Math 249B. Relative Bruhat decomposition and relative Weyl group

1. Overview

Let G be a connected reductive group over a field k, S a maximal k-split torus in G, and P a minimal parabolic k-subgroup of G. Let $N = N_G(S)$ and $Z = Z_G(S)$, so N/Z is a finite étale k-group (as for any torus centralizer in any smooth affine group). The ordinary group $N(k)/Z(k)$ is called the *relative Weyl group* and is usually denoted ${}_kW$. (Later it will naturally be identified with the "combinatorial" Weyl group of the root system attached to (G, S).)

In class we proved most of the (relative) Bruhat decomposition: the natural map

$${}_kW := N(k)/Z(k) \to P(k)\backslash G(k)/P(k)$$

is bijective. In other words, if we choose $Z(k)$-coset representatives $n_w \in N(k)$ for each $w \in {}_kW$ then every $P(k)$-double coset in $G(k)$ has the form $P(k)n_wP(k)$ for a unique $w \in {}_kW$. In §2 we fill in the one loose end of that argument (which is also the step where the Bruhat decomposition over $\overline{k}$ is used!).

Then we establish a geometric refinement: if $n, n' \in N(k)$ are distinct modulo $Z(k)$ then $PnP \neq Pn'P$ as locally closed subschemes of G. (Equivalently, $P(\overline{k})nP(\overline{k}) \neq P(\overline{k})n'P(\overline{k})$ inside $G(\overline{k})$.) We also address the question of how the finite group $N(k)/Z(k)$ is related to the finite étale k-group N/Z, ultimately showing that every connected component of N has a k-point (so N/Z is the constant group associated to $N(k)/Z(k)$), and in §5ff. we wrap up our discussion of these matters by giving examples with commutative Z for which $\mathrm{H}^1(k, Z) \neq 0$.

2. Intersection of parabolics

To complete the proof of the relative Bruhat decomposition, we have to prove:

Theorem 2.1. *Let P, Q be parabolic k-subgroups of G. There exists a maximal split k-torus of G contained in both P and Q.*

To prove this we may shrink P to be minimal, so the quotient $\overline{P} := P/\mathscr{R}_{u,k}(P)$ has a central maximal split k-torus. Consequently, if T is a maximal k-torus of P then its maximal torus isomorphic image $\overline{T}$ in the *connected reductive* group $\overline{P}$ is its own centralizer and thus must contain the unique central maximal split k-torus. It follows that T contains a split k-torus of the same dimension as the maximal split k-tori of G; i.e., T contains a maximal split k-torus of G. Consequently, it suffices to show that P and Q share a common maximal k-torus of G.

Suppose we knew that $P \cap Q$ is *smooth* (and connected). Then the dimension of its maximal k-tori coincides with the dimension of the maximal tori of $P_{\overline{k}} \cap Q_{\overline{k}} = (P \cap Q)_{\overline{k}}$; we want this to coincide with the dimension of the maximal tori of $G_{\overline{k}}$. The property of an affine k-group scheme of finite type being k-smooth and connected is also checkable over $\overline{k}$, so we are brought to a geometric problem (applied to $G_{\overline{k}}$):

Proposition 2.2. *Let H be a connected reductive group over an algebraically closed field k. For any parabolic subgroups $P, Q \subset H$, the intersection $P \cap Q$ is smooth and connected and it contains a maximal torus of H.*

Proof. First we find a maximal torus $T \subset H$ contained in each of P and Q, and then we use this T as a crutch to prove $P \cap Q$ is smooth and connected. For the purpose of finding such a T it is harmless to shrink P and Q to be minimal; i.e., we may focus on Borel subgroups $B, B' \subset H$, and seek a maximal torus of H contained in each.

Fix a maximal torus $T \subset B$. Note that $B' = hBh^{-1}$ for some $h \in H$. By the Bruhat decomposition over algebraically closed fields, applied to the borus (B, T), we have $h = b_1 n b_2$ for some $b_1, b_2 \in B$ and $n \in N_H(T)$. Hence,

$$B' = b_1 n B n^{-1} b_1^{-1},$$

so $B' \cap B = b_1(nBn^{-1} \cap B)b_1^{-1} \supset b_1 T b_1^{-1}$ since $n \in N_H(T)$. Thus, $b_1 T b_1^{-1}$ is a maximal torus of H contained in B and B'.

