Math 249B. Basics of reductivity and semisimplicity

In the previous course, we have proved the important fact that over any field k, a non-solvable connected reductive group containing a 1-dimensional split maximal k-torus is k-isomorphic to SL_2 or PGL_2. That proof relied on Grothendieck's theorem that maximal k-tori remain maximal after a ground field extension to $\overline{k}$. But for algebraically closed fields there is no content to Grothendieck's theorem, so for $k = \overline{k}$ this rank-1 classification is simpler to prove.

The aim of this handout is first to use the rank-1 classification (usually just over algebraically closed fields) to prove some important results on the behavior of unipotent radicals and the property of reductivity with respect to two ubiquitous operations on smooth connected affine groups over an arbitrary field k: the formation of quotient k-groups (modulo normal k-subgroup schemes) and the formation of centralizers of k-tori (which we have seen are always smooth and connected).

Notation. In what follows, G always denotes a smooth connected affine group over an arbitrary field k, unless we indicate otherwise. Also, following tradition, we often denote characters and cocharacters of tori in *additive* notation, for instance writing $-\lambda$ rather than λ^{-1} for the composition of a homomorphism $\lambda : \mathbf{G}_m \to T$ with inversion and likewise writing 0 to denote the trivial character of T. The reason for doing this is that it is convenient to work with the $\mathbf{Q}$-vector space $\mathrm{X}(T)_{\mathbf{Q}}$ and to view the collections of characters and cocharacters as $\mathbf{Z}$-lattices.

1. Preliminary results

We recall the following important fact (proved in an earlier handout):

Lemma 1.1. *Assume $k = \overline{k}$, and let S be a k-torus in G. The Borel subgroups of $Z_G(S)$ are precisely the subgroups $Z_B(S) = B \cap Z_G(S)$ (scheme-theoretic intersection, as always) for Borel subgroups B of G which contain S.*

This was deduced from a rather more general result (proved in the same handout):

Proposition 1.2. *If H is a smooth closed subgroup of G (not necessarily connected or solvable) that is normalized by a torus $S \subset G$, then under the resulting left multiplication action on $(G/H)^S$ by $Z_G(S)$ all orbit maps $Z_G(S) \to (G/H)^S$ through points $g_0 \in (G/H)^S(k)$ are smooth. In particular, the orbits are open and hence coincide with the connected components of $(G/H)^S$. More specifically, the natural map of smooth varieties*

$$f : Z_G(S)/Z_H(S) \to (G/H)^S$$

(induced by the orbit map through $1 \bmod H$, with $\mathrm{Stab}_{Z_G(S)}(1 \bmod H) = Z_H(S)$) is an isomorphism onto the identity component of the target.

Here is a new lemma that we shall need (and which is useful rather generally):

Lemma 1.3. *For any torus T over any field F and any closed F-subgroup scheme $M \subset T$, $M^0_{\mathrm{red}} \subset T$ is an F-torus (in particular, smooth and an F-subgroup). Moreover, its formation commutes with any extension F'/F; i.e., $((M^0)_{\mathrm{red}})_{F'} = (M^0_{F'})_{\mathrm{red}}$ inside $M_{F'}$.*

Likewise, M_{red} is an F-smooth subgroup and its formation commutes with any extension on F.

Over every imperfect field there exist affine group schemes H of finite type such that H_{red} is either not a subgroup scheme (see [SGA3, $\mathrm{VI_A}$, 1.3.2] for a *connected* example), or is a subgroup but is not smooth (e.g., the norm-1 hypersurface relative to a nontrivial purely inseparable finite extension of the ground field), and in such cases its formation does not commute with scalar extension to $\overline{F}$! Thus, this lemma is a special property of subgroups of tori when the ground field is possibly imperfect.

Proof. Since a smooth connected F-subgroup of a torus is a torus, it suffices to show that $M_{\rm red}$ is a smooth F-subgroup whose formation commutes with extension on F. Once it is proved to be smooth in general, the compatibility with ground field extension F'/F is immediate. Indeed, $(M_{\rm red})_{F'}$ and $(M_{F'})_{\rm red}$ are then smooth closed subgroups of $T_{F'}$ with the same underlying space, so they must coincide inside $T_{F'}$. Hence, the problem is to prove (for any F) that $M_{\rm red}$ is a smooth F-subgroup of M.

The case $\mathrm{char}(F) = 0$ is immediate via Cartier's theorem (i.e., $M_{\rm red} = M$ and it is smooth), so we may assume $\mathrm{char}(F) = p > 0$. The formation of $M_{\rm red}$ commutes with finite separable extension on F (as such extensions are étale) and thus with scalar extension to F_s, so we can assume $F = F_s$. Hence, $T = \mathbf{G}_{\rm m}^r$ for some $r \geq 0$. The key point is that since T is a *split* torus, subtori of T correspond bijectively to saturated subgroups of $\mathrm{X}(T)$. Since $\mathrm{X}(T) \to \mathrm{X}(T_{\overline{F}})$ is bijective (split tori do not acquire new characters after a ground field extension), it follows that *every $\overline{F}$-torus in $T_{\overline{F}}$ descends to an F-subtorus of T*. Hence, the torus $(M_{\overline{F}})^0_{\rm red} \subset T_{\overline{F}}$ descends to an F-torus $S \subset T$, and $S \subset M$ since $S_{\overline{F}} \subset M_{\overline{F}}$. By considering character lattices (with trivial Galois action), the subtorus S in T splits off as a direct factor, say $T = S \times S'$, so $M = S \times M'$ inside T with $M' = M \bigcap S'$. Hence, $M_{\rm red} = S \times M'_{\rm red}$ since S is smooth, so we can pass to (M', S') in place of (M, T) to reduce to the case that $(M_{\overline{F}})^0 = 1$; i.e., M is *finite*.

Any finite commutative group scheme over a field is killed by a positive integer (proof: pass to an algebraically closed ground field and use the connected-étale sequence to pass to the infinitesimal case, which is treated by hand using induction on the order and the vanishing of p on the Lie algebra), so $M \subset T[n] = \mu_n^r$ for some $n \geq 1$. By Cartier duality for finite commutative group schemes, such subgroups M are Cartier dual to quotients of $\mathbf{D}(\mu_n)^r = (\mathbf{Z}/n\mathbf{Z})^r$. Hence, $\mathbf{D}(M)$ is a finite constant group Λ. Decomposing $\Lambda = \Lambda' \times \Lambda''$ where $\Lambda' = \prod \mathbf{Z}/p^{n_j}\mathbf{Z}$ and $\Lambda'' = \prod \mathbf{Z}/d_i\mathbf{Z}$ with $p \nmid d_i$ for all i shows that

$$M \simeq \mathbf{D}(\mathbf{D}(M)) = (\prod \mu_{p^{n_j}}) \times (\prod \mu_{d_i}).$$

Since $\mu_{p^{n_j}}$ is infinitesimal and μ_{d_i} is étale, we get $M_{\rm red} = \mathbf{D}(\Lambda'')$. By inspection this is an étale subgroup of M. ■

For a smooth connected affine group G over an algebraically closed field, since $\mathscr{R}_u(G)$ is normal and solvable in G it is contained in every Borel subgroup B of G. (Indeed, it is contained in some Borel subgroup, hence in all by conjugacy and normality arguments.) Hence, $\mathscr{R}_u(G)$ is contained in $\mathscr{R}_u(B)$ for every B, since such B are solvable and the unipotent radical is *functorial* for solvable smooth k-groups. The following result goes much deeper, and the proof will take a long time.

Theorem 1.4. *Let T be a maximal torus in a smooth connected affine group G over an algebraically closed field k. As B varies through the Borel subgroups which contain T, the resulting smooth connected unipotent subgroup*

$$I(T) := \left(\bigcap_{B \supseteq T} \mathscr{R}_u(B) \right)^0_{\rm red}$$

coincides with $\mathscr{R}_u(G)$. In particular, if G is reductive then $I(T) = 1$.

This result is quite striking, since a-priori it isn't evident that $I(T)$ is even normal in G. (In fact, this is the only problem, since $\mathscr{R}_u(G)$ certainly lies in $I(T)$, and by construction $I(T)$ is smooth, connected, and unipotent). But there is a reason to expect this result: experience with many examples in the reductive case (for which the assertion is that $I(T) = 1$). In fact, it will be easy to reduce the general case to the reductive case, and once the structure theory of connected reductive groups is set up (in terms of root systems and root groups) it will follow that for any single Borel subgroup B containing a maximal torus T in a connected reductive group G there is a (unique) B' containing T such that $\mathscr{R}_u(B) \cap \mathscr{R}_u(B') = 1$ scheme-theoretically (one calls B' the "opposite" Borel subgroup to B relative to T; for $G = \mathrm{GL}_n$ and the diagonal T and upper-triangular B, the lower-triangular Borel is B'). Thus, for a general smooth connected affine group G over $k = \overline{k}$, we may apply this to $G/\mathscr{R}_u(G)$ to get a pair of Borel subgroups B and B' containing T such that $\mathscr{R}_u(B) \cap \mathscr{R}_u(B') = \mathscr{R}_u(G)$ scheme-theoretically. This is a much stronger assertion than that $I(T) = \mathscr{R}_u(G)$, but it rests upon finer structure theory of connected reductive groups.

Proof. The torus T maps isomorphically onto a torus in $G/\mathscr{R}_u(G)$, and its image must be a maximal torus for dimension reasons (as the preimage in G of any torus in $G/\mathscr{R}_u(G)$ is clearly smooth connected and solvable). Thus, it is harmless to replace G with $G/\mathscr{R}_u(G)$ to reduce to the case when G is reductive. We aim to prove $I(T) = 1$.

If we can prove that $I(T)$ is *normal* in G then it must lie in $\mathscr{R}_u(G) = 1$, so we would be done. Such normality is not at all obvious, since $G(k)$-conjugations move T all over the place! The crux of the matter is to prove that G is generated by some finite collection of smooth connected subgroups that each normalize $I(T)$ (so G does as well). We will achieve this by using the classification of connected reductive groups with a 1-dimensional maximal torus over algebraically closed fields: such groups are either SL_2 or PGL_2, for which we can do some concrete calculations. (The intuition, for those familiar with the structure theory of complex semisimple Lie algebras, is that already for a single B and its "opposite" Borel with respect to T we should get a trivial intersection. The problem is that this intuition rests on the structure theory for such Lie algebras in terms of root systems, and the analogous structure theory for connected reductive groups rests on what we are presently trying to prove!)

Let $\Phi = \Phi(G, T)$ denote the set of *nontrivial* weights for the adjoint action of T on $\mathfrak{g} = \mathrm{Lie}(G)$. We may (and do) assume Φ is non-empty. Indeed, otherwise $Z_G(T)$ has Lie algebra $\mathfrak{g}^T = \mathfrak{g}$ and thus $Z_G(T) = G$. But any smooth connected affine group over $k = \overline{k}$ with a central maximal torus must be solvable (since the quotient by the central maximal torus has no nontrivial tori and hence is unipotent). Thus, by reductivity we'd have $G = T$, leaving nothing to do.

