Math 249B. Compactness and anisotropicity

1. Introduction

Consider an affine scheme X of finite type over a field k equipped with a rank-1 valuation, so $X(k)$ has a natural topology using a closed immersion $j : X \hookrightarrow \mathbf{A}_k^n$ (the choice of which does not matter). We say that a subset $\Sigma \subset X(k)$ is *bounded* if for some j as above its image in k^n is bounded in the usual sense. This is independent of j because it is equivalent to say that any finite subset (or just one finite generating set) of the coordinate ring $k[X]$ is pointwise bounded on $\Sigma \subset X(k)$. This notion certainly depends on X (i.e., it is not intrinsic to the topological space $X(k)$), and if $Y \subset X$ is closed and $X(k)$ is bounded then clearly $Y(k)$ is bounded.

Example 1.1. Taking X to be the affine k-group GL_d, we claim that a subgroup of $\mathrm{GL}_d(k)$ is bounded if the matrix entries are bounded functions on the subgroup. This is not a tautology since the matrix functions don't generate $k[\mathrm{GL}_d]$ as a k-algebra, but it is easy to verify by using the composition of *closed immersions* $\mathrm{GL}_d \hookrightarrow \mathrm{SL}_{2d} \subset \mathrm{Mat}_{2d}$, where the first map is

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

(using $n \times n$ blocks) and we note that subgroups are stable under inversion.

Using matrix entries in GL_d to check boundedness on a subset of the k-points of this affine variety is not valid for subsets more general than subgroups; this is related to the reason that one cannot correctly topologize the adelic points of GL_d just by using the matrix entries. The problem already occurs for $d = 1$: if $\pi \in R$ is a nonzero non-unit then $S := \{\pi^m\}_{m \geq 1}$ is *unbounded* in $\mathrm{GL}_1(k) = k^\times$ (even though it is bounded in $\mathbf{G}_{\mathrm{a}}(k) = k$; the specified affine k-variety affects the notion of boundedness!) since the identification with the closed subscheme $\{xy = 1\} \subset \mathbf{A}_k^2$ via $t \mapsto (t, 1/t)$ has unbounded k-valued 2nd coordinate on S. In particular, if G is an affine k-group scheme of finite type that contains $\mathbf{G}_m$ or $\mathbf{G}_{\mathrm{a}}$ as a closed k-subgroup then $G(k)$ cannot be bounded since $\mathbf{G}_m(k) = k^\times$ and $\mathbf{G}_{\mathrm{a}}(k) = k$ are unbounded.

Here are two elementary properties of boundedness:

- (i) If k'/k is a finite extension field equipped with a rank-1 valuation extending the one on k (so a subset of k is bounded if and only if its image in k' is bounded) then a subset of $X(k)$ is bounded if and only if its image in $X(k')$ is bounded.
- (ii) If $f : X \to Z$ is a k-morphism between two affine k-schemes of finite type then the map $X(k) \to Z(k)$ on k-points carries bounded sets into bounded sets. This is seen either by describing such a map in terms of polynomials over k or by pulling back $k[Z]$ into $k[X]$.

A converse to (ii) is available (and useful!) in the finite case:

Lemma 1.2. *If $f : X \to Z$ is a finite k-morphism between affine k-schemes of finite type then a bounded subset $\Sigma \subset Z(k)$ has bounded preimage in $X(k)$.*

Proof. Let $h_1, \ldots, h_n$ be k-algebra generators of $k[X]$. It suffices to show that each h_j is bounded on $f^{-1}(\Sigma)$. By finiteness, each h_j satisfies a monic polynomial over $k[Z]$. The coefficient functions on Z in these monic relations are bounded on Σ, so their compositions with f are bounded on $f^{-1}(\Sigma)$. Hence, each $h_j|_\Sigma$ satisfies a monic polynomial with coefficients that are bounded k-valued functions, so clearly each $h_j|_\Sigma$ is bounded. ■

If k is locally compact then boundedness of a *closed* subset of $X(k)$ (with the valuation topology) is equivalent to compactness. In particular, for such k the set $X(k)$ is bounded if and only if it is compact. Hence, boundedness (or not) of $X(k)$ makes sense for general k as above and coincides with compactness when k is locally compact.