Returning to the original setup with parabolic subgroups $P, Q \subset H$, we have found a maximal torus $T \subset H$ contained in P and Q. The dynamic description of P involves a cocharacter of any desired maximal torus of P, such as T. Hence, $P = P_G(\lambda)$ for some $\lambda : \mathrm{GL}_1 \to T$. But we rigged T to be contained in Q also, so the good behavior of dynamic constructions relative to intersections gives

$$P \cap Q = P_G(\lambda) \cap Q = P_Q(\lambda),$$

and that inherits smoothness and connectedness from Q. ■

3. Geometric refinement

As explained above, for $n, n' \in N(k)$ distinct modulo $Z(k)$, we wish to show that $(PnP)(\overline{k})$ and $(Pn'P)(\overline{k})$ are disjoint. Since $P = Z \ltimes U$ for the k-unipotent radical $U = \mathscr{R}_{u,k}(P)$ and n and n' normalize Z, we have

$$(PnP)(\overline{k}) = Z(\overline{k}) \cdot (UnU)(\overline{k})$$

and similarly for n'. Thus, if $(PnP)(\overline{k})$ and $(Pn'P)(\overline{k})$ have a common point then there exists $z \in Z(\overline{k})$ such that $z(UnU)(\overline{k})$ meets $(Un'U)(\overline{k})$. But Z normalizes U, so $z(UnU)(\overline{k}) = (U(zn)U)(\overline{k})$ and $zn \neq n'$ since n and n' are distinct modulo $Z(\overline{k})$ (as $N(k)/Z(k)$ injects into $(N/Z)(k) \subseteq (N/Z)(\overline{k}) = N(\overline{k})/Z(\overline{k})$). By Corollary 7.4.5 in the course notes, there exists $\lambda : \mathrm{GL}_1 \to S$ such that $P = P_G(\lambda)$ and $Z_G(S) = Z_G(\lambda)$. Hence, it suffices to prove the following result applied over $\overline{k}$ upon renaming such a hypothetical zn as n:

Proposition 3.1. *Let G be a smooth connected affine group over a field, S a torus of G, and $\lambda : \mathrm{GL}_1 \to S$ a cocharacter of S such that $Z_G(\lambda) = Z_G(S)$. Let $N = N_G(S)$, $Z = Z_G(S) = Z_G(\lambda)$, $U = U_G(\lambda)$, and $P = P_G(\lambda) = Z \ltimes U$.*

For $n, n' \in N(k)$, if UnU meets $Un'U$ then $n = n'$.

Proof. Without loss of generality $k = \overline{k}$, and since the double cosets meet we must have $nu = u'n'$ for some $u, u' \in U(k)$. Hence, ${n'}^{-1}n = {n'}^{-1}(nu)u^{-1} = {n'}^{-1}u'n'u^{-1}$. We aim to eventually prove that ${n'}^{-1}n = 1$, so as a preliminary step we will investigate the structure of the solvable (even unipotent) smooth connected group $V := {n'}^{-1}Un'$.

Note that $Sn' = n'S$, so V is normalized by S (since $S \subset Z \subset P$ and U is normal in P), so it is normalized by λ. Thus, the dynamic method applies to V (or rather to the semidirect product $\mathrm{GL}_1 \ltimes V$ defined via the GL_1-action on V through λ), so we obtain an open subscheme

$$U_V(-\lambda) \times Z_V(\lambda) \times U_V(\lambda) \hookrightarrow V$$

via multiplication. But in the connected solvable case this open immersion is always an equality! (Indeed, in the commutative case it follows from the fact that a smooth connected group has no proper open subgroups, and in general one can bootstrap from the commutative case to the solvable case via the derived series and appropriate dimension induction: see Proposition 2.1.12 and its proof in [CGP].) Hence, this says that we have an equality of schemes via multiplication

$$(V \cap U') \times (V \cap Z) \times (V \cap U) = V,$$

where $U' := U_G(-\lambda)$. By definition of V we have $V \cap Z = n'^{-1}(U \cap Z)n' = 1$, so

$$(V \cap U') \times (V \cap U) = V$$

via multiplication. Thus, the element $n'^{-1}u'n' \in V(k)$ can be written as $u_1'u_1$ for $u_1' \in U'(k)$ and $u_1 \in U(k)$.

Recall that $n'^{-1}n = n'^{-1}u'n'u^{-1}$, so this is equal to $u_1'(u_1u^{-1}) \in U'(k) \times U(k)$. Thus, $n'^{-1}n$ lies in $N(k) \cap (U'(k) \times U(k))$. We claim that the intersection of N with the open subscheme $U' \times Z \times U$ of G is equal to Z, so this would force $u_1', u_1u^{-1} = 1$ and hence $n' = n$ as desired. It now remains to prove rather generally that for any smooth closed subgroup H of Z (such as S), the inclusion

$$(Z_G(H) \cap U') \times N_Z(H) \times (Z_G(H) \cap U) \subseteq N_G(H) \cap (U' \times Z \times U)$$

inside G is an equality. (Indeed, $N_Z(H) = Z_G(H) \cap Z$ and if $H = S$ then by hypothesis $Z_G(S) = Z_G(\lambda) = Z$ and we know that $Z \cap U', Z \cap U = 1$, so we would be done.) We will prove the equality by computing with points valued in arbitrary k-algebras R.