By Lemma **??**, for each $a \in \Phi$ the reduced subscheme $T_a := (\ker a)^0_{\mathrm{red}}$ is a codimension-1 subtorus in T whose formation commutes with extension of the ground field. Also, $G_a := Z_G(T_a)$ is a smooth connected subgroup of G containing T with $\mathfrak{g}_a := \mathrm{Lie}(G_a) = \mathfrak{g}^{T_a}$. In other words, $\mathfrak{g}_a$ is the span of the weight spaces in $\mathfrak{g}$ for those T-weights which kill T_a, or in other words are rational multiples of a in $\mathrm{X}(T)_{\mathbf{Q}}$ (as $\mathrm{X}(T/T_a)_{\mathbf{Q}}$ is 1-dimensional and contains $a \neq 0$). In particular, the trivial weight space $\mathfrak{g}^T = \mathrm{Lie}(Z_G(T))$ is contained in every $\mathfrak{g}_a$, as is the a-weight space, so $\mathfrak{g}$ is spanned by the $\mathfrak{g}_a$'s due to the complete reducibility of the T-action on $\mathfrak{g}$. Thus, G is generated by the subgroups G_a. It therefore suffices to prove that each G_a normalizes $I(T)$.

Note that by its definition, each G_a does contain $Z_G(T)$. In particular, T is a maximal torus in every G_a. We claim that each G_a is generated by its Borel subgroups that contain T. If G_a is solvable (which is actually impossible, but we do not know that yet) then it is its own Borel subgroup and there is nothing to do. In the non-solvable case, passing to the non-solvable connected reductive quotient $G_a/\mathscr{R}_u(G_a)$ in which T maps isomorphically onto a maximal torus allows us to apply:

Proposition 1.5. *Let H be a non-solvable connected reductive group over an algebraically closed field, and assume H contains a maximal torus S such that all nontrivial S-weights occurring on $\mathfrak{h}$ are $\mathbf{Q}$-multiples of each other.*

The quotient of H modulo its maximal central torus is either SL_2 *or* PGL_2 *with the image of S going over to the diagonal torus, there are exactly two Borel subgroups of H that contain S, and these Borel subgroups generate H.*

Note that it is automatic that the set $\Phi(H,S)$ of non-trivial S-weights on $\mathfrak{h}$ is non-empty, as otherwise the equality $\mathfrak{h} = \mathfrak{h}^S = \mathrm{Lie}(Z_H(S))$ would force $H = Z_H(S)$, and then H/S makes sense as a k-group and must be *unipotent* (as it has no nontrivial tori, due to maximality of S), contradicting that H is *non-solvable.*

Proof. Consider the maximal smooth connected solvable normal subgroup R in H. This is reductive (since H is), so it is a torus. Being a normal torus in the connected H, it must be central. Thus, it is contained in S (as well as in every Borel) and is killed by all S-weights on $\mathfrak{h}$, so replacing H and S with H/R and S/R respectively is harmless. Thus, we may assume that there is no nontrivial central torus in H. We will next prove that $\dim S = 1$ (so we can apply the classification of non-solvable connected reductive groups with a 1-dimensional maximal torus!).

We have seen that $\Phi(H,S)$ is non-empty, so by the hypotheses $\Phi(H,S)$ spans a single line in $\mathrm{X}(S)_{\mathbf{Q}}$. Hence, if we pick $a \in \Phi(H,S)$ then $S' := (\ker a)^0_{\mathrm{red}}$ is a codimension-1 torus in S on which all elements of $\Phi(H,S) \subset \mathbf{Q} \cdot a$ are trivial, so the containment $Z_H(S') \subset H$ is an equality due to comparison of the Lie algebras. This forces $S' = 1$ since H has no nontrivial central torus, so $\dim S = 1$.

It follows from our classification of non-solvable connected reductive groups with a 1-dimensional maximal torus that necessarily H is isomorphic to either SL_2 or PGL_2. By conjugacy of maximal tori, we can choose this isomorphism so that S goes over to the diagonal torus. The two standard Borel subgroups containing S in each case then generate H: for SL_2 we know that even their unipotent radicals do the job, and so the same holds for the quotient PGL_2. To prove that these two Borel subgroups are the only ones containing S, we first observe that in both SL_2 and PGL_2 the diagonal torus D is its own centralizer, and that $D(k)$ has index 2 in its normalizer. (The case of PGL_2 can be reduced to SL_2 since the kernel of $\mathrm{SL}_2 \twoheadrightarrow \mathrm{PGL}_2$ is contained in the diagonal torus.) We may then conclude by applying the lemma below. ∎

Lemma 1.6. *For any smooth connected affine group G over an algebraically closed field k and any maximal torus T in G, $Z_G(T)$ is contained in every Borel subgroup B of G that contains T, and the resulting "conjugation" action of $N_G(T)(k)/Z_G(T)(k)$ on the set of such B is transitive.*

Proof. The smooth connected subgroup $Z_G(T)$ is solvable (since $Z_G(T)/T$ is a connected linear algebraic group with no nontrivial tori, so it is unipotent). Hence, $Z_G(T)$ is contained in *some* Borel subgroup B_0 of G, and visibly B_0 contains T. Since $Z_G(T)$ is certainly normalized by $N_G(T)(k)$, once it is shown that $N_G(T)(k)$-conjugation is transitive on the set of $B \supset T$ it will follow that all such B contains $Z_G(T)$ and we will be done.

Consider any two $B, B' \supset T$, so $gBg^{-1} = B'$ for some $g \in G$. Observe that gTg^{-1} and T are maximal torus in B', so for some $b' \in B'$ we have $b'gTg^{-1}b'^{-1} = T$. Hence, $b'g \in N_G(T)(k)$ does the job. (Also see HW9 Exercise 6(i) of the previous course.) ∎

Returning to our setup of interest, we have shown that G is generated by the Borel subgroups $B \supset T$ in the groups $G_a = Z_G(T_a)$, so it suffices to prove that $I(T)$ is normalized by each such Borel subgroup. According to Lemma **??**, the Borel subgroups of G_a are precisely $Z_B(T_a)$ for Borel subgroups B of G containing T_a, and such a subgroup contains T if and only if B does (as T obviously centralizes T_a!). Hence, G is generated by its subgroups $Z_B(T_a)$ as B varies through the Borel subgroups containing T. For such B, the smooth connected solvable group $Z_B(T_a)$ is $T \ltimes \mathscr{R}_u(B)^{T_a}$, so its unipotent radical is $\mathscr{R}_u(B)^{T_a}$.

If G_a is non-solvable then the maximal central torus in G_a is T_a (as this has codimension 1 in T and certainly T cannot be central as otherwise G_a/T would be unipotent, forcing G_a to be solvable). Continuing to assume G_a is non-solvable, the reductive quotient $G_a/\mathscr{R}_u(G_a)$ must have the central codimension-1 torus $T_a \subset T$ as its maximal central torus, and Proposition **??** implies that the resulting quotient $G_a/(T_a \times \mathscr{R}_u(G_a))$ by this central torus is either SL_2 or PGL_2 carrying T/T_a over to the diagonal torus. In each of SL_2 and PGL_2 there are *exactly two* Borel subgroups containing the diagonal torus (Proposition **??**). Moreover, each such Borel subgroup supports (in the Lie algebra of its unipotent radical) exactly one of two nontrivial T-weights $\pm q_a \cdot a$ for some rational $q_a > 0$, *both* signs actually occur, and the corresponding weight spaces are 1-dimensional. Since T_a (and hence G_a) is insensitive to replacing a with a nonzero rational multiple (among the T-weights on $\mathfrak{g}$), it follows that each of $\pm q_a \cdot a$ is insensitive to replacing a with a positive rational multiple (among the T-weights on $\mathfrak{g}$).

If some G_a is equal to G then it is non-solvable and T_a is central in G and $\mathscr{R}_u(G_a) = 1$, so G/T_a is either SL_2 or PGL_2, making it evident by inspection that G has exactly two Borel subgroups containing T and that their intersection is trivial. Hence, we may assume that all G_a are proper subgroups of G, so by induction on $\dim G$ (!) each unipotent radical $\mathscr{R}_u(G_a)$ is the reduced identity component of the intersection of the $\mathscr{R}_u(B)^{T_a}$ for B containing T. By the noetherian condition, this intersection over all B stabilizes at a finite set of B's, and likewise for the definition of $I(T)$. Since torus centralizers are compatible with smoothness and with identity components (in the sense that they preserve connectedness), it follows that $\mathscr{R}_u(G_a) = I(T)^{T_a}$ for every a.

For a such that G_a is solvable we have $G_a = T \ltimes \mathscr{R}_u(G_a) = T \ltimes I(T)^{T_a}$, in which case it is clear that G_a normalizes $I(T)$ and the nonzero a-weight space in $\mathfrak{g}$ is contained in $\mathrm{Lie}(I(T))$. (This also shows that once the proof of Theorem **??** is finished, so $I(T) = 1$, no G_a can be solvable.) Thus, we now consider a for which G_a is non-solvable.

Under surjective homomorphisms between smooth connected affine groups over an algebraically closed field, Borel subgroups map onto Borel subgroups (since images of solvable groups are solvable and images of complete varieties are complete) and hence likewise for their unipotent radicals (due to the structure of smooth connected solvable groups over $k = \overline{k}$). Thus, as we vary $B \supset T$, for a such that G_a is non-solvable the image of each $\mathscr{R}_u(B)^{T_a}$ in $G_a/(T_a \times \mathscr{R}_u(G_a))$ is one of two 1-dimensional possibilities. Hence, $\mathscr{R}_u(B)^{T_a}$ contains $\mathscr{R}_u(G_a) = I(T)^{T_a}$ as a normal subgroup with codimension 1 and quotient whose Lie algebra supports a T-weight $\pm q_a \cdot a$ that is insensitive to replacing a with a positive rational multiple (among the T-weights on $\mathfrak{g}$). Moreover, this 1-dimensional quotient as a T-normalized subvariety of the coset space $G/I(T)^{T_a}$ depends *only* on the sign of the multiplier against a. Among *all* nonzero rational multiples of a which arise as T-weights on the tangent space at the identity for the coset space $G/I(T)$ it follows from Proposition **??** (with $S = T_a$ and $H = I(T)$) that *exactly* two have weight space in $\mathfrak{g}$ not entirely contained in $\mathrm{Lie}(I(T))$, and that these two weights are negatives of each other and have weight spaces meeting $\mathrm{Lie}(I(T))$ with codimension 1.

