Math 249B. Lang's theorem and dynamic methods

In the previous course, we punted to §16 of Borel's textbook for Lang's technique on how to establish certain results for *connected* linear algebraic groups over finite fields that are proved over infinite fields via Zariski-density considerations with rational points (as cannot work over a finite field). This handout gives a complete treatment of Lang's theorem, along with some important applications (filling in the case of finite fields for Grothendieck's theorem on geometrically maximal tori). We also review dynamic techniques, complementing those worked out in HW10 of the previous course, to prove the absolutely essential fact that a torus centralizer in a connected linear algebraic group is not only smooth but especially *connected*; this underlies a vast range of arguments that proceed via dimension induction, such as the proof of Chevalley's all-important self-normalizing property of parabolic subgroups of connected linear algebraic groups (shown in the first lecture).

1. Lang's theorem

The following result of Lang is useful to circumvent the absence of Zariski-density techniques with rational points in the study of *connected* linear algebraic groups over finite fields:

Theorem 1.1 (Lang)**.** *Let G be a connected group scheme of finite type over a finite field k, and let X be a non-empty finite type k-scheme equipped with a left G-action $G \times X \to X$ such that $G(\overline{k})$ acts transitively on $X(\overline{k})$. Then $X(k)$ is non-empty.*

Lang's theorem is stated and proved in §16 of Borel's "Linear algebraic groups" book in the affine setting, but the proof works without affineness and so we will proceed in that generality.

In practice, Lang's theorem is often applied to a special class of X, namely G-torsors. The torsor property is the condition that the action is "simply transitive". To be precise, if H is a group scheme of finite type over a field K and E is a finite type K-scheme equipped with a left H-action then E is an *H-torsor* (for the fppf topology) if the natural map

$$H \times E \to E \times E$$

defined by $(h, x) \mapsto (h.x, x)$ is an isomorphism. The techniques of descent theory imply that in such cases, E inherits many "nice" properties of H (smoothness, properness, geometric connectedness, etc.). In case H is smooth, such an E is necessarily smooth and so $E(K_s)$ is non-empty; we then say E is a torsor for the étale topology (over K).

Example 1.2. A typical source of H-torsors is fibers over K-points for the faithfully flat quotient map $G \to H\backslash G$ with a finite type K-group scheme G containing H as a closed K-subgroup scheme, especially when G is smooth (and H is often but not always smooth). Lang's theorem is the key tool to use for lifting rational points of $H\backslash G$ to rational points of G when working over a finite field, in the presence of connectedness of H. Such connectedness is an essential assumption, as we see by trying to lift k-rational points to k-rational points through the quotient map $\mathrm{GL}_1 \to \mathrm{GL}_1$ defined by $t \mapsto t^n$ for an integer $n > 1$ (using $H = \mu_n$).

To prove Lang's theorem, first observe that since X is non-empty, we may and do choose $x_0 \in X(\overline{k})$. We seek a k-point in X, and over $\overline{k}$ such a point must have the form $g_0(x_0)$ for some $g_0 \in G(\overline{k})$. For any k-scheme Z, denote the q-Frobenius morphism $F_{Z/k,q} : Z \to Z$ over k by the notation $z \mapsto z^{[q]}$ functorially on points. (Note that $F_{Z/k,q}$ is functorial in Z over k.) This is the identity on the underlying topological space and the q-power endomorphism of the structure sheaf. The k-rationality of a point $g_0(x_0) \in X(\overline{k})$ amounts to the "Galois-invariance" property $g_0(x_0)^{[q]} = g_0(x_0)$.

The G-action on X is defined over k, so

$$g_0(x_0)^{[q]} = (g_0)^{[q]}(x_0^{[q]}).$$

Hence, we can recast our problem is that of finding $g_0 \in G(\overline{k})$ such that

$$(g_0^{-1} \cdot (g_0)^{[q]})(x_0^{[q]}) = x_0.$$

Since the $G(\overline{k})$-action on $X(\overline{k})$ is transitive by hypothesis, $x_0^{[q]} = g'(x_0)$ for some $g' \in G(\overline{k})$. Thus, it suffices to show that for any $g' \in G(\overline{k})$, ${g'}^{-1}$ has the form $g_0^{-1} \cdot (g_0)^{[q]}$ for some $g_0 \in G(\overline{k})$, or equivalently $g' = (g_0^{-1})^{[q]} \cdot g_0$. In other words, we are reduced to proving that the k-scheme morphism $L : G \to G$ defined by

$$L(g) = g^{[q]} \cdot g^{-1}$$

is surjective on $\overline{k}$-points (or equivalently, is a surjective map of k-schemes, as G is finite type over k). In the special case $G = \mathbf{G}_{\mathrm{a}}$ this is the Artin-Schreier map $t \mapsto t^q - t$, so the map L is a generalization of the Artin-Schreier homomorphism; it is called the *Lang map* (and is generally not a homomorphism when G is non-commutative). Our problem is now entirely about G and has nothing anymore to do with X.