The aim of this handout is to give Gopal Prasad's elementary proof of the following result originally due to Bruhat–Tits (in the complete discretely-valued case) and Rousseau:

Theorem 1.3. *Assume k is henselian (i.e., its valuation ring R is henselian) and that G is a k-anisotropic connected reductive k-group. Then $G(k)$ is bounded.*

In particular, if k is a non-archimedean local field then $G(k)$ is compact if and only if G is k-anisotropic.

A reader who is unfamiliar with henselian rings may assume instead that k is complete (a stronger condition). The relevance of either of these conditions on k (henselian or complete) to the proof of Theorem 1.3 is that it ensures every finite extension of k admits a unique valuation (moreover still henselian or complete respectively) extending the given one on k.

Also, a variant of the method of proof below can be applied to the local field $k = \mathbf{R}$ to show that if a connected reductive $\mathbf{R}$-group H is anisotropic then $H(\mathbf{R})$ is compact (the converse being obvious). This (self-contained) argument is given in the proof of Theorem D.2.4 in the Luminy SGA3 notes on reductive group schemes.

Remark 1.4. As a warm-up to the main argument, we relate boundedness to eigenvalues for cyclic groups of invertible matrices. Consider henselian (or complete) k. For $\gamma \in \mathrm{GL}_n(k)$, let $\Gamma = \gamma^{\mathbf{Z}}$. We claim that Γ is bounded if and only if the eigenvalues of γ are units in the valuation ring of a finite extension of k. It is harmless to replace k with a finite extension. The Zariski closure C of Γ in GL_n is smooth and commutative, and it is harmless to replace γ with γ^m for $m > 0$ Taking $m = \#C/C^0$, we may arrange that C is connected. Thus, $C_{\overline{k}} = T \times U$ for a $\overline{k}$-torus T and unipotent smooth connected commutative $\overline{k}$-group U, so by replacing k with a finite extension we can arrange that $C = T \times U$ for a k-torus T and unipotent smooth connected commutative k-group U. Since C is closed in GL_n, so a subset of $C(k)$ is bounded if and only if its image in $\mathrm{GL}_n(k)$ is bounded, we see that Γ is bounded in $\mathrm{GL}_n(k)$ if and only if its images under the projections $C \rightrightarrows T, U$ are bounded. This allows us to treat separately the cases when γ is diagonal and when γ is unipotent.

First assume γ is diagonal. Using the evident closed immersion

$$\mathrm{GL}_n \hookrightarrow \mathrm{SL}_{2n} \hookrightarrow \mathrm{Mat}_{2n} = \mathbf{A}_k^{4n^2}$$

as in Example 1.1, the group Γ of diagonal elements is bounded if and only if every eigenvalue λ of every element of Γ has the property that $\{\lambda^m\}_{m\in\mathbf{Z}}$ is bounded in k. But we are allowing $m > 0$ and $m < 0$, so this says exactly that λ is a unit in the valuation ring R. Hence, if Γ is bounded then the eigenvalues of γ are in $R^\times$. Conversely, if the eigenvalues of γ lie in $R^\times$ then clearly the same holds for γ^m for all $m \in \mathbf{Z}$.

Now assume γ is unipotent. In this case we need to show that Γ is bounded. It is harmless to replace γ with γ^m for some $m > 0$ (at worst Γ may be replaced with a subgroup of finite index, so the boundedness of Γ is unaffected). If $\mathrm{char}(k) = p > 0$ then γ has finite order and so everything is clear. Suppose instead that $\mathrm{char}(k) = 0$, so we have an isomorphism of varieties $\log(1+t)$ from U onto the affine space of nilpotent upper triangular matrices. This turns our problem into the boundedness of the set of $\mathbf{Z}$-multiples of a fixed (nilpotent) matrix M, and that boundedness is obvious.

2. Proof of main result

We assume $G(k)$ is unbounded and will build a k-torus $T \subset G$ admitting a nontrivial character over k, so it also admits a nontrivial cocharacter over k (due to the Galois-equivariant duality between $\mathrm{X}(T_{k_s})_{\mathbf{Q}}$ and $\mathrm{X}_*(T_{k_s})_{\mathbf{Q}}$) and thus G contains $\mathbf{G}_m$ as a k-subgroup. The essential step is:

Lemma 2.1. *Let G be a connected reductive group over a field k equipped with a rank-1 valuation. If a subgroup $\Gamma \subset G(k)$ is Zariski-dense in G and it is unbounded then there exists $\gamma \in \Gamma$ such that $\gamma^{\mathbf{Z}}$ is unbounded.*

This lemma applies to $\Gamma = G(k)$ since G is unirational and k is infinite. This is the only Γ we will need for the main argument, but to prove the lemma the generality of any Zariski-dense Γ is convenient.