Now we may focus on a such that G_a is non-solvable and (by replacing a with a uniquely determined positive rational multiple if necessary) the a-weight space is not entirely contained in $\mathrm{Lie}(I(T))$. (As we have seen above, this latter condition on a already forces G_a to be non-solvable.) The only other nonzero rational multiple of a which occurs in this way is $-a$ (and it *does* occur). We have seen that as we vary through all Borel subgroups B of G containing T, the groups $\mathscr{R}_u(B)^{T_a}/\mathscr{R}_u(G_a)$ vary through precisely the two Borel subgroups $B_{\pm a}$ of $G_a/(T_a \times \mathscr{R}_u(G_a))$ containing T/T_a (i.e., Borels of SL_2 or PGL_2 containing the diagonal torus), distinguished by which of a or $-a$ occurs as the T-weight on its Lie algebra. Correspondingly the preimage Borel subgroups $B^{T_a} = T \ltimes \mathscr{R}_u(B)^{T_a}$ in G_a vary through exactly two possibilities which are distinguished by which of a or $-a$ occurs as a T-weight on its Lie algebra outside of $\mathrm{Lie}(I(T)^{T_a}) = \mathrm{Lie}(I(T))^{T_a}$. (Keep in mind that $T_{-a} = T_a$, and $I(T)^{T_a} = \mathscr{R}_u(G_a)$ has nothing to do with the choice of B.) But $\mathscr{R}_u(B)^{T_a}$ is the unipotent radical of $T \ltimes \mathscr{R}_u(B)^{T_a}$. Thus, as we vary though all Borel subgroups B of G

containing T, the groups $\mathscr{R}_u(B)^{T_a}$ vary through exactly two possibilities, distinguished by which of a or $-a$ occurs as the T-weight on the 1-dimensional $\mathscr{R}_u(B)^{T_a}/\mathscr{R}_u(G_a)$.

Our remaining task is to prove that both possibilities for $\mathscr{R}_u(B)^{T_a}$ normalize $I(T)$. It is harmless to rename $-a$ as a, due to the symmetry of the situation (as $T_{-a} = T_a$ and both $\pm a$ occur as T-weights outside of $\mathrm{Lie}(I(T))$), so we may fix a and focus on $B_0 \supset T$ such that the 1-dimensional subgroup $\mathscr{R}_u(B_0)^{T_a}/\mathscr{R}_u(G_a)$ in $G_a/(T_a \times \mathscr{R}_u(G_a))$ coincides with B_a. But for all such B_0 the groups $\mathscr{R}_u(B_0)^{T_a} \subset G_a$ are *the same*, whence $\mathscr{R}_u(B_0)^{T_a}$ lies in $\mathscr{R}_u(B)$ as B varies through *all* Borel subgroups of G containing T for which $\mathscr{R}_u(B)^{T_a}/\mathscr{R}_u(G_a) = B_a$.

Define $I_a(T)$ as the reduced identity component of an intersection similar to $I(T)$, except that we restrict to those $B \supset T$ such that $\mathscr{R}_u(B)^{T_a}/\mathscr{R}_u(G_a) = B_a$ (and not B_{-a}). Since the formation of T_a-centralizers preserves connectedness and smoothness, $I_a(T)^{T_a} = \mathscr{R}_u(B_0)^{T_a}$. Hence, $\mathscr{R}_u(B_0)^{T_a} \subset I_a(T)$, so to prove that $\mathscr{R}_u(B_0)^{T_a}$ normalizes $I(T)$ it suffices to prove that $I_a(T)$ normalizes its subgroup $I(T)$.

The preceding considerations yield the following very important consequence (especially after we finish the proof of Theorem **??**, so for reductive G we have $I(T) = 1$ and G_a is always non-solvable when some a exist, which is to say $G \neq T$):

Lemma 1.7. *Assume G is connected reductive and is not a torus. The finite collection $\Psi(G,T) \subset \mathrm{X}(T)$ of non-trivial T-weights a on $\mathfrak{g}$ whose weight space is not contained in* $\mathrm{Lie}(I(T))$ *is non-empty and stable under negation, with each such weight having a* 1*-dimensional weight space in the tangent space at the identity on the coset space $G/I(T)$. Moreover, for any $a \in \Psi(G,T)$, the set of* $\mathbf{Q}$*-multiples of a in $\Psi(G,T)$ is $\{\pm a\}$.*

Proof. For any such weight a, apply the preceding arguments and Proposition **??** with $S = T_a$ and $H = I(T)$. ■

The normality of $I(T)$ in $I_a(T)$ is reduced to a dimension property, due to:

Lemma 1.8. *For any inclusion $U \hookrightarrow U'$ between smooth connected unipotent groups over a field, if $U \neq U'$ then $N_{U'}(U)$ is strictly larger than U. In particular, if* $\dim(U'/U) = 1$ *then U is normal in U'.*

Proof. We may assume the ground field is algebraically closed. The descending central series of U' (or consideration of upper-triangular unipotent matrices) forces U' to contain a central $\mathbf{G}_\mathrm{a}$ (here we use that the ground field is algebraically closed). If this is not contained in U then we win. Otherwise we can replace U and U' with their quotients modulo this common central subgroup and proceed by induction on $\dim U'$. ■

It now suffices to prove that $\dim I_a(T)/I(T) \leq 1$. The coset space $I_a(T)/I(T)$ has a natural T-action (as $I_a(T)$ and $I(T)$ are normalized by T), so its tangent space at the identity point is a direct sum of weight spaces for some T-weights; by the way we have chosen a, one such weight is a itself. Explicitly, the elements of $\mathbf{Q} \cdot a$ that arise as such weights must show up as T-weights on $I_a(T)^{T_a} = \mathscr{R}_u(B_0)^{T_a}$, yet $I_a(T)^{T_a} \bigcap I(T) = I(T)^{T_a} = \mathscr{R}_u(G_a)$ and $\mathscr{R}_u(B_0)^{T_a}/\mathscr{R}_u(G_a) = B_a$ is 1-dimensional with T-weight a. Hence, the only T-weight in $\mathrm{Tan}_1(I_a(T)/I(T))$ lying in $\mathbf{Q} \cdot a$ is a itself, with a 1-dimensional weight space. To prove the 1-dimensionality of $I_a(T)/I(T)$, it therefore suffices to prove that for any $b \in \Phi(G,T) \subset \mathrm{X}(T)_{\mathbf{Q}}$ *linearly independent* from a over $\mathbf{Q}$, b does *not* arise as a T-weight on $\mathrm{Tan}_1(I_a(T)/I(T))$.

We assume to the contrary that such a b exists, and we seek a contradiction. The hypothesis on b implies that the b-weight space in $\mathfrak{g}$ is not entirely contained in $\mathrm{Lie}(I(T))$, so G_b is non-solvable and hence the preceding results for a may be applied to b as well. Hence, for any $B \supset T$, the quotient

$\mathscr{R}_u(B)^{T_b}/\mathscr{R}_u(G_b)$ is 1-dimensional with b or $-b$ as the unique T-weight on its Lie algebra, and $\mathscr{R}_u(B)^{T_b}$ is the unipotent radical of the preimage in G_b of one of the two Borel subgroups $B_{\pm b} \subset G_b/(T_b \times \mathscr{R}_u(G_b))$ containing T/T_b. More specifically, the Borel subgroup $B^{T_b} = T \ltimes \mathscr{R}_u(B)^{T_b} \subset G_b$ maps onto exactly one of $B_{\pm b}$ in $G_b/(T_b \times \mathscr{R}_u(G_b))$ and is uniquely determined by that image group.

Now we return to our choice of $B_0 \supset T$ such that $B_0^{T_a} \twoheadrightarrow B_a$ inside $G_a/(T_a \times \mathscr{R}_u(G_a))$. By definition we have $I_a(T) \subset \mathscr{R}_u(B_0)^{T_a}$, so $I_a(T) \subset \mathscr{R}_u(B_0)$ and hence $I_a(T)^{T_b} \subset \mathscr{R}_u(B_0)^{T_b}$. By hypothesis $\mathrm{Tan}_1(I_a(T)/I(T))$ has b as a T-weight, so passing to T_b-fixed points implies that $\mathrm{Tan}_1(I_a(T)^{T_b}/I(T)^{T_b})$ has b as a T-weight, yet the equality $I(T)^{T_b} = \mathscr{R}_u(G_b)$ yields the closed immersion

$$I_a(T)^{T_b}/I(T)^{T_b} = I_a(T)^{T_b}/\mathscr{R}_u(G_b) \subset \mathscr{R}_u(B_0)^{T_b}/\mathscr{R}_u(G_b) = B_{\pm b}!$$

The source has b as a T-weight on its tangent space at 1, hence the right side must be B_b and *not* B_{-b}.

Summarizing, for any Borel subgroup $B \supset T$, if $B^{T_a} \twoheadrightarrow B_a$ then necessarily $B^{T_b} \twoheadrightarrow B_b$ (and *not* B_{-b}). To get a contradiction, it suffices to show that we can actually build Borel subgroups $B \supset T$ for which B^{T_a} and B^{T_b} may be "arbitrarily" assigned. More precisely, Recall that T_a uniquely determines the pair $\{a, -a\}$. Call a codimension-1 torus S in T *singular* if there is a T-weight on $\mathfrak{g}$ which kills S and whose weight space is not entirely contained in $\mathrm{Lie}(I(T))$. To get a contradiction and complete the proof of Theorem **??**, we apply the following lemma. ■

Lemma 1.9. *Let $a, b \in \Phi$ be linearly independent over $\mathbf{Q}$ such that their weight spaces in $\mathfrak{g}$ are not contained in* $\mathrm{Lie}(I(T))$. *Then there exist Borel subgroups B, B' in G containing T such that $B^{T_a}, B^{T_a} \twoheadrightarrow B_a$ but $B^{T_b} \twoheadrightarrow B_b$ and $B'^{T_b} \twoheadrightarrow B_{-b}$.*

Proof. We bring in the "dynamic approach" to algebraic groups (discussed in an earlier handout, and in March 8 and March 10 lectures from the previous course). Call a cocharacter $\lambda : \mathbf{G}_m \to T$ *regular* if is not killed by any of the weights in $\Phi(G,T)$. This amounts to requiring that $\lambda \in \mathrm{X}_*(T) = \mathrm{X}(T)^\vee$ avoids finitely many "hyperplanes", so there are many such λ. In particular, for all $c \in \Phi(G,T)$ the pairing $\langle c, \lambda \rangle = c \circ \lambda \in \mathrm{End}(\mathbf{G}_m) = \mathbf{Z}$ is nonzero. For any regular λ (or even any 1-parameter subgroup of G at all), we obtained smooth connected unipotent subgroups $U_G(\lambda)$ and $U_G(-\lambda)$, as well as a smooth connected subgroup $Z_G(\lambda) = Z_G(-\lambda)$, such that all are normalized by T and their Lie algebras are the respective weight spaces in $\mathfrak{g}$ for the weights $c \in \Phi(G,T) \cup \{0\}$ satisfying $\langle c, \lambda \rangle > 0$, $\langle c, \lambda \rangle < 0$, and $\langle c, \lambda \rangle = 0$. The final case occurs precisely for $c = 0$ since λ is regular, so $Z_G(\lambda)$ and $Z_G(T)$ have the same Lie algebra and hence the containment $Z_G(T) \subseteq Z_G(\lambda)$ (which follows from the functorial characterization of $Z_G(\lambda)$ because λ is valued in T) is forced to be an equality due to connected and dimension reasons. Hence, we have an open immersion

$$U_G(\lambda) \times Z_G(T) \times U_G(-\lambda) \to G$$

via multiplication (see §2 of the handout "Lang's theorem and dynamic methods", and HW10 Exercise 3 of the previous course), and $Z_G(T) = Z_G(\lambda) = Z_G(-\lambda)$ normalizes both $U_G(\lambda)$ and $U_G(-\lambda)$.