Since k is perfect, G_{red} is a closed k-subgroup scheme of G whose formation commutes with any extension of the ground field, and it has the same $\overline{k}$-points as G. Thus, by functoriality of the q-Frobenius morphism with respect to the inclusion of G_{red} into G we may replace G with G_{red} so that G is reduced and hence smooth (as for any reduced group scheme of finite type over a perfect field). Now we may use tangent space considerations to investigate the local structure of k-morphisms from G to itself.

Consider the action of G on itself via $g.x = g^{[q]}xg^{-1}$. Clearly $L(G)$ is the orbit of the identity point. To show that this is the only orbit (and so L is surjective), by the connectedness of G and the disjointness of distinct orbits it is enough to prove that *every* G-orbit on $G_{\overline{k}}$ is open. To do this, it is enough to show that the orbit map $g \mapsto g^{[q]}g_0g^{-1}$ through every $g_0 \in G(\overline{k})$ is an open map. We will show that all of these orbit maps $G \to G$ are étale. By smoothness of G, it is equivalent to check that such orbit maps are isomorphisms on tangent spaces at all points of the source. The homogeneity of orbit maps reduces this assertion to the isomorphism property at a single point of $G(\overline{k})$, such as the identity point.

More specifically, the map $g \mapsto g^{[q]}g_0g^{-1}$ carries e to g_0 and we claim that the induced map $\mathrm{T}_e(G) \to \mathrm{T}_{g_0}(G)$ is an isomorphism. It is harmless to post-compose with right translation by g_0^{-1}, so we are analyzing the map $f_{g_0} : g \mapsto L(g) \cdot (gg_0g^{-1}g_0^{-1})$. Recall the following differential identities: the group law $m : G \times G \to G$ induces addition $\mathrm{T}_e(G) \oplus \mathrm{T}_e(G) \to \mathrm{T}_e(G)$, inversion induces negation on $\mathrm{T}_e(G)$, and $F_{G/k,q}$ induces the zero map on $\mathrm{T}_e(G)$. Thus, $\mathrm{d}L(e)$ is negation and so

$$\mathrm{d}f_{g_0}(e) = \mathrm{d}L(e) + \mathrm{id} - \mathrm{Ad}_G(g_0) = -\mathrm{Ad}_G(g_0).$$

Hence, $\mathrm{d}f_{g_0}(e)$ is an isomorphism, so f_{g_0} is étale. This proves that every G-orbit is open, so we are done. In particular, the Lang map L is an étale surjection. In fact, we can do a bit better:

Proposition 1.3. *The fibers of L are right $G(k)$-cosets inside G. In particular, L has* constant *fiber rank (namely, the size of $G(k)$) over its open image $f(G)$.*

Proof. It suffices to check that L is étale on $G_{\overline{k}}$ at any $\overline{k}$-point g_0, with each fiber $L^{-1}(L(g_0))$ a right $G(k)$-coset inside $G(\overline{k})$. For the étale property of L at g_0, it is equivalent to prove the étale property of $g \mapsto L(g_0g)$ at the identity. But

$$L(g_0g) = g_0^{[q]}(g^{[q]}g^{-1})g_0^{-1} = g_0^{[q]} \cdot L(g) \cdot g_0^{-1}.$$

Since L is étale at the identity, and post-composing with left translation by $g_0^{[q]}$ and right translation by g_0^{-1} amounts to applying automorphisms on the target, the desired étaleness is established.

Analyzing the fibers amounts to an elementary computation: for $g, g_0 \in G(\overline{k})$, the equality $L(g) = L(g_0)$ says exactly that $g_0^{-1}g$ is fixed by $F_{G/k,q}$ on $G(\overline{k})$. But the set of such fixed points is exactly $G(k)$, so $L(g) = L(g_0)$ if and only if $g_0^{-1}g \in G(k)$, which is to say $g \in g_0G(k)$. ■

Remark 1.4. By using Zariski's Main Theorem, one can show that a quasi-compact separated étale map with constant fiber rank is necessarily *finite*, so $L : G \to G$ is not merely surjective étale but even finite. In fact, $L : g \mapsto g^{[q]}g^{-1}$ is visibly invariant under the right $G(k)$-action on G, so L actually exhibits G as a right $G(k)$-torsor over itself. For commutative G, the map L is usually called the *Lang isogeny*.