Granting the lemma, let's now prove that $G(k)$ is bounded when G is k-anisotropic and k is henselian. Assuming otherwise, the lemma with $\Gamma = G(k)$ provides $g \in G(k)$ such that $g^{\mathbf{Z}}$ is unbounded. We next reduce to the case that g is semisimple.

We can replace g with g^m for any $m > 0$, so if $\mathrm{char}(k) = p > 0$ we can replace g with g^{p^r} for $r \geq 0$ to ensure that g is semisimple. If $\mathrm{char}(k) = 0$ then the Jordan decomposition $g = su = us$ is k-rational. By Remark 1.4, relative to a choice of faithful representation $G \hookrightarrow \mathrm{GL}_n$ the eigenvalues of g (in a finite extension of k) are not all units of the valuation ring. But the same property is then inherited by s, so $\langle s \rangle$ is unbounded. In other words, we can rename s as g to again arrange that g is semisimple.

Now that g is semisimple, we have $g \in Z_G(g)^0$ with $Z_G(g)^0$ reductive, by Proposition 3.1 in the handout on applications of Grothendieck's covering theorem. Thus, we can replace G with is k-anisotropic connected reductive k-subgroup $Z_G(g)^0$ (as $Z_G(g)^0(k)$ is certainly unbounded due to its subgroup $g^{\mathbf{Z}}$) to reduce to the case that g is central. Thus, g lies in a maximal k-torus $T \subset G$, so we can further replace G with T to finally arrive at the case that $G = T$ is an anisotropic k-torus. In this case we seek a contradiction from the existence of $g \in T(k)$ for which $g^{\mathbf{Z}}$ is unbounded.

Let k'/k be a finite Galois extension that split T, and let $R' \subset k'$ be the valuation ring. Fix an isomorphism $T_{k'} \simeq \mathbf{G}_m^e$, so the subgroup $g^{\mathbf{Z}} \subset T(k') = (k'^{\times})^e$ is unbounded. It follows that some component of $g \in (k'^{\times})^e$ is not a unit in R', so we get $\chi \in \mathrm{X}(T_{k'})$ such that $\chi(g) \notin R'^{\times}$.

Consider

$$\psi = \prod_{\sigma} \sigma^*(\chi) \in \mathrm{X}(T_{k'})^{\mathrm{Gal}(k'/k)} = \mathrm{Hom}_k(T, \mathbf{G}_m),$$

where σ varies through $\mathrm{Gal}(k'/k)$. We claim that $\psi \neq 1$, which would contradict that T is k-anisotropic. More specifically, we claim that $\psi(g) \notin R'^{\times}$. Indeed, since $g \in T(k)$ we have $\psi(g) = \prod_{\sigma} \sigma(\chi(g)) \in k'^{\times}$, and the Galois action on k' over k must preserve the unique valuation on k' extending the given one on k. Hence, all elements $\sigma(\chi(g))$ have the same *non-trivial* valuation, so their product has valuation equal to the $\#\mathrm{Gal}(k'/k)$-th power of that, so indeed $\psi(g) \notin R'^{\times}$.

It remains to prove Lemma 2.1:

Proof. If $f : G \to G'$ is a quotient by a (possibly non-central and non-étale) finite subgroup scheme then G' is connected reductive and $f(\Gamma)$ is Zariski-dense in G'. Moreover, $f(\Gamma)$ is unbounded due to Lemma 1.2 and the assumption that Γ is unbounded. Likewise, if there exists $\gamma \in \Gamma$ such that $f(\gamma)^{\mathbf{Z}}$ is unbounded in $G'(k)$ then $\gamma^{\mathbf{Z}}$ is unbounded in $G(k)$. Hence, in any such situation we may pass to working with G' and $f(\Gamma)$.