The centralizer $Z_G(T)$ is solvable (since $Z_G(T)/T$ must be unipotent, due to maximality of T), so by the centrality of T it has the form $Z_G(T) = T \times U$ for a unipotent radical U. Let us show that $U \subset I(T)$. The smooth connected solvable subgroup $Z_G(T)$ is contained in *some* Borel subgroup B of G, and $T \subset B$ since $T \subset Z_G(T)$, so $U = \mathscr{R}_u(Z_G(T)) \subset \mathscr{R}_u(B)$. But $N_G(T)(k)$ acts transitively on the set of Borel subgroups of B containing T (Lemma **??**), yet it clearly normalizes $Z_G(T)$, so it follows that $Z_G(T)$ is contained in *every* Borel subgroup of G containing T. Hence, U lies in the unipotent radical of all such Borel subgroups, so $U \subset I(T)$ as claimed. In particular, we see that $Z_G(T) \subset T \ltimes I(T)$.

Fix a regular cocharacter λ, so all nontrivial T-weights c on $\mathfrak{g}$ have nonzero pairing against λ and hence have weight space meeting the Lie algebra of $U_G(\lambda)$ or $U_G(-\lambda)$ nontrivially. Since $Z_G(T) \subset T \ltimes I(T)$, it follows that $\mathfrak{g}$ is spanned by $\mathrm{Lie}(I(T))$ and $\mathrm{Lie}(U_G(\pm\lambda))$. Define the smooth connected *solvable* subgroup

$$H(\lambda) := T \ltimes U_G(\lambda).$$

We claim that the subgroup

$$B(\lambda) := \langle H(\lambda), I(T) \rangle$$

is a Borel subgroup containing T, and that $\mathrm{Lie}(B(\lambda)) = \mathrm{Lie}(H(\lambda)) + \mathrm{Lie}(I(T))$. (Since $N_G(T)(k)$-conjugation permutes the subgroups $B(\lambda)$ via the $N_G(T)(k)$-action on $\mathrm{X}_*(T)$, it would then follow from the transitivity of the $N_G(T)(k)$-action on the set of Borels containing T that *every* Borel subgroup of G containing T has the form $B(\lambda)$ for some regular λ!)

To prove that every $B(\lambda)$ is a Borel subgroup, and moreover has the predicted Lie algebra, we argue indirectly. Since $H(\lambda)$ is smooth, connected, and solvable, it is contained in *some* Borel subgroup B, so $T \subset B$ since $H(\lambda)$ contains T. In particular, $I(T) \subset B$, so $B(\lambda) \subset B$. We shall prove that the containment $\mathrm{Lie}(B(\lambda)) \subset \mathrm{Lie}(B)$ is an equality, forcing $B(\lambda) = B$, so $B(\lambda)$ is a Borel subgroup. Since $Z_G(T) \subset T \times I(T) \subset B(\lambda)$, if the T-stable subspace $\mathrm{Lie}(B) \subset \mathfrak{g}$ is strictly larger than $\mathrm{Lie}(B(\lambda))$ then it must support a weight $c \in \Phi(G,T)$ such that the c-weight space in $\mathfrak{g}$ is *not* entirely contained in $\mathrm{Lie}(I(T))$ and $\langle c, \lambda \rangle < 0$.

For any such c the group G_c must be non-solvable (since the c-weight space of $\mathfrak{g}$ is not entirely inside $\mathrm{Lie}(I(T))$, due to how c was chosen). It follows from our study of $G_c/(T_c \times \mathscr{R}_u(G_c))$ for such c that $-c \in \Phi(G,T)$ as well, and that moreover the $-c$-weight space of $\mathfrak{g}$ is not entirely contained in $\mathrm{Lie}(I(T))$. More specifically, since $\langle -c, \lambda \rangle > 0$ and B contains $U_G(\lambda)$, the entire $-c$-weight space is contained in $\mathrm{Lie}(B)$, so in fact $\mathrm{Lie}(B^{T_c}) = \mathrm{Lie}(B)^{T_c}$ supports both $\pm c$-weight lines *outside* of $\mathrm{Lie}(I(T))$. But then the map

$$f : B^{T_c} \to G_c/(T_c \times \mathscr{R}_u(G_c))$$

induced by the inclusion $B^{T_c} \subset G_c$ has T-equivariant Lie algebra map that hits *both* nontrivial weight lines on the target as well as $\mathrm{Lie}(T/T_c)$, so $\mathrm{Lie}(f)$ is *surjective.* That forces f to be surjective, which is absurd because f is a map from a solvable group to a non-solvable group. This completes the proof that $B(\lambda)$ is a Borel subgroup, and the argument also shows that $\mathrm{Lie}(B(\lambda)) = \mathrm{Lie}(H(\lambda)) + \mathrm{Lie}(I(T))$ (because it $\mathrm{Lie}(B(\lambda))$ were any larger then it would admit a T-weight c of exactly the type from which a contradiction was deduced above).

Now we can construct the desired Borel subgroups containing T. Let $S := T_a$ and $S' := T_b$. For any regular $\lambda \in \mathrm{X}_*(T)$ and the Borel subgroup $B = B(\lambda)$ (with Lie algebra $\mathrm{Lie}(H(\lambda)) + \mathrm{Lie}(I(T))$), the Lie algebra of B^S is spanned by the Lie algebras of $H(\lambda)^S$ and $I(T)^S$. Hence, the Borel subgroup B^S in $G^S = G_a$ is generated by $H(\lambda)^S = T \ltimes U_G(\lambda)^S = T \ltimes U_{G^S}(\lambda)$ and $I(T)^S$. Observe that $I(T)^S = I(T)^{T_a} = \mathscr{R}_u(G_a) = \mathscr{R}_u(G^S)$ is determined by S: it has nothing to do with the choice of B! Thus, the good behavior of the "$U_H(\mu)$" construction with respect to *surjections* implies that the common image of B^S and $U_{G^S}(\lambda)$ in $G^S/(S \times \mathscr{R}_u(G^S))$ is the unipotent radical of one of the two Borel subgroups $B_{\pm a}$ containing T/S, depending on which of $\langle \pm a, \lambda \rangle$ is positive. If we replace λ with $-\lambda$ then we get the "opposite" one (since $U_{G^S}(-\lambda)$ supports the entire $-a$-weight space in $\mathfrak{g}$, which is *not* entirely contained in the Lie algebra of $\mathscr{R}_u(G^S) = I(T)^S$, due to the occurrence in opposite pairs in Lemma **??**). The same conclusions apply to (b, S') in place of (a, S).

Since a and b are linearly independent over $\mathbf{Q}$, we may pick $\lambda, \lambda' \in \mathrm{X}_*(T) \subset \mathrm{X}_*(T)_{\mathbf{Q}} = \mathrm{X}(T)_{\mathbf{Q}}^{\vee}$ such that

$$\langle a, \lambda \rangle, \langle b, \lambda \rangle > 0, \ \ \langle a, \lambda' \rangle > 0 > \langle b, \lambda' \rangle.$$

Then the Borel subgroups $B = B(\lambda)$ and $B' = B(\lambda')$ containing T satisfy $B^S, {B'}^S \twoheadrightarrow B_a$ but $B^{S'} \twoheadrightarrow B_b$ and ${B'}^{S'} \twoheadrightarrow B_{-b}$. ■

Corollary 1.10. *Let k be a field and G a connected reductive k-group that is not a torus. Assume G contains a split maximal k-torus T.*

The set $\Phi(G,T)$ of non-trivial T-weights occurring on $\mathfrak{g}$ is non-empty and stable under negation in $\mathrm{X}(T)$, and for each $a \in \Phi(G,T)$ the weight space $\mathfrak{g}_a$ is 1-dimensional and the only $\mathbf{Q}$-multiples of a in $\Phi(G,T)$ are $\pm a$.

Proof. If $\Phi(G,T)$ is empty then T acts trivially on $\mathfrak{g}$, so $Z_G(T)$ has full Lie algebra in $\mathfrak{g}$ and hence $Z_G(T) = G$. But $Z_G(T) = T$ by reductivity, and G is not a torus by hypothesis. Hence, $\Phi(G,T)$ is non-empty. By Theorem **??**, $I(T) = 1$. Thus, we may apply Lemma **??** to conclude. ■

In view of the triviality of $I(T)$, so $B(\lambda) = H(\lambda)$ in the proof of Lemma **??**, we get an important consequence:

Corollary 1.11. *Let G be a connected reductive group over $k = \overline{k}$, T a maximal torus. As λ varies through the regular cocharacters in $\mathrm{X}_*(T)$ (i.e., $\langle a, \lambda \rangle \neq 0$ for all $a \in \Phi(G,T)$), the subgroups $B(\lambda) = T \ltimes U_G(\lambda)$ vary through precisely the Borel subgroups of G containing T.*

Remark 1.12. This dynamic description has an extremely interesting consequence: the dimensions of Borel subgroups B and maximal tori T of G satisfy $\dim B = (1/2)(\dim G + \dim T)$. Indeed, fix $T \subset B$, so $B = B(\lambda) = T \ltimes U_G(\lambda)$ for a regular $\lambda \in \mathrm{X}_*(T)$. Consider the T-weight space decomposition $\mathfrak{g} = \mathfrak{t} \oplus (\oplus_{a \in \Phi} \mathfrak{g}_a)$ with $\langle a, \lambda \rangle \neq 0$ for all $a \in \Phi = \Phi(G,T)$ and all $\mathfrak{g}_a$ of dimension 1 (see Corollary **??**). We know that $\mathrm{Lie}(U_G(\lambda))$ is spanned by the lines $\mathfrak{g}_a$ with $\langle a, \lambda \rangle > 0$, so for each pair $\{a, -a\}$ in Φ, exactly one of the associated root lines is in $\mathrm{Lie}(U_G(\lambda))$. In other words,

$$\dim U_G(\lambda) = (1/2)\#\Phi = (1/2)(\dim \mathfrak{g} - \dim \mathfrak{t}) = (1/2)(\dim G - \dim T).$$

Since $\dim B = \dim T + \dim U_G(\lambda)$, the asserted formula for $\dim B$ follows.