Lang's theorem has very useful consequences. We record the most basic one here:

Corollary 1.5. *Let G be a smooth connected affine group over a finite field k. There exists a k-torus $T \subset G$ that is maximal over $\overline{k}$ and there exists a Borel k-subgroup $B \subset G$.*

The proof (in the previous course) of Grothendieck's theorem on the existence of (geometrically!) maximal tori over the ground field was carried out almost entirely over infinite fields via Zariski-density considerations with "rational points" in Lie algebras over the ground field. The case of finite fields was punted to Lang's theorem via exactly the torus part of Corollary 1.5, so that fills in the loose end for the treatment of finite fields in the proof of Grothendieck's theorem in the previous course.

The "right" proof of Corollary 1.5 is to construct moduli schemes $\mathrm{Tor}_{G/k}$ and $\mathrm{Bor}_{G/k}$ whose sets of rational points over any extension field k'/k are respectively naturally identified with the set of geometrically maximal k'-tori and the set of Borel k'-subgroups, and to apply Lang's theorem with X taken to be either of these moduli schemes (equipped with the natural *transitive* G-action via conjugation). To be precise, over an arbitrary field k there exist closed subschemes

$$\mathscr{T} \subset G \times \mathrm{Tor}_{G/k},\ \ \mathscr{B} \subset G \times \mathrm{Bor}_{G/k}$$

that respectively represent functors of maximal tori and Borel subgroups in a relative setting, or in a weaker (but sufficient) sense recover exactly the geometrically maximal k'-tori and the Borel k'-subgroups as we vary through all k'-point fibers of $\mathscr{T} \to \mathrm{Tor}_{G/k}$ and $\mathscr{B} \to \mathrm{Bor}_{G/k}$ inside $G_{k'}$ with any extension field k'/k.

For our purposes, rather than construct such moduli schemes (which is a game of Galois descent for perfect ground fields, and very delicate over imperfect fields), we will instead adapt the *method* of proof of Lang's theorem rather than apply the *statement* of the theorem.

Proof. We give the argument for geometrically maximal tori, and the case of Borel subgroups goes similarly. Pick a maximal torus $T' \subset G_{\overline{k}}$, so we seek $g \in G(\overline{k})$ such that $gT'g^{-1}$ is $\mathrm{Gal}(\overline{k}/k)$-invariant. In other words, if k has size q then we want that $F_{G/k,q}(gT'g^{-1}) = gT'g^{-1}$. This says that $(g^{[q]})^{-1}gT'g^{-1}g^{[q]} = T'^{[q]}$. Since $T'^{[q]}$ and T' are maximal tori in $G_{\overline{k}}$, by the conjugacy of maximal tori over $\overline{k}$ there exists $g' \in G(\overline{k})$ such that $g'T'g'^{-1} = T'^{[q]}$. Hence, it suffices to find $g \in G(\overline{k})$ such that $(g^{[q]})^{-1}g = g'$. This says that the Lang map $L : G \to G$ carries g^{-1} onto g'. Since the Lang map is surjective, we are done. ■

2. Subgroups associated to a 1-parameter subgroup

Let G be a smooth affine group over a field k, and $\lambda : \mathbf{G}_m \to G$ a k-homomorphism (possibly trivial, though that case is not interesting). One often calls λ a 1-*parameter k-subgroup* of G, even when $\ker \lambda \neq 1$. Such a homomorphism defines a left action of $\mathbf{G}_m$ on G via the functorial procedure $t.g = \lambda(t)g\lambda(t)^{-1}$ for $g \in G(R)$ and $t \in R^\times$ for any k-algebra R. In lecture we introduced the following associated subgroup functors of G: for any k-algebra R,

$$P_G(\lambda)(R) = \{g \in G(R) \mid \lim_{t \to 0} t.g \text{ exists } \},\ \ U_G(\lambda)(R) = \{g \in G(R) \mid \lim_{t \to 0} t.g = 1\},$$

and

$$Z_G(\lambda)(R) = \{g \in G(R) \mid \lambda_R \text{ centralizes } g\}.$$

In the March 10 lecture of the previous course it was proved that these are all represented by closed k-subgroup schemes of G, with $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$.

By a direct calculation with graded modules over the dual numbers, it is shown in Proposition 2.1.8(1) of "Pseudo-reductive groups" that when using the $\mathbf{Z}$-grading $\oplus_{n \in \mathbf{Z}} \mathfrak{g}_n$ of $\mathfrak{g} = \mathrm{Lie}(G)$ defined by the $\mathbf{G}_m$-action induced by conjugation through λ (i.e., $\mathfrak{g}_n$ is the space of $v \in \mathfrak{g}_n$ such that $\mathrm{Ad}_G(\lambda(t))(v) = t^n v$ for all $t \in \mathbf{G}_m$), we have

$$\mathrm{Lie}(Z_G(\lambda)) = \mathfrak{g}_0,\ \ \mathrm{Lie}(U_G(\lambda)) = \mathfrak{g}^+ := \bigoplus_{n>0} \mathfrak{g}_n.$$

