the unique $\mathbf{Z}_\ell$-extension. ■

*Remark* 4.4. Let $G$ be a connected semisimple $\mathbf{R}$-group with maximal split $\mathbf{R}$-torus $S$ of dimension $r$. We have seen that $\pi_0(G(\mathbf{R})) = (\mathbf{Z}/2\mathbf{Z})^e$ for some $e \leq \dim S$. By using Cartan's connectedness theorem, we can control $e$ as follows. Consider the simply connected central cover $f : \widetilde{G} \to G$. The induced map $\widetilde{G}(\mathbf{R}) \to G(\mathbf{R})$ is a local analytic isomorphism (by the Inverse Function Theorem) with $\widetilde{G}(\mathbf{R})$ connected, so $f(\widetilde{G}(\mathbf{R})) = G(\mathbf{R})^0$. The identity component $\widetilde{S} = f^{-1}(S)^0$ is a maximal split $\mathbf{R}$-torus in $\widetilde{G}$, and the invariance of relative root systems under a central isogeny implies via Theorem 3.1 that $\mathrm{X}(\widetilde{S}) = P({}_{\mathbf{R}}\Phi)$. Beware that the finite central subgroup $\ker f$ might not lie entirely inside $\widetilde{S}$, so $f^{-1}(S) = \widetilde{S} \cdot \ker f$.

We now prove Corollary 4.7 in the Borel–Tits IHES 27 paper, which asserts

$$\#\pi_0(G(\mathbf{R})) = \#(\ker f)(\mathbf{R})/((\ker f)/(\widetilde{S} \cap \ker f))(\mathbf{R}). \tag{4}$$

In terms of a maximal $\mathbf{R}$-torus $T \supset S$, the absolute root system $\Phi = \Phi(G_{\mathbf{C}}, T_{\mathbf{C}})$, and the preimage $\widetilde{T} = f^{-1}(T)$ in $\widetilde{G}$, we have $\mathrm{X}(\widetilde{T}) = P(\Phi)$ and $\mathrm{X}(\widetilde{S}) = P({}_{\mathbf{R}}\Phi)$ (by Theorem 3.1), so

$$(\ker f)(\mathbf{C}) = \mathrm{Hom}(\mathrm{X}(T)/P(\Phi), \mathbf{C}^\times), \;\; (\widetilde{S} \cap \ker f)(\mathbf{C}) = \mathrm{Hom}(\mathrm{X}(S)/P({}_{\mathbf{R}}\Phi), \mathbf{C}^\times)$$

as modules over $\Gamma = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$, so

$$((\ker f)/(\widetilde{S} \cap \ker f))(\mathbf{C}) = \mathrm{Hom}(\mathrm{X}(T)/(\mathrm{X}(S) + P(\Phi)), \mathbf{C}^\times).$$

Thus, an equivalent formulation of (4) in more combinatorial terms is

$$\#\pi_0(G(\mathbf{R})) \stackrel{?}{=} \#\mathrm{Hom}_\Gamma(\mathrm{X}(T)/P(\Phi), \mathbf{C}^\times)/\#\mathrm{Hom}_\Gamma(\mathrm{X}(T)/(\mathrm{X}(S) + P(\Phi)), \mathbf{C}^\times)$$

where $\mathrm{X}(S)$ is the maximal torsion-free quotient of $\mathrm{X}(T)_\Gamma$.

Proposition 2.2 gives that $G(\mathbf{R})/G(\mathbf{R})^0 = S(\mathbf{R})/(S(\mathbf{R}) \cap G(\mathbf{R})^0)$. Hence, since $S(\mathbf{R})$ and $S(\mathbf{R}) \cap G(\mathbf{R})^0$ share the same identity component and $S(\mathbf{R}) = S(\mathbf{R})^0 \times \{\pm 1\}^r$ for $r = \dim S = \dim \widetilde{S}$ (the $\mathbf{R}$-rank of $G$), we see that

$$\#\pi_0(G(\mathbf{R})) = 2^r/\#\pi_0(S(\mathbf{R}) \cap G(\mathbf{R})^0).$$

The surjectivity of $f : \widetilde{G}(\mathbf{R}) \to G(\mathbf{R})^0$ implies that

$$S(\mathbf{R}) \cap G(\mathbf{R})^0 = f((\widetilde{S} \cdot \ker f)(\mathbf{R})) \simeq (\widetilde{S} \cdot \ker f)(\mathbf{R})/(\ker f)(\mathbf{R}).$$

But $(\widetilde{S} \cdot \ker f)(\mathbf{R})$ has *torsion-free* identity component, so the quotient of $(\widetilde{S} \cdot \ker f)(\mathbf{R})$ modulo a finite subgroup has $\#\pi_0$ equal to $\#\pi_0((\widetilde{S} \cdot \ker f)(\mathbf{R}))$ divided by the size of that finite subgroup. Hence,

$$\#\pi_0(G(\mathbf{R})) = 2^r \#(\ker f)(\mathbf{R})/\#\pi_0((\widetilde{S} \cdot \ker f)(\mathbf{R})). \tag{5}$$

The exact sequence

$$1 \to \widetilde{S}(\mathbf{R}) \to (\widetilde{S} \cdot \ker f)(\mathbf{R}) \to ((\widetilde{S} \cdot \ker f)/\widetilde{S})(\mathbf{R}) = (\ker f/(\widetilde{S} \cap \ker f))(\mathbf{R}) \to 1$$

with *finite* target has kernel with $2^r$ connected components, so

$$\#\pi_0((\widetilde{S} \cdot \ker f)(\mathbf{R})) = 2^r \#(\ker f/(\widetilde{S} \cap \ker f))(\mathbf{R}). \tag{6}$$

Combining (5) and (6) yields (4).