One reason that this formula for $\dim B$ is interesting is that it provides a criterion to identify when an explicitly constructed connected solvable subgroup in G is maximal as such (i.e., is a Borel subgroup): it is necessary and sufficient that its dimension is $(1/2)(\dim G + \dim T)$. We know how to identity when a candidate torus T in G is maximal: just check that $\dim T = \dim Z_G(T)$, or in other words that $\mathrm{Lie}(T)$ is the entire 0-weight space for the T-action on $\mathfrak{g}$. Once such a T has been found, we can then compute $(1/2)(\dim G + \dim T)$ to know what the dimension of the Borel subgroups must be!

We will later vastly generalize Corollary **??**, giving a dynamical description of all parabolic subgroups and deducing an analogue over any ground field. Inspired by Theorem **??** let's now analyze the set of all Borel subgroups B containing a fixed maximal torus T in a smooth connected affine group G over an algebraically closed field k, going beyond the reductive case as in Corollary **??**. By Lemma **??**, we know that all such B contain $Z_G(T)$ and that the finite constant group $W(G,T) = N_G(T)/Z_G(T)$ acts transitively on this collection. Even better:

Proposition 1.13. *The transitive action by $W(G,T) = N_G(T)/Z_G(T)$ on the set of Borel subgroups containing T is simply transitive. In particular, the number of such Borel subgroups is finite, and in fact equal to $\#W(G,T)$.*

As an example, if $G = \mathrm{GL}_n$ and T is the diagonal torus D then the subgroup $S_n \subset G(k)$ of permutation matrices lies in $N_G(T)$ and maps isomorphically onto $W(G,T)$. Thus, the S_n-orbit of the standard upper-triangular subgroup $B \subset G$ is the set of Borel subgroups of G containing D

(i.e., these correspond to choices of enumeration of the standard ordered basis, each enumeration giving rise to a different flag, the stabilizer of which is the corresponding Borel subgroup).

Proof. We have to show that if $n \in N_G(T)$ satisfies $nBn^{-1} = B$ for some $n \in N_G(T)$ then $n \in Z_G(T)$. In the March 5 lecture of the previous course we discussed the important theorem of Chevalley that every parabolic subgroup is its own normalizer at the level of field-valued points, and its proof (resting on dimension induction and especially the *connectedness* of torus centralizers in connected linear algebraic groups) was given at the start of the present course. As a consequence of that result, $n \in B$, so $n \in N_B(T)$. The problem is reduced to a general property of *solvable* connected linear algebraic groups H over a field: the normalizer of a maximal torus T in H is the centralizer of T. We may assume the ground field k is algebraically closed, so $H = T \ltimes U$ for a smooth connected unipotent U, and we just need to show that if $u \in U(k)$ normalizes T then it centralizes T. We will not even use the unipotence of U.

It suffices to show that if $u \in U(k)$ and $utu^{-1} \in T$ for all $t \in T$ then u is centralized by the T-action. It is harmless to multiply on the right by t^{-1}, so it is equivalent to say $u(tu^{-1}t^{-1}) \in T$ for all $t \in T$. But $tu^{-1}t^{-1} \in U$, so $u(tu^{-1}t^{-1}) \in U$. Thus, membership in T is equivalent to the condition $u(tu^{-1}t^{-1}) = 1$ which says exactly that u commutes with every $t \in T$; i.e., $u \in Z_G(T)$. ■

2. Torus centralizers and unipotent radicals

The following theorem is the key miracle.

Theorem 2.1. *For any k-torus S in G, we have*

$$Z_G(S)_{\overline{k}} \cap \mathscr{R}_u(G_{\overline{k}}) = \mathscr{R}_u(Z_G(S)_{\overline{k}})$$

inside $G_{\overline{k}}$. In particular, if G is reductive then so is $Z_G(S)$.

The preservation of reductivity under passage to torus centralizers in connected reductive groups is a powerful inductive technique to prove general theorems by dimension induction.

Proof. We may and do assume $k = \overline{k}$. The S-conjugation on G preserves the normal subgroup $\mathscr{R}_u(G)$, and the scheme-theoretic intersection $Z_G(S) \cap \mathscr{R}_u(G)$ is simply the S-centralizer $\mathscr{R}_u(G)^S$ in $\mathscr{R}_u(G)$ under this action. But functorial considerations make it clear that

$$Z_{S \ltimes \mathscr{R}_u(G)}(S) = S \times \mathscr{R}_u(G)^S,$$

and the left side is smooth and connected since it is a torus centralizer in the smooth connected affine group $S \ltimes \mathscr{R}_u(G)$! Thus, it follows that the direct factor (as a k-scheme) $\mathscr{R}_u(G)^S$ is also smooth and connected. (This same argument shows more generally that for *any* smooth connected subgroup H in G normalized by S, $Z_G(S) \cap H$ is smooth and connected.)

We conclude that $Z_G(S) \cap \mathscr{R}_u(G)$ is a smooth connected unipotent subgroup of $Z_G(S)$, and it is visibly normal (as $\mathscr{R}_u(G)$ is normal in G), whence $Z_G(S) \cap \mathscr{R}_u(G) \subseteq \mathscr{R}_u(Z_G(S))$. It remains to prove the reverse inclusion, which is to say that $\mathscr{R}_u(Z_G(S)) \subseteq \mathscr{R}_u(G)$.

The unipotent radical of any smooth connected affine group H (over $k = \overline{k}$) is smooth connected solvable and thus lies in *some* Borel subgroup. By conjugacy of Borel subgroups and normality of the unipotent radical, it follows that $\mathscr{R}_u(H)$ lies in *all* Borel subgroups of H, and thus (by solvability of Borel subgroups) in the *unipotent radicals* of all of these Borel subgroups. Taking $H = Z_G(S)$, we obtain from Lemma **??** that

$$\mathscr{R}_u(Z_G(S)) \subseteq \bigcap_{B \supseteq S} \mathscr{R}_u(Z_G(S) \cap B) \subseteq \bigcap_{B \supseteq S} \mathscr{R}_u(B)$$

since the formation of the unipotent radical is functorial in smooth connected solvable groups (such as with respect to the inclusion $Z_G(S) \cap B \to B$). Thus,

$$\mathscr{R}_u(Z_G(S)) \subseteq \left(\bigcap_{B \supseteq S} \mathscr{R}_u(B) \right)^0_{\mathrm{red}}.$$

If we pick a maximal torus T containing S, then the intersection can only grow if we restrict to those B that contain T. But restricting to such B yields the group $I(T) = \mathscr{R}_u(G)$ by Theorem **??**. ∎

Recall the elementary fact that reductivity is inherited by smooth connected normal k-subgroups. More specifically, if $N \subseteq G$ is a smooth connected normal k-subgroup then the $G(\overline{k})$-conjugation action on $N_{\overline{k}}$ must preserves $\mathscr{R}_u(N_{\overline{k}})$, so this unipotent radical is normal in $G_{\overline{k}}$. Hence, $\mathscr{R}_u(N_{\overline{k}}) \subset \mathscr{R}_u(G_{\overline{k}})$, so reductivity of G implies that of N. In fact, the inclusion $\mathscr{R}_u(N_{\overline{k}}) \subseteq N_{\overline{k}} \cap \mathscr{R}_u(G_{\overline{k}})$ of subgroup schemes of $G_{\overline{k}}$ (using scheme-theoretic intersection) is always an *equality*, but the proof rests on some non-trivial structural properties of reductive groups which have not yet been proved and we will not need this result. (A proof is given in Proposition A.4.8 of "Pseudo-reductive groups", working over $\overline{k}$ there.) The main input is the non-obvious fact that the scheme-theoretic center of a connected reductive group is a subgroup scheme of a torus (see Corollary **??** below), and so has no nontrivial subgroup schemes which can arise as subgroup schemes of smooth unipotent groups (see HW5, Exercise 1 of the previous course).

Corollary 2.2. *If G is a connected reductive group over a field k and T is a maximal k-torus then $Z_G(T) = T$; in particular, the scheme-theoretic center Z_G is contained in all such T.*

Also, for any surjective k-homomorphism $\pi : G \twoheadrightarrow G'$, $\pi(\mathscr{R}_u(G_{\overline{k}})) = \mathscr{R}_u(G'_{\overline{k}})$. In particular, if G is reductive then so is G'.

Our proof of the first assertion in this corollary will rest on Grothendieck's theorem concerning the existence of a maximal k-torus which remains maximal over $\overline{k}$, as that ensures $T_{\overline{k}}$ is maximal in $G_{\overline{k}}$. But we only apply the equality $Z_G(T) = T$ in the setup where $k = \overline{k}$ (e.g., in the proof of the behavior of unipotent radicals under quotient maps). Special cases were seen in HW3 Exercise 4(i) and HW4 Exercise 1 of the previous course.

Proof. We may and do assume $k = \overline{k}$. By Theorem **??**, $Z_G(T)$ is reductive since G is reductive. But its maximal torus T is central, so the quotient $Z_G(T)/T$ is unipotent. Hence, $Z_G(T)$ is a solvable connected reductive group, so it is a torus (due to the structure of smooth connected solvable groups over algebraically closed fields). By maximality, the inclusion $T \hookrightarrow Z_G(T)$ must then be an equality.

Now consider the scheme-theoretic preimage of $\mathscr{R}_u(G')$ under the quotient map $G \to G'$. This is a normal subgroup scheme of G (since $\mathscr{R}_u(G')$ is normal in G'), so the identity component N of its underlying reduced scheme is as well. Then N inherits reductivity from G and admits $\mathscr{R}_u(G')$ as a quotient, so we can replace G with N to reduce to showing that for any connected reductive group G, a smooth connected unipotent quotient U of G must be trivial. Let T be a maximal torus in G. Its image in U is trivial, so by the compatibility of torus centralizers with respect to surjective homomorphisms between smooth connected affine groups (Corollary 2.5 in the handout "Lang's theorem and dynamic methods") it follows that $U = Z_U(1)$ is the image of $Z_G(T) = T$. This forces $U = 1$ since U is unipotent and T is a torus. ∎

Example 2.3. Consider a smooth affine group G over an algebraically closed field k, and any quotient $\pi : G \twoheadrightarrow G'$ with G' reductive. Thus, G^0 maps onto the reductive ${G'}^0$, so by Corollary **??**

the unipotent radical $\mathscr{R}_u(G) = \mathscr{R}_u(G^0)$ is killed by this quotient map. Hence, π factors uniquely through the natural quotient map $G \to G/\mathscr{R}_u(G)$, and conversely any quotient of G which factors through this latter map must be a quotient of $G/\mathscr{R}_u(G)$ and hence is reductive. For these reasons, the quotient $G/\mathscr{R}_u(G)$ is sometimes called the *maximal reductive quotient* of G.