For example, if $T \subset G$ is a split k-torus and λ is valued in T, then using the resulting T-weight space decomposition $\mathfrak{g} = \mathrm{Lie}(Z_G(T)) \oplus (\oplus_{a \in \Phi} \mathfrak{g}_a)$ (with Φ the set of nontrivial T-weights on $\mathfrak{g}$) we see that for any $n \in \mathbf{Z} - \{0\}$,

$$\mathfrak{g}_n = \bigoplus_{\langle a, \lambda \rangle = n} \mathfrak{g}_a$$

since the adjoint action of $\lambda(t)$ on $\mathfrak{g}_a$ is multiplication by $a(\lambda(t)) = t^{\langle a, \lambda \rangle}$. Hence,

$$\mathfrak{g}_0 = \mathrm{Lie}(Z_G(T)) \oplus (\oplus_{\langle a, \lambda \rangle = 0} \mathfrak{g}_a),\ \ \mathrm{Lie}(U_G(\lambda)) = \mathfrak{g}_+ = \oplus_{\langle a, \lambda \rangle > 0} \mathfrak{g}_a.$$

We write λ^{-1} to denote the reciprocal homomorphism $t \mapsto \lambda(t)^{-1} = \lambda(1/t)$. In Exercise 3 of HW10 of the previous course you were led through an elementary proof that if $G = \mathrm{GL}(V)$ then the multiplication map

$$\mu = \mu_{G,\lambda} : U_G(\lambda^{-1}) \times P_G(\lambda) \to G$$

is an open immersion, with $P_G(\lambda)$ a subgroup of "block upper-triangular matrices" and $U_G(\lambda)$ its unipotent radical (even over $\overline{k}$). We first wish to deduce the open immersion property for *general* G from that special case, which immediately implies:

Corollary 2.1. *The k-groups $U_G(\lambda)$, $P_G(\lambda)$, and $Z_G(\lambda)$ are all smooth, and all are connected when G is connected. Likewise, $U_G(\lambda)$ is unipotent in general.*

Indeed, the inheritance of smoothness and connectedness by direct factors gives the first assertion of this corollary, and the final assertion is immediate via functoriality with respect to an inclusion $G \hookrightarrow \mathrm{GL}_n$ (as that reduces it to the settled case of GL_n). Note that by iterating the connectedness of $Z_G(\lambda)$ several times (using λ's that generate a given torus in $G_{\overline{k}}$) we obtain the important:

Corollary 2.2. *If G is a connected linear algebraic group then $Z_G(S)$ is smooth and connected for any k-torus S in G.*

To establish the open immersion property for μ in general, consider any pair (G, λ) and a k-subgroup inclusion $j : G \hookrightarrow G'$ into another smooth affine k-group (the case of interest being $G' = \mathrm{GL}(V)$). Let $\lambda' = j \circ \lambda$. By the functorial definition,

$$P_G(\lambda) = G \cap P_{G'}(\lambda'), U_G(\lambda^{\pm 1}) = G \cap U_{G'}({\lambda'}^{\pm 1}), Z_G(\lambda) = G \cap Z_{G'}(\lambda').$$

In particular, if $U_{G'}(\lambda'^{-1}) \cap P_{G'}(\lambda') = 1$ then $U_G(\lambda^{-1}) \cap P_G(\lambda) = 1$. In other words, if $\mu' = \mu_{G',\lambda'}$ is a monomorphism then so is μ. This monicity hypothesis on μ for $G' = \mathrm{GL}(V)$ (and any 1-parameter k-subgroup λ' of $\mathrm{GL}(V)$) was verified in HW10 of the previous course, so μ is monic in general. But is it an open immersion?

If μ' is an open immersion (as is proved on HW10 from the previous course for $G' = \mathrm{GL}(V)$!) then the same holds for μ due to the following non-obvious lemma:

Lemma 2.3. *With notation as above, if μ' is monic then*

$$G \cap (U_{G'}(\lambda'^{-1}) \times P_{G'}(\lambda')) = U_G(\lambda^{-1}) \times P_G(\lambda)$$

as subfunctors of G.

Proof. Since $P_{G'}(\lambda') = U_{G'}(\lambda') \rtimes Z_{G'}(\lambda')$, by evaluating on points valued in k-algebras R we have to show that if

$$u'_- \in U_{G'}(\lambda'^{-1})(R), \;\; u'_+ \in U_{G'}(\lambda')(R), \;\; z' \in Z_{G'}(\lambda')(R)$$

and $u'_- u'_+ z' = g \in G(R)$ then that $u'_+, u'_-, z' \in G(R)$.

As usual, we can pick a finite-dimensional k-vector space V, a k-homomorphism $\rho : G' \to \mathrm{GL}(V)$, and a line L in V such that G is the scheme-theoretic stabilizer of L in G'. Let $v \in L$ be a basis element, so $\rho(g)(v) = cv$ in $V_R = R \otimes_k V$ for a unique $c \in R^\times$. Since $g = u'_- u'_+ z'$, we get

$$\rho(u'_+ z')(v) = c\rho((u'_-)^{-1})(v) \tag{1}$$

in V_R.