To apply the preceding maneuver, pick a faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$ (i.e., $\ker \rho = 1$ scheme-theoretically), and let $\{V_i\}$ be a G-stable flag in V with irreducible successive quotients $W_i = V_i/V_{i-1}$. (These might not be absolutely irreducible.) Thus, we get a representation $\overline{\rho} : G \to \prod \mathrm{GL}(W_i)$ whose kernel has only unipotent geometric points (ρ-preimage of a unipotent subgroup of $\mathrm{GL}(V)$), so $(\ker \overline{\rho})(\overline{k})$ is finite due to the reductivity of G. It follows that the normal subgroup scheme $\ker \overline{\rho}$ in G is finite, so we can replace G with $\overline{\rho}(G) = G/(\ker \overline{\rho})$ to reduce to the case $\ker \overline{\rho} = 1$; i.e., we may rename $\overline{\rho}$ as ρ to arrange that the faithful ρ is *semisimple*; i.e., (V, ρ) is a direct sum of irreducible representations (W_i, ρ_i).

Since $\rho : G \to \prod \mathrm{GL}(W_i)$ is a closed immersion, boundedness of a subset of $G(k)$ is equivalent to that of its image under ρ! Hence, unboundedness of Γ in $G(k)$ implies that for some i_0 the image $\rho_{i_0}(\Gamma) \subset \mathrm{GL}(W_{i_0})(k)$ is unbounded. Thus, by Example 1.1, $\rho_{i_0}(\Gamma)$ is unbounded inside the Euclidean space $\mathrm{End}_k(W_{i_0})$.

We will find $\gamma \in \Gamma$ such that $\rho_{i_0}(\gamma)^{\mathbf{Z}}$ is unbounded inside $\mathrm{End}_k(W_{i_0})$, so it is also unbounded in $\mathrm{GL}(W_{i_0})(k)$ (by Example 1.1), and hence $\rho(\gamma)^{\mathbf{Z}}$ is unbounded in $\prod_i \mathrm{GL}(W_i)(k)$, so $\gamma^{\mathbf{Z}}$ is unbounded in $G(k)$ as desired.

Suppose to the contrary that no such γ exists, so for each $\gamma \in \Gamma$ the cyclic subgroup generated by $\rho_{i_0}(\gamma)$ is bounded (in $\mathrm{GL}(W_{i_0})(k)$ or in $\mathrm{End}_k(W_{i_0})$, which come to the same thing by Example 1.1). Hence, by Remark 1.4, for every $\gamma \in \Gamma$ the endomorphism $\rho_{i_0}(\gamma)$ of W_{i_0} has all eigenvalues (in a finite extension of k) lying in the unit group of the valuation ring (of a finite extension of k), so the expression for the matrix trace in terms of eigenvalues implies that $\mathrm{Tr}(\rho_{i_0}(\gamma)) \in R$ for all $\gamma \in \Gamma$.

To apply this trace integrality, we now use the irreducibility of W_{i_0}: the Zariski-density of Γ in G and the G-irreducibility of W_{i_0} implies that Γ acts irreducibly on W_{i_0} too (why?). Burnside's theorem then gives that the matrix algebra $\mathrm{End}_k(W_{i_0})$ is generated as a k-algebra by its subset $\rho_{i_0}(\Gamma)$ that we have seen is unbounded inside that matrix algebra. In particular, since ρ_{i_0} is multiplicative it follows that $\rho_{i_0}(\Gamma)$ spans $\mathrm{End}_k(W_{i_0})$ as a k-vector space, so there exist elements $\gamma_\alpha \in \Gamma$ such that $\{\rho_{i_0}(\gamma_\alpha)\}_\alpha$ is a k-basis of $\mathrm{End}_k(W_{i_0})$. Letting L be the R-submodule $\bigoplus_\alpha R\rho_{i_0}(\gamma_\alpha) \subset \mathrm{End}_k(W_{i_0})$, the integrality of the trace on $\rho_{i_0}(\Gamma)$ implies that under the non-degenerate k-bilinear trace pairing

$$\mathrm{End}_k(W_{i_0}) \times \mathrm{End}_k(W_{i_0}) \to k$$

we have that $L \times \rho_{i_0}(\Gamma)$ is carried into R. In other words, $\rho_{i_0}(\Gamma)$ as a subset of $\mathrm{End}_k(W_{i_0})$ lies inside the R-dual L^* (i.e., the R-span of the k-basis of $\mathrm{End}_k(W_{i_0})$ dual to an R-basis of L). But this contradicts that $\rho_{i_0}(\Gamma)$ is unbounded in $\mathrm{End}_k(W_{i_0})$! ■