Corollary 2.4. *Let G be a connected reductive group over a separably closed field k. Then G is generated by its maximal k-tori, and Z_G is the scheme-theoretic intersection of such tori.*

This corollary is actually true over any field, but the proof requires deeper structure theory (and extra care when k is finite).

Proof. Let N be the smooth connected k-subgroup generated by the maximal k-tori. Since $G(k)$ is Zariski-dense in G (as $k = k_s$) and it normalizes N, it follows that N is normal in G. Thus, G/N makes sense as a smooth connected group, and by construction it contains no nontrivial k-tori. By Grothendieck's theorem, such a group is unipotent. But G/N is reductive by Corollary **??**, so it is trivial. Hence, $G = N$, so G is generated by its maximal k-tori T. It follows that Z_G is defined (functorially) by the condition of centralizing all such T. But $Z_G(T) = T$, so Z_G is the (scheme-theoretic) intersection of all such T. ■

We end this section with a surprisingly useful and non-obvious fact:

Corollary 2.5. *Let G be a connected reductive group over a field k of characteristic $p > 0$, and Z_G its scheme-theoretic center. Then $(Z_G)_{\overline{k}}$ cannot contain α_p or $\mathbf{Z}/p\mathbf{Z}$ as subgroup schemes. In particular, if U is a smooth unipotent k-subgroup of G then $Z_G \cap U = 1$ scheme-theoretically, and so $U \to G/Z_G$ is a closed k-subgroup inclusion.*

Beware that it *can* happen that a connected reductive group G contains a normal non-central infinitesimal subgroup scheme U having a composition series by α_p's, though the resulting so-called *unipotent isogenies* $G \to G/U$ only exist in characteristic 2. The simplest example is the weird purely inseparable isogeny $\mathrm{PGL}_2 \to \mathrm{SL}_2$ obtained by factors the Frobenius isogeny $\mathrm{SL}_2 \to \mathrm{SL}_2$ through the central quotient $\mathrm{SL}_2 \to \mathrm{PGL}_2$ whose kernel μ_2 is killed by Frobenius.

Proof. By Corollary **??**, Z_G is a k-subgroup of a torus. Thus, we just have to check that $\mathbf{G}_m$ does not contain α_p or $\mathbf{Z}/p\mathbf{Z}$. Since $\mathbf{G}_m[p] = \mu_p$, this amounts to the assertion that α_p and $\mathbf{Z}/p\mathbf{Z}$ are not isomorphic to μ_p. The case of $\mathbf{Z}/p\mathbf{Z}$ is clear since μ_p is not étale in characteristic p, and for α_p we can use the comparison of their p-Lie algebras to rule out an isomorphism (though Cartier duality provides another way, once one has computed that α_p is its own Cartier dual, which is not entirely trivial to verify directly.) ■

3. Derived groups and semisimple groups

Now we reap the fruit of our labors. A smooth connected affine group H over a field is called *perfect* if $H = \mathscr{D}(H)$. For example, if G is connected reductive over a field k then $\mathscr{D}(G)$ is a smooth connected normal k-subgroup of G, so it is also reductive. How about its own derived group $\mathscr{D}(\mathscr{D}(G))$? Can this decreasing chain involve several steps before it terminates (for dimension reasons), as happens for solvable groups? No, the process ends immediately:

Lemma 3.1. *Let G be a connected reductive group over a field k. The derived group $\mathscr{D}(G)$ is perfect.*

Proof. Let $N = \mathscr{D}(G)$. To prove that N is perfect, first note that $\mathscr{D}(N)$ is normal in G, so we may replace G with the (reductive!) quotient $G/\mathscr{D}(N)$ to reduce to the case when $\mathscr{D}(N) = 1$. In other words, the connected reductive group N is commutative, so it is a torus. But G/N is also commutative and reductive, hence a torus, so G must itself be a torus! But then obviously $N = 1$, which is perfect. ■

Recall that a solvable connected reductive k-group is just a k-torus by another name. By HW5 Exercise 4(iii) of the previous course, these are the same as the "Galois lattices" via the functor $T \rightsquigarrow \mathrm{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m)$. The non-solvable case is much more interesting, and requires the apparatus of root systems to get a handle on the structure. The following result, refining Corollary **??**, explains the importance of SL_2 and PGL_2 in the general theory of connected reductive groups (very similarly to the reason for the importance of $\mathfrak{sl}_2$ in the general theory of semisimple Lie algebras in characteristic 0).

Proposition 3.2. *Let k be a field and G a connected reductive k-group that is not a torus. Assume G contains a split maximal k-torus T.*

For $a \in \Phi(G,T)$ and $G_a := Z_G(T_a)$, the natural map $T_a \times \mathscr{D}(G_a) \to G_a$ is a central isogeny (*i.e., isogeny with central kernel*) *and $\mathscr{D}(G_a)$ is k-isomorphic to* SL_2 *or* PGL_2 *with the scheme-theoretic intersection $T \bigcap \mathscr{D}(G_a)$ as a 1-dimensional split maximal k-torus.*

In characteristic $p > 0$, general isogenies between connected linear algebraic groups need not be central (unlike in characteristic 0); e.g., n-fold Frobenius isogenies $F_{G/k,n} : G \to G^{(p^n)}$ for any $n \geq 1$ and any smooth affine k-group G, or $\mathrm{PGL}_p = \mathrm{SL}_p/\mu_p \to \mathrm{SL}_p$ through which $F_{\mathrm{SL}_p/k,1}$ factors.

Proof. By Theorem **??** we know that the smooth connected k-subgroup G_a is reductive. Each G_a is *non-solvable*, because otherwise by reductivity such a G_a would be a torus, and hence equal to its maximal torus T, contradicting that $\mathrm{Lie}(G_a) = \mathfrak{g}^{T_a}$ contains the *non-zero* weight space in $\mathfrak{g}$ for the nontrivial T-weight a. Thus, the central quotient G_a/T_a is a connected reductive non-solvable k-group in which T/T_a is a *split* 1-dimensional maximal k-torus. Hence, by the classification of non-solvable connected reductive groups with a 1-dimensional split maximal torus (see the March 12 lecture from the previous course), the quotient G_a/T_a must be *k-isomorphic* to SL_2 or PGL_2. This is its own derived group by inspection (classical for SL_2, hence inherited by the quotient PGL_2), so $\mathscr{D}(G_a) \to G_a/T_a$ is surjective.

It follows that $T_a \times \mathscr{D}(G_a) \to G_a$ is surjective, with central kernel given by the anti-diagonally embedded $T_a \cap \mathscr{D}(G_a)$. We claim that this intersection is finite, which would imply that $\mathscr{D}(G_a) \to G_a/T_a$ is an isogeny, so the maximal tori in $\mathscr{D}(G_a)$ are then 1-dimensional. This finiteness is a special case of:

Lemma 3.3. *A central torus C in a linear algebraic group H has* finite *intersection with $\mathscr{D}(H)$.*

Proof. We can assume the ground field is algebraically closed, so C is split. Pick a faithful linear representation $H \hookrightarrow \mathrm{GL}(V)$, and form the weight decomposition $V = \oplus V_{\chi_i}$ with respect to the *faithful* C-action, so the χ_i generate $\mathrm{X}(C)$ up to finite index. Then by centrality, H lands in $\prod \mathrm{GL}(V_{\chi_i})$, so $\mathscr{D}(H)$ projects to have determinant 1 in each factor. Thus, $C \cap \mathscr{D}(H)$ maps into each $\mathrm{GL}(V_{\chi_i})$ with scalar image killed by the determinant, hence inside the diagonal μ_{d_i} with $d_i = \dim V_{\chi_i}$. It follows that for $n = \prod d_i$ we have that χ_i^n kills $C \cap \mathscr{D}(H)$ for all i. But the χ_i generate a finite-index subgroup of $\mathrm{X}(C)$, so $C \cap \mathscr{D}(H)$ is killed by a finite-index subgroup of $\mathrm{X}(C)$. Hence, $C \cap \mathscr{D}(H)$ cannot contain any tori of positive dimension, so it is finite. ■

It remains to show that $\mathscr{T} := T \bigcap \mathscr{D}(G_a)$ is a maximal k-torus of $\mathscr{D}(G_a)$, as such a torus inherits the k-split property from T and so forces $\mathscr{D}(G_a)$ to be k-isomorphic to SL_2 or PGL_2 via the rank-1 classification theorem. We have seen that all maximal tori in $\mathscr{D}(G_a)$ are 1-dimensional, and $\mathscr{T}$ must have positive dimension since $G_a/\mathscr{D}(G_a)$ is isogenous to the codimension-1 torus $T_a \subset T$. Hence, we just have to show that $\mathscr{T}$ is a torus. But $T = Z_G(T)$, so $T \bigcap \mathscr{D}(G_a) = \mathscr{D}(G_a)^T$. This is smooth and connected (as for centralizers of torus actions on smooth connected affine groups in general), so it must be a torus because it is a subgroup of T. ■

Example 3.4. For $G = \mathrm{SL}_n$ and T the diagonal torus, it is straightforward to check by inspection that each $\mathscr{D}(G_a)$ is SL_2 and not PGL_2. We claim that for $n \geq 3$, in $G' := \mathrm{PGL}_n$ every $\mathscr{D}(G'_{a'})$ is SL_2 (and not PGL_2). To see this, first note that $q : G = \mathrm{SL}_n \to \mathrm{PGL}_n = G'$ has kernel equal to the *central* diagonal μ_n, so (by centrality) for the diagonal torus $T' = T/\mu_n \subset G'$ we have $\Phi(G', T') = \Phi(G, T) =: \Phi$ inside $\mathrm{X}(T') \subset \mathrm{X}(T)$ (for both groups, by inspection the roots are $t \mapsto t_i/t_j$ for $i \neq j$). For each $a \in \Phi$, the isogeny $T_a \to T'_a$ induces a central isogeny $G_a = Z_G(T_a) \twoheadrightarrow Z_{G'}(T'_a) = G'_a$ (central since its kernel is contained in $\ker q \subset Z_G$), so we get a central isogeny $q_a : \mathscr{D}(G_a) \twoheadrightarrow \mathscr{D}(G'_a)$ between derived groups. But $\mathscr{D}(G_a) = \mathrm{SL}_2$ has center μ_2 that is clearly not contained in the diagonal $\mu_n = \ker q \subset G$ when $n \geq 3$. Thus, for $n \geq 3$ we have $\ker q_a = 1$ and so q_a is an isomorphism.