For any point t of $\mathbf{G}_m$ valued in an R-algebra R', the point $\lambda'(t)$ of $G'(R')$ lies in $G(R')$ and so acts on v (through ρ) by some $R'^\times$-scaling. Hence, we can replace v with $\rho(\lambda'(t)^{-1})(v)$ on both sides of (1). Now act on both sides of (1) by $\rho(\lambda'(t))$, and then commute $\rho(\lambda'(t)^{-1})$ past $\rho(z')$ (as we may, since $z' \in Z_{G'}(\lambda')(R)$) to get the identity

$$\rho((t.u'_+)z')(v) = c\rho(t.(u'_-)^{-1})(v) \tag{2}$$

as points of the affine space $\underline{V}_R$ over R covariantly associated to V_R.

Viewing the two sides of (2) as R-scheme maps $(\mathbf{G}_m)_R \to \underline{V}_R$, the left side extends to an R-map $\mathbf{P}^1_R - \{\infty\} = \mathbf{A}^1_R \to \underline{V}_R$ and the right side extends to an R-map $\mathbf{P}^1_R - \{0\} \to \underline{V}_R$. By combining these, we arrive at an R-map $\mathbf{P}^1_R \to \underline{V}_R$ from the projective line to an affine space over R. The only such map is a constant R-map to some $v_0 \in \underline{V}_R(R) = V_R$ (concretely, $R[t] \cap R[1/t] = R$ inside of $R[t, 1/t]$), so both sides of (2) are independent of t (and equal to v_0). Passing to the limit as $t \to 0$ on the left side and as $t \to \infty$ on the right side yields $\rho(z')(v) = v_0 = cv$. We have proved that z' carries v to an $R^\times$-multiple of itself. Thus, the point $z' \in G'(R)$ is an R-point of the functorial stabilizer of L inside of V. This stabilizer is exactly G, by the way we chose ρ, so z' is an R-point of $G \cap Z_{G'}(\lambda') = Z_G(\lambda)$.

Since $\rho(z')(v) = cv$, by cancellation of c on both sides of the identity (2) we get

$$\rho(t.u'_+)(v) = \rho(t.(u'_-)^{-1})(v)$$

with both sides independent of t and equal to $c^{-1}v_0 = v$. Taking $t = 1$, this says that $u'_\pm$ lies in the stabilizer G of v, so $u'_\pm$ is an R-point of $G \cap U_{G'}(\lambda'^{\pm 1}) = U_G(\lambda^{\pm 1})$, as required. ■

At the end of the March 10 lecture of the previous course, we used the open immersion property for μ to prove the following crucial result:

Proposition 2.4. *Let $f : G \twoheadrightarrow G'$ be a surjective k-homomorphism between smooth connected affine k-groups, and let $\lambda : \mathbf{G}_m \to G$ be a k-homomorphism. For $\lambda' = f \circ \lambda$, the natural maps $P_G(\lambda) \to P_{G'}(\lambda')$, $U_G(\lambda) \to U_{G'}(\lambda')$, and $Z_G(\lambda) \to Z_{G'}(\lambda')$ are surjective.*

We have shown that surjective homomorphisms between smooth connected affine k-groups carry maximal k-tori onto maximal k-tori and Borel k-subgroups onto Borel k-subgroups. Another related important compatibility is the good behavior of *torus centralizers* under surjective homomorphisms. This follows from the preceding proposition:

Corollary 2.5. *Let $f : G \twoheadrightarrow G'$ be a surjective k-homomorphism between smooth connected affine k-groups. Let S be a k-torus in G, and $S' = f(S)$. Then $f(Z_G(S)) = Z_{G'}(S')$.*

This result is Corollary 2 to 11.14 in Borel's book. You may find it instructive to compare the proofs.

Proof. We may assume $k = \overline{k}$. If S_1 and S_2 are k-subtori in S such that $S_1 \cdot S_2 = S$, which is to say that the k-homomorphism $S_1 \times S_2 \to S$ is surjective, it is an exercise (do it!) to check that $Z_G(S) = Z_{Z_G(S_1)}(S_2)$. (Note that since torus centralizers in smooth affine groups are smooth, this equality may be checked by computing with geometric points.) Hence, by induction on $\dim S$, we may and do assume $S \simeq \mathbf{G}_m$.