Here is a nontrivial example in which some $\mathscr{D}(G_a)$ are PGL_2 and some are SL_2. Let $Q = x_1x_5 + x_2x_4 + x_3^2$, so $G := \mathrm{SO}(Q) = \mathrm{SO}_5$ has diagonal maximal torus

$$m : \mathbf{G}_{\mathrm{m}}^2 \simeq T \subset \mathrm{SO}(Q)$$

via $m(t, t') = \mathrm{diag}(t', t, 1, 1/t, 1/t')$. (Use weight space considerations and pairwise distinctness of the diagonal characters of T to prove that $T = Z_G(T)$, so T is indeed maximal in G.) It is easy to check that the closed immersion $u : \mathbf{G}_{\mathrm{a}} \hookrightarrow G$ defined by

$$u(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & y & -y^2 & 0 \\ 0 & 0 & 1 & -2y & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a homomorphism normalized by T, with $m(t, t')u(y)m(t, t')^{-1} = u(ty)$. (This does land in G, since $x_2x_4 + x_3^2$ is invariant under $(x_2, x_3, x_4) \mapsto (x_2, x_3 + yx_2, x_4 - 2yx_3 - y^2x_2)$.) This subgroup $U \subset G$ satisfies $\mathrm{Lie}(U) = \mathfrak{g}_a$ where $a : T \to \mathbf{G}_{\mathrm{m}}$ is $a(m(t, t')) = t$.

We claim that in this example, $\mathscr{D}(G_a) = \mathrm{PGL}_2$. In fact, by inspection the "opposite" T-normalized unipotent subgroup $U' = \mathbf{G}_{\mathrm{a}}$ using the "lower triangular" analogue has the opposite weight on its Lie algebra, so these are the unipotent radicals of the two Borel subgroups of $\mathscr{D}(G_a)$ that contain the 1-dimensional diagonal maximal torus $T \bigcap \mathscr{D}(G_a)$. Together they visibly generate the 3-dimensional $\mathrm{SO}(x_2x_4 + x_3^2) = \mathrm{SO}_3 \simeq \mathrm{PGL}_2$, so this is $\mathscr{D}(G_a)$ for dimension reasons.

In contrast, the subgroup $\mathrm{SO}(x_1x_5 + x_2x_4) \subset G$ is an SO_4 in which T is a maximal torus, and $\mathrm{SO}_4 \simeq \mathrm{SL}_2 \times^{\mu_2} \mathrm{SL}_2$. The two evident $\mathbf{G}_{\mathrm{m}}$-factors of T are the respective 1-dimensional diagonal maximal tori of these "factor" subgroups $\mathrm{SL}_2 \subset \mathrm{SO}_4 \subset G$, and the standard weight spaces in each $\mathrm{sl}_2 \subset \mathfrak{g}$ are T-weight spaces for some pairs of opposite roots $\{\pm b\}, \{\pm b'\}$. Hence, we see that each of these SL_2's centralizes one of the codimension-1 tori T_b or $T_{b'}$ and must therefore exhaust the 3-dimensional $\mathscr{D}(G_b)$ or $\mathscr{D}(G_{b'})$ respectively.

Remark 3.5. For any smooth connected normal k-subgroup N in any connected reductive k-group H (e.g., $N = \mathscr{D}(G_a)$ in $H = G_a$) and any maximal k-torus T in H, the scheme-theoretic intersection $N \cap T$ is a maximal k-torus of N. To prove this, we first note (as in the proof of Proposition **??**) that $T \bigcap N$ is smooth and connected, since $T \bigcap N = N^T$ (as $T = Z_H(T)$). Thus, $T \bigcap N$ is a torus, as for any smooth connected subgroup of T.

The maximality requires a different argument from what was done in the "1-dimensional" setting of Proposition **??**. Let $S \subset N$ is a k-torus containing $T \bigcap N$. Let $S' \supset S$ be a maximal torus of G containing S, so $S'_{\overline{k}}$ and $T_{\overline{k}}$ are $G(\overline{k})$-conjugate (as they are maximal over $\overline{k}$, by Grothendieck's theorem!). By normality of N in G, it follows that both $(S' \bigcap N)_{\overline{k}} = S'_{\overline{k}} \bigcap N_{\overline{k}}$ and $(T \bigcap N)_{\overline{k}} = T_{\overline{k}} \bigcap N_{\overline{k}}$ are $G(\overline{k})$-conjugate. The resulting equality of dimensions then forces the containments $T \bigcap N \subset S \subset S' \bigcap N$ among smooth connected groups to be equalities.

Theorem 3.6. *Let G be a connected reductive group over a field k, and Z the maximal central torus. Then $Z = (Z_G)^0_{\mathrm{red}}$ and the formation of Z commutes with any extension of the ground field, and the multiplication homomorphism*

$$Z \times \mathscr{D}(G) \to G$$

is a central isogeny (i.e., an isogeny with central kernel).

In particular, $\mathscr{D}(G) \to G/Z$ and $Z \to G/\mathscr{D}(G)$ are central isogenies.

The basic example of this theorem is the central isogeny $\mathbf{G}_m \times \mathrm{SL}_n \to \mathrm{GL}_n$, whose kernel is the central anti-diagonal μ_n, along with the induced central isogenies $\mathrm{SL}_n \to \mathrm{PGL}_n$ and $\mathbf{G}_m \to \mathrm{GL}_n/\mathrm{SL}_n = \mathbf{G}_m$ given by $t \mapsto \det(\mathrm{diag}(t)) = t^n$.

Proof. Since Z_G lies in any maximal k-torus T (as $T = Z_G(T)$), any central torus must lie in $(Z_G)^0_{\mathrm{red}}$. If we can prove that this is itself a torus, then it must be Z. Moreover, if that is established, then for any K/k we have that $(Z_{G_K})^0_{\mathrm{red}}$ is the maximal central torus in G_K yet it contains the base change $((Z_G)^0_{\mathrm{red}})_K$ that is also a *torus* and visibly has the same geometric points, so the inclusion $((Z_G)^0_{\mathrm{red}})_K \subset (Z_{G_K})^0_{\mathrm{red}}$ must be an equality. Thus, for the assertions concerning Z we just have to check that $(Z_G)^0_{\mathrm{red}}$ is a k-torus. Pick a maximal k-torus $T \subset G$, so $Z_G \subset Z_G(T) = T$. By Lemma **??**, it follows that $(Z_G)^0_{\mathrm{red}}$ is a torus whose formation commutes with any extension on k, settling the assertions concerning Z alone.

By Lemma **??**, $Z \cap \mathscr{D}(G)$ is finite. Hence, $Z \times \mathscr{D}(G) \to G$ has finite central kernel. To prove that this is an isogeny, it suffices to prove that G/Z is perfect. We claim that there are no nontrivial central tori in G/Z. If S is a central torus in G/Z then its preimage S' in G is a torus (being an extension of the torus S by the torus Z) and is also normal in G, forcing centrality in G (since G is connected). That implies $S' \subseteq Z$, so $S = 1$ as claimed. We may now rename G/Z as G to reduce to showing that if $Z = 1$ then $\mathscr{D}(G) = G$. In particular, we may and do assume that k is algebraically closed (since we already proved that the formation of Z commutes with extension of the ground field).

Pick a maximal torus T in G. By Corollary **??**, for each $a \in \Phi(G,T)$, the a-weight space is 1-dimensional and the $\mathbf{Q}$-multiples of a in $\Phi(G,T)$ are precisely $\pm a$. The set $\Phi(G,T)$ generates a finite-index subgroup of $\mathrm{X}(T)$. Indeed, otherwise there would be a nontrivial torus S in T killed by all elements of $\Phi(G,T)$, so $\mathfrak{g} = \mathfrak{g}^S = \mathrm{Lie}(Z_G(S))$, forcing $Z_G(S) = G$ and so contradicting that we arranged for G to contain no nontrivial central tori.

For each $a \in \Phi(G,T)$, Proposition **??** ensures that for $T_a = (\ker a)^0_{\mathrm{red}}$, the natural map

$$T_a \times \mathscr{D}(Z_G(T_a)) \to Z_G(T_a)$$

is a central isogeny. More specifically, $\mathscr{D}(Z_G(T_a))$ equipped with its T/T_a-action has exactly $\pm a$ as the nontrivial weights on its Lie algebra, and $S_a := T \cap \mathscr{D}(Z_G(T_a))$ is a 1-dimensional maximal torus of $\mathscr{D}(Z_G(T_a))$. Thus, the smooth connected subgroups $\mathscr{D}(Z_G(T_a))$ of $\mathscr{D}(G)$ generate a smooth connected subgroup H of $\mathscr{D}(G)$ whose Lie algebra supports all weight spaces for the nontrivial T-weights on $\mathfrak{g}$. Since $\mathfrak{h}$ is a T-stable subspace of $\mathfrak{g}$ which contains all weight spaces for nontrivial weights, whereas

$$\mathrm{Lie}(T) = \mathrm{Lie}(Z_G(T)) = \mathfrak{g}^T$$

is the weight space for the trivial weight, to prove that $G = \mathscr{D}(G)$ it remains to show that $T \subseteq \mathscr{D}(G)$.

We will prove that T is equal to the group $(N_G(T), T)$ generated by commutators $ntn^{-1}t^{-1}$ for $n \in N_G(T)(k)$ and $t \in T(k)$. Let $W = N_G(T)(k)/Z_G(T)(k) = N_G(T)(k)/T(k)$ denote the usual Weyl group which acts on T, so $(N_G(T), T)$ is the smooth connected subgroup of T generated by the images of the maps $T \to T$ defined by $t \mapsto (w.t)t^{-1}$ for $w \in W$. There is a natural action of W on the lattice $\mathrm{X}_*(T)$ of cocharacters $\lambda : \mathbf{G}_m \to T$, and the sublattice $\mathrm{X}_*((N_G(T), T))$ contains all elements $w.\lambda - \lambda$. Hence, to prove that the subtorus $(N_G(T), T)$ in T is full, it suffices to show that the elements $w.\lambda - \lambda$ generate a finite-index sublattice of $\mathrm{X}_*(T)$, or equivalently that the $\mathbf{Q}[W]$-module $\mathrm{X}_*(T)_{\mathbf{Q}}$ has vanishing space of coinvariants. Since W is *finite* (HW8 Exercise 4(iii) of the previous course), so $\mathbf{Q}[W]$ is semisimple, it is equivalent to have a vanishing space of W-invariants, which is to say that $\mathrm{X}_*(T)^W = 0$. In other words, we claim that T^W is *finite*.