With $S \simeq \mathbf{G}_m$, the inclusion of S into G is given by a k-homomorphism $\lambda : \mathbf{G}_m \to G$ with image S. Likewise, $\lambda' = f \circ \lambda : \mathbf{G}_m \to G'$ has image S'. Hence, $Z_G(S) = Z_G(\lambda)$ and $Z_{G'}(S') = Z_{G'}(\lambda')$. Thus, the map $Z_G(S) \to Z_{G'}(S')$ that we wish to prove is surjective is identified with the natural map $Z_G(\lambda) \to Z_{G'}(\lambda')$. By Proposition 2.4, this latter map is surjective! ■

3. Conjugacy for split tori

It is a deep fact that in smooth connected affine groups G over any field k, all maximal k-split k-tori S in G (not to be confused with k-split maximal k-tori, which may not exist!) are $G(k)$-conjugate. Their common dimension is called the *k-rank* of G; it could be considerably smaller than the common dimension of the maximal k-tori (which may be called the *geometric rank*, since it is the $\overline{k}$-rank of $G_{\overline{k}}$).

The proof of this conjugacy result rests on the theory of reductive groups (and pseudo-reductive groups when k is imperfect), and we do not need it in general; later in this course we will prove that result for reductive G. The special case of PGL_2 plays a role in getting the structure theory of reductive groups off the ground, so we now give an elementary direct proof in the special case of PGL_n and GL_n:

Proposition 3.1. *Let V be a finite-dimensional vector space over a field, and $G = \mathrm{GL}(V)$ or $\mathrm{PGL}(V)$. The maximal k-split k-tori in G are $G(k)$-conjugate to each other.*

Proof. Using the quotient map $\mathrm{GL}(V) \to \mathrm{PGL}(V)$ whose kernel is $\mathbf{G}_m$ and which is surjective on k-points (!), it is easy to reduce to the case of $\mathrm{GL}(V)$ in place of $\mathrm{PGL}(V)$ (check!). By HW5, Exercise 5, such k-tori correspond precisely to commutative k-subalgebras $A \subseteq \mathrm{End}(V)$ of the form $A \simeq k^n$ with $n = \dim V$. Such a k-subalgebra amounts to a k^n-module structure on an n-dimensional vector space V, which is nothing more or less than a decomposition of V into a direct sum of lines. But any two such decompositions are clearly related via the action of $\mathrm{Aut}_k(V) = \mathrm{GL}(V)(k)$, so we are done. ■

Now we turn out attention to an "axiomatic" $G(k)$-conjugacy result. The axioms turn out to hold for all connected reductive k-groups containing a split maximal k-torus, as we will show later (see Remark 3.4), but we note here that it will rest on the dynamic method (which is why we mention the topic in this handout, to illustrate how useful the dynamic viewpoint is).

Theorem 3.2. *Let G be a smooth connected affine k-group such that for every maximal torus T in $G_{\overline{k}}$, $Z_{G_{\overline{k}}}(T) = T$ and the finite group $W_{G_{\overline{k}}}(T)$ acts transitively on the set of Borel subgroups of $G_{\overline{k}}$ containing T. Also assume that any Borel subgroup B of $G_{\overline{k}}$ satisfies $N_{G(\overline{k})}(B) = B(\overline{k})$.*

Assume that G contains a k-split maximal k-torus, and that for all such k-tori T there is a Borel k-subgroup B containing T. All such pairs (T, B) are $G(k)$-conjugate to each other.

The centralizer hypothesis on the maximal tori of $G_{\overline{k}}$ is invariant under conjugation, so by the $G(\overline{k})$-conjugacy of all maximal tori of $G_{\overline{k}}$ it suffices to check this condition for one maximal torus of $G_{\overline{k}}$. The same holds for the normalizer hypothesis on Borel subgroups.

Remark 3.3. In the homework for the previous course we have seen many examples of G for which $Z_G(T) = T$ for some maximal k-torus T, such as GL_n, SL_n, Sp_{2n}, and SO_n with their "diagonal" (split) maximal k-tori. But don't forget that there are plenty of interesting nontrivial k-anisotropic connected reductive groups, such as $\mathrm{SL}(D)$ for a finite-dimensional central division algebra $D \neq k$ and $\mathrm{SO}(q)$ for an anisotropic quadratic space (V, q) over k with $\dim V \geq 3$, and in such cases there is *no nontrivial k-split torus* at all, let alone one which is maximal as a k-torus.

Remark 3.4. It is a general fact that $Z_G(T) = T$ for *every* maximal torus T in any connected reductive group G, but this is not at all obvious from the definitions; it will be proved later as part of a general development of basic structure theory of connected reductive groups. Likewise, the general development will verify the transitivity axiom on Weyl groups in Theorem 3.2 for connected reductive groups, as well as the fact that any k-split maximal k-torus (if one exists!) in a connected reductive k-group lies in a Borel k-subgroup. Finally, the self-normalizing property of Borel subgroups is a fundamental result of Chevalley for any smooth connected affine $\overline{k}$-group, as mentioned at the start of this handout.