We will prove that $T^W \subseteq \ker(2a)$ (scheme-theoretically) for all $a \in \Phi(G, T)$, so $(T^W)^0_{\mathrm{red}} \subseteq (\ker(2a))^0_{\mathrm{red}} = (\ker a)^0_{\mathrm{red}} = T_a$. This is sufficient because the subtorus $(\cap_a T_a)^0_{\mathrm{red}}$ in T is killed by all $a \in \Phi(G, T)$ and hence is trivial (as we have seen that $\Phi(G, T)$ generates $\mathrm{X}(T)_{\mathbf{Q}}$, due to the arranged property $Z = 1$). Consider the group $G_a = Z_G(T_a)$ and its derived group $H_a = \mathscr{D}(G_a)$ (which is isomorphic to SL_2 or PGL_2), so T_a is the maximal *central* torus in G_a and $S_a := T \cap H_a$ is a maximal torus of H_a. Pick any representative $h_a \in H_a$ of the nontrivial element in $N_{H_a}(S_a)/S_a$, so h_a acts on S_a via inversion. It also centralizes T_a, and so normalizes $T_a \cdot S_a = T$. Thus, the class $w_a \in W$ of h_a centralizes T_a and swaps the weight spaces $\pm a$ for S_a, which in turn are the weight spaces for T/T_a acting on $\mathrm{Lie}(G_a)$ (since $S_a \to T/T_a$ is an isogeny). In other words, the W-action on $\mathrm{X}(T)$ negates $a \in \mathrm{X}(T/T_a)$. It follows that if $t \in T^W(R)$ for a k-algebra R then

$$a(t) = a(w_a.t) = (w_a.a)(t) = (-a)(t),$$

so $(2a)(t) = 1$ (i.e., $a(t)^2 = 1$). In other words, $t \in \ker(2a)$, as desired. ■

Now we prove the equivalence of several different ways to characterize *semisimple* groups:

Corollary 3.7. *Let G be a smooth connected affine group over a field k. The following are equivalent:*

1. *The maximal smooth connected solvable normal subgroup $\mathscr{R}(G_{\overline{k}})$ of $G_{\overline{k}}$ is trivial.*
2. *The group G is reductive and has finite center.*
3. *The group G is reductive and perfect.*

Condition (1) is the usual definition of semisimplicity, but sometimes one sees (2) or (3) used as cheap definitions. In practice it is important to know the equivalence among all of these conditions. Also, the connectedness condition on G cannot be removed, since the semi-direct product $G = \mathbf{G}_{\mathrm{m}} \rtimes (\mathbf{Z}/2\mathbf{Z})$ via inversion has $G^0 = \mathbf{G}_{\mathrm{m}}$ a positive-dimensional torus but $Z_G = \mu_2 \times (\mathbf{Z}/2\mathbf{Z})$ is finite.

Proof. Under all hypotheses G is reductive, so we now may assume G is connected reductive. By Lemma **??** and Theorem **??**, there is then is a central isogeny

$$f : Z \times \mathscr{D}(G) \to G$$

where $Z = (Z_G)^0_{\mathrm{red}}$ is the maximal central k-torus and $\mathscr{D}(G)$ is perfect. Thus, (2) implies (3). Likewise, if (3) holds then the isogeny property for f implies $Z = 1$, so the central torus $(Z_G)^0_{\mathrm{red}}$ is trivial. Hence, Z_G is finite, so (2) holds. This proves the equivalence of (2) and (3).

It is clear that (1) implies $Z = 1$, and hence implies (3). Conversely, if (3) holds then $R = \mathscr{R}(G_{\overline{k}})$ is a smooth connected solvable normal subgroup of the perfect connected reductive group $G_{\overline{k}}$. Normality forces R to be reductive, and solvability forces it to be a torus. Normality in the connected $G_{\overline{k}}$ then forces this torus to be central. But (3) is equivalent to (2), so this central torus is trivial. Thus, (1) holds. ■

4. Relations between subgroups of G and $\mathscr{D}(G)$

Let G be a connected reductive group over a field k, and Z its maximal central k-torus, so we have a central isogeny $Z \times \mathscr{D}(G) \to G$. This underlies a link between maximal k-tori (resp. parabolic k-subgroups, resp. Borel k-subgroups) of G and $\mathscr{D}(G)$. For example, G admits a Borel k-subgroup if and only if $\mathscr{D}(G)$ does. First we explain the dictionary for relating such subgroups through central quotient maps in general, and then we deduce the consequences for passing between such k-subgroups of G and $\mathscr{D}(G)$.

Proposition 4.1. *Let $f : G' \to G$ be a central surjective homomorphism between connected reductive k-groups. Then the operations $T' \mapsto f(T')$ and $T \mapsto f^{-1}(T)$ are inverse bijections between the sets of maximal k-tori in G' and G.*

The same operations (*image and preimage*) *define inverse inclusion-preserving bijections between the sets of parabolic k-subgroups* (*and hence between the sets of minimal parabolic k-subgroups*), *as well as between the sets of Borel k-subgroups, in G' and G.*

In this result, we use scheme-theoretic preimages; e.g., part of the assertion is that $f^{-1}(H)$ is k-smooth when H is a maximal k-torus or parabolic k-subgroup of G. Before we prove Proposition **??**, we present an example that highlights the importance of centrality of $\ker f$: an example of a non-central isogeny $f : G' \to G$ between connected semisimple groups such that G admits a Borel k-subgroup B and G' does *not*! (What "goes wrong" in the following example is that $f^{-1}(B)$ is not smooth, and in fact $f^{-1}(B)_{\rm red}$ is *not* a smooth k-subgroup of G'.)

Example 4.2. Let k be a local function field of equicharacteristic $p > 0$ (with finite residue field). By local class field theory, $\mathrm{Br}(k) = \mathbf{Q}/\mathbf{Z}$ and the Brauer class $[D]$ of a central division algebra of rank n^2 over k has order exactly n. Pick D of rank p^2, so $[D]$ has order p.

Let $G = \mathrm{SL}_1(D)$. We claim that G has no Borel k-subgroup, nor any proper parabolic k-subgroup whatsoever. Indeed, we will later show (via the dynamic method) that for any proper parabolic F-subgroup P of a connected reductive group over any field F, the geometric unipotent radical $\mathscr{R}_u(P_{\overline{F}})$ descends to a unipotent smooth connected F-subgroup $U \subset P$ that is "split": admits a composition series with successive quotients F-isomorphic to $\mathbf{G}_{\rm a}$. Hence, if G contains a proper parabolic k-subgroup P (so $\mathscr{R}_u(P_{\overline{k}}) \neq 1$) then G contains $\mathbf{G}_{\rm a}$ as a k-subgroup, so $G(k)$ contains nontrivial p-torsion elements. But this is impossible, since $G(k) \subset D^\times$, and any element of $D^\times$ lies in a commutative extension field of k, yet multiplicative groups of commutative fields of characteristic $p > 0$ never have nontrivial p-torsion!

The reason for interest in G is that it admits a (non-centrally) isogenous quotient that *does* have Borel k-subgroups. To see this, consider the Frobenius isogeny $F_{G/k} : G \to G^{(p)}$, where $G^{(p)} = G \otimes_{k,\phi_k} k$ with $\phi_k : k \to k$ the p-power endomorphism. Clearly $G^{(p)} = \mathrm{SL}_1(D^{(p)})$, where $D^{(p)} = D \otimes_{k,\phi_k} k$. We claim that $D^{(p)} \simeq \mathrm{Mat}_p(k)$ as k-algebras, so $G^{(p)} \simeq \mathrm{SL}_p$ as k-groups (so $G^{(p)}$ visibly has a Borel k-subgroup).

In other words, we claim that the central simple k-algebra D is split by the field extension $\phi_k : k \to k$. That is, we claim that the induced map $\mathrm{Br}(\phi_k) : \mathrm{Br}(k) \to \mathrm{Br}(k)$ kills $[D]$. Explicitly, $k \simeq \mathbf{F}_q(\!(t)\!)$, so ϕ_k is a degree-p extension field. But it is a general fact in local class field theory (essentially proved in Serre's "Local Fields") that the canonical isomorphism $\mathrm{inv}_L : \mathrm{Br}(L) \simeq \mathbf{Q}/\mathbf{Z}$ for non-archimedean local fields L is functorial with respect to arbitrary finite extensions $j : L \to L'$ (no separability hypotheses!) via multiplication by $[L' : L]$. That is,

$$\mathrm{inv}_{L'} \circ \mathrm{Br}(j) = [L' : L] \cdot \mathrm{inv}_L.$$

Applying this to $\phi_k : k \to k$ shows that $\mathrm{Br}(\phi_k)$ kills $\mathrm{Br}(k)[p]$, and so kills $[D]$.

Now we prove Proposition **??**.

Proof. The operations in both directions are compatible with change of the ground field, and whether or not a closed subgroup is smooth or a maximal torus or a Borel subgroup or a parabolic subgroup can be checked after a ground field extension. Hence, it is harmless to replace k with $\overline{k}$ (!), so we may assume k is algebraically closed. In particular, all maximal tori in G are $G(k)$-conjugate, and similarly with Borel subgroups.

Let $T' \subset G'$ be a maximal torus, so $\ker f \subset Z_{G'} \subset Z_{G'}(T') = T'$. Hence, $T := f(T') = T'/(\ker f)$ inside $G = G'/(\ker f)$. Thus, $f^{-1}(T) = T'$. It follows that $T' \mapsto f(T')$ is an injective map from the set of maximal tori in G' to the set of those of G, and that f^{-1} inverts this map on its image in the set of maximal tori of G. But this map between sets of maximal tori is surjective because every maximal torus of G is $G(k)$-conjugate to T and $G'(k) \to G(k)$ is surjective. This settles the case of maximal tori. The case of Borel subgroups goes exactly the same way.

For parabolic subgroups, conjugacy no longer holds for all possibilities, so instead we use group theory by collecting the parabolics containing a fixed Borel subgroup. Let $B' \subset G'$ be a Borel subgroup and $B = B'/(\ker f) \subset G$, so $G'/B' \simeq G/B$ as schemes. Consideration of geometric points shows that $P' \mapsto f(P')$ defines a bijection between sets of parabolic subgroups containing B' and B respectively, with f^{-1} defining an inverse operation. ■

Corollary 4.3. *Let G be a connected reductive group over a field k, and $Z \subset G$ the maximal central k-torus. Then $T \mapsto T \bigcap \mathscr{D}(G)$ defines a bijection between the sets of maximal k-tori of G and $\mathscr{D}(G)$, with $\mathscr{T} \mapsto Z \cdot \mathscr{T}$ the inverse bijection.*

Likewise, the operations $H \mapsto H \bigcap \mathscr{D}(G)$ and $\mathscr{H} \mapsto Z \cdot \mathscr{H}$ define inverse inclusion-preserving bijections between the sets of parabolic k-subgroups of G and $\mathscr{D}(G)$ (hence between sets of minimal parabolic k-subgroups), and also inverse bijections between the sets of Borel k-subgroups of G and $\mathscr{D}(G)$.

Proof. As in the proof of Proposition **??**, we may replace k with $\overline{k}$ (since the formation of Z commutes with extension on k). We apply Proposition **??** to the central isogeny $Z \times \mathscr{D}(G) \to G$ and observe that (by conjugacy considerations over $k = \overline{k}$) the maximal tori of $Z \times \mathscr{D}(G)$ are precisely $Z \times \mathscr{T}$ for maximal tori $\mathscr{T} \subset \mathscr{D}(G)$, and similarly for Borel subgroups. In the case of parabolic subgroups the same description holds, but rather than appeal to conjugacy considerations (which no longer apply) we use group theory considerations relative to the containment of Borel subgroups (much as in the treatment of parabolic subgroups in the proof of Proposition **??**). ■