To begin the proof of Theorem 3.2, let T and T' be k-split maximal k-tori in G, and choose Borel k-subgroups $B \supset T$ and $B' \supset T'$. We have $T = Z_G(T)$ and $T' = Z_G(T')$, since such equality among k-subgroups may be checked over $\overline{k}$ (where it follows from the hypotheses). The proof goes in two steps: conjugacy over k_s, and then a Galois cohomology argument to get down to k. But we follow the usual "reduction step" style and argue in reverse, by first showing that the general case can be reduced to the separably closed case, and then handling the case $k = k_s$.

Let's first reduce to the case of maximal tori over separably closed k: we will prove that if T_{k_s} and T'_{k_s} are $G(k_s)$-conjugate then they are $G(k)$-conjugate by an element carrying B_{k_s} to B'_{k_s}. Pick $g \in G(k_s)$ such that $T'_{k_s} = gT_{k_s}g^{-1}$, so $gB_{k_s}g^{-1}$ and B'_{k_s} are Borel k_s-subgroups containing T'_{k_s}. We first seek to choose g so that also these Borel k_s-subgroups coincide.

By hypothesis, the group $W_{G_{\overline{k}}}(T'_{\overline{k}})$ acts transitively on the set of Borel $\overline{k}$-subgroups containing $T'_{\overline{k}}$. But $W_G(T')$ is a finite étale k-group, so its geometric points are defined over k_s. Thus,

$$N_G(T')(k_s)/T'(k_s) = W_G(T')(k_s) = W_G(T')(\overline{k}) = W_{G_{\overline{k}}}(T'_{\overline{k}}).$$

In other words, the group $N_G(T')(k_s) = N_{G(k_s)}(T'_{k_s})$ acts transitively on the set of Borel $\overline{k}$-subgroups of $G_{\overline{k}}$ containing $T'_{\overline{k}}$. Hence, replacing $g \in G(k_s)$ with its left-translate by some element of $N_G(T')(k_s)$ (which doesn't affect the condition that $gT_{k_s}g^{-1} = T'_{k_s}$!) brings us to the case that the Borel k_s-subgroups $gB_{k_s}g^{-1}$ and B'_{k_s} containing T'_{k_s} coincide over $\overline{k}$ and hence coincide over k_s.

Now we can carry out a Galois cohomology argument to push down the $G(k_s)$-conjugacy to $G(k)$-conjugacy. For any $\gamma \in \mathrm{Gal}(k_s/k)$ we apply γ to both sides of the equalities

$$T'_{k_s} = gT_{k_s}g^{-1}, \;\; B'_{k_s} = gB_{k_s}g^{-1}.$$

This gives

$$T'_{k_s} = \gamma(g)T_{k_s}\gamma(g)^{-1},\ \ B'_{k_s} = \gamma(g)B_{k_s}\gamma(g)^{-1},$$

so $\gamma(g)^{-1}g$ normalizes T_{k_s} as well as B_{k_s}. By hypothesis $N_{G(\overline{k})}(B_{\overline{k}}) = B(\overline{k})$, so

$$\gamma(g)^{-1}g \in B(\overline{k}) \cap G(k_s) = B(k_s)$$

and likewise $\gamma(g)^{-1}g \in N_G(T)(k_s)$.

If we did not have available the Borel k-subgroups and only worked with the split maximal k-tori, we would only have $\gamma(g)^{-1}g \in N_G(T)(k_s)$ and then we would get hopelessly stuck due to possible obstructions in $\mathrm{H}^1(k_s/k, W_G(T))$. Now the importance of using the Borel k-subgroups emerges: $B(k_s) \cap N_G(T)(k_s) = T(k_s)$! Indeed, since $T = Z_G(T)$ (by our hypotheses) we can express this as the statement that $N_B(T)(k_s) = Z_B(T)(k_s)$, and this in turn is a special case of:

Lemma 3.5. *Let H be a connected solvable smooth affine group over a field k, and let T be a maximal k-torus in H. Then $N_H(T)(k) = Z_H(T)(k)$.*

Proof. Since $T_{\overline{k}}$ is a maximal torus in $H_{\overline{k}}$, and the problem of showing a k-point of H lies in the closed subset $Z_H(T)$ may be checked over $\overline{k}$, it is harmless to extend the ground field to $\overline{k}$ so that k is *algebraically closed*. Hence, the structure theorem for connected solvable groups becomes available: $H = T \ltimes U$ for $U = \mathscr{R}_u(H)$. To show that any $h \in H(k)$ normalizing T actually centralizes T, we may assume $h = u \in U(k)$. Hence, for any $t \in T(k)$ we have

$$utu^{-1} = t(t^{-1}ut)u^{-1}.$$

But $(t^{-1}ut)u^{-1} \in U(k)$ since U is normal in $H = T \ltimes U$, so the condition that $utu^{-1} \in T(k)$ forces it to equal t. ■

Thus, we have obtained a function $c : \gamma \mapsto \gamma(g)^{-1}g$ from $\mathrm{Gal}(k_s/k)$ to $T(k_s)$. This functor factors through the quotient $\mathrm{Gal}(K/k)$ for a finite Galois extension K/k inside of k_s such that $g \in G(K)$. It is therefore easy to check that $c \in \mathrm{Z}^1(k_s/k, T(k_s))$. Consider the cohomology class $[c] \in \mathrm{H}^1(k_s/k, T)$. Since $T \simeq \mathbf{G}_m^r$, this cohomology group vanishes by Hilbert 90. Hence, $c = \gamma(t)t^{-1}$ for some $t \in T(k_s)$. Thus, if we replace g with gt (as we may!), we get to the case when $\gamma(g) = g$ for all γ, so $g \in G(k)$. That does the job. (This idea adapts to pull down the result from $\overline{k}$ by using Hilbert 90 for the fppf topology, but we give a more hands-on procedure below to get down to k_s from $\overline{k}$.)

Now we can assume that $k = k_s$, and it remains to show:

Proposition 3.6. *If T and T' are maximal tori in a smooth connected affine group G over a separably closed field k then T and T' are $G(k)$-conjugate.*

This says that the general conjugacy result over algebraically closed fields actually holds over separably closed fields. I think it is due to Grothendieck. Regardless, the argument we give is a version of the method he used in SGA3 for smooth affine groups over any scheme (working locally for the étale topology). The idea is similar to the trick with Isom-schemes in HW4 Exercise 5 of the previous course.

Proof. Consider the functor I on k-algebras defined by

$$I(R) = \{g \in G(R) \,|\, T'_R = gT_Rg^{-1}\}.$$

This is a subfunctor of G, and its restriction $I_{\overline{k}}$ to $\overline{k}$-algebras is represented by a *smooth* closed subscheme of $\overline{k}$: since $T'_{\overline{k}} = g_0T_{\overline{k}}g_0^{-1}$ for some $g_0 \in G(\overline{k})$ by the known "geometric" case over $\overline{k}$, we see that $I_{\overline{k}}(R)$ consists of points $g \in G(R)$ such that $g_0^{-1}g \in Z_{G(R)}(T_R)$. In other words, $I_{\overline{k}}$

is represented by $g_0 Z_G(T)_{\overline{k}}$. By HW8, Exercise 3 of the previous course, this is smooth and non-empty. Thus, if we can prove that I is represented by a closed k-subscheme of G then its $\overline{k}$-fiber represents $I_{\overline{k}}$ and hence is smooth (and non-empty)! But we know that a smooth non-empty scheme over a *separably closed* field always has a k-point, so it would follow that $I(k) \neq \emptyset$, so the desired $G(k)$-conjugacy of T and T' would follow.

It remains to prove that I is represented by a closed k-subscheme of G. We will do this by approaching tori through their torsion-levels. For each $n \geq 1$ not divisible by $\mathrm{char}(k)$, define a functor on k-algebras as follows:

$$I_n(R) = \{g \in G(R) \,|\, T'[n]_R = gT[n]_R g^{-1}\}.$$

Clearly I is a subfunctor of I_n. Since $T[n]$ and $T'[n]$ are finite étale, each is just a finite set of k-points in G (as $k = k_s$). Thus, it is rather elementary to check that I_n is represented by a closed subscheme of G (verify!). The infinite intersection $\cap_n I_n$ as subfunctors of G is likewise represented by a closed subscheme of G (form the infinite intersection of representing closed subschemes for the I_n's). Thus, we just have to check that the inclusion $I(R) \subseteq \cap_n I_n(R)$ is an equality for all k-algebras R.

Equivalently, picking a point g of $G(R)$ lying in $\cap_n I_n(R)$ and conjugating T_R by this point, we are reduced to proving that $gT_R g^{-1}$ and T'_R coincide if their n-torsion subgroups coincide for all $n \geq 1$ not divisible by $\mathrm{char}(k)$. By the same "relative schematic density" argument used in your solution to HW3 Exercise 3(iii) of the previous course, since the union of the $T[n](k)$ is Zariski-dense in T (why?) and likewise for T' it follows that a closed subscheme of G_R which contains all $T[n]_R$'s (resp. all $T'[n]_R$'s) must contain T_R (resp. T'_R). The automorphism of g-conjugation on G_R then implies likewise that a closed subscheme of G_R which contains every $gT[n]_R g^{-1}$ must contain $gT_R g^{-1}$. We conclude that if $gT[n]_R g^{-1} = T'[n]_R$ for all $n \geq 1$ not divisible by $\mathrm{char}(k)$ then the two closed subschemes $gT_R g^{-1}$ and T'_R of G_R each contain the other and hence are equal as such. ■