Choose a k-algebra R and points $u' \in U'(R)$, $z \in Z(R)$, and $u \in U(R)$ such that $u'zu$ lies in $N_G(H)(R)$, which is to say that it normalizes H_R inside G_R. The aim is to prove that u, u' centralize H_R and z normalizes H_R. Such properties are sufficient to check on R'-valued points for every R-algebra R', so upon choosing an R' and renaming it as R (as we may do), the task is to prove that u, u' centralize $H(R)$ and z normalizes $H(R)$ inside $G(R)$. That is, if $h \in H(R)$ we want to show that it commutes with u and u' and also that $zhz^{-1} \in H(R)$.

Consider the automorphism f of H_R induced by conjugation by $u'zu \in N_G(H)(R)$. For all $h \in H(R)$,

$$u' \cdot zh \cdot h^{-1}uh = u'zuh = f(h)u'zu = (f(h)u'f(h)^{-1}) \cdot f(h)z \cdot u.$$

But $H \subset Z$ and so it normalizes U and U'. Hence, the outer terms in this equality visibly correspond to decompositions in the subset $U'(R) \times Z(R) \times U(R) \subset G(R)$ (inclusion via multiplication), so corresponding terms coincide. This says exactly that

$$u' = f(h)u'f(h)^{-1}, zh = f(h)z, h^{-1}uh = u.$$

As we vary h through $H(R)$, $f(h)$ likewise sweeps out $H(R)$ (as f is an automorphism of H_R), so the first and third equalities give that u and u' are centralized by $H(R)$, whereas the second says $zhz^{-1} = f(h)$, so z normalizes $H(R)$ inside $G(R)$. ∎

4. Split Weyl group

Now we study the relationship between $N(k)/Z(k)$ and N/Z. As a first step, we show:

Lemma 4.1. *The finite étale k-group N/Z is constant. Equivalently, the natural* $\mathrm{Gal}(k_s/k)$*-action on $(N/Z)(k_s) = N(k_s)/Z(k_s)$ is trivial.*

Note that the equality $(N/Z)(k_s) = N(k_s)/Z(k_s)$ rests on the k-smoothness of Z.

Proof. The slick proof is to observe that the N-action on S identifies N/Z with a k-subgroup of the automorphism scheme $\mathrm{Aut}_{S/k}$ that is constant since S is split. We now give a more down-to-earth version of the same idea, avoiding automorphism schemes of tori.

Choose $n \in N(k_s)$ and $\gamma \in \mathrm{Gal}(k_s/k)$. We want to show that $\gamma(n)^{-1}n \in Z(k_s)$, as that says $\gamma(n)$ and n have the same image in $N(k_s)/Z(k_s) = (N/Z)(k_s)$, so the triviality of the Galois action would be proved.

Consider the k_s-automorphism of S_{k_s} defined by conjugation against the element $n \in N(k_s)$: this is $x \mapsto nxn^{-1}$. For any two split k-tori T and T', all k_s-homomorphisms $T_{k_s} \to T'_{k_s}$ are defined over k. Thus, n-conjugation on S_{k_s} is defined over k, which is to say that this k_s-automorphism is equivariant with respect to the application of any γ. That is, for $x \in S(k_s)$ we have $\gamma(nxn^{-1}) = n\gamma(x)n^{-1}$, but $\gamma(nxn^{-1}) = \gamma(n)\gamma(x)\gamma(n)^{-1}$, so $\gamma(n)^{-1}n$ centralizes all such x and hence $\gamma(n)^{-1}n \in Z(k_s)$ as desired. ■

Since the cosets of Z_{k_s} inside N_{k_s} are the connected components of N_{k_s}, the triviality of the Galois action in the preceding lemma says exactly that each of these components is defined over k inside N. In other words, the connected components of N are *geometrically connected* over k. Rather more subtle is that each of these components actually contains a k-point. That property is equivalent to the assertion that the inclusion $N(k)/Z(k) \hookrightarrow (N/Z)(k)$ is an equality, and it is most remarkable since $\mathrm{H}^1(k, Z)$ is utterly mysterious (see §5ff.). Let us now prove it:

Proposition 4.2. *The natural map $N(k)/Z(k) \to (N/Z)(k)$ is surjective.*

Proof. Let W denote the constant finite k-scheme N/Z. The idea for proving that the subgroup $N(k)/Z(k) \subseteq W(k)$ is full is to show that $W(k)$ acts freely on a set whose resulting $N(k)/Z(k)$-action is transitive. Motivated by the bijective correspondence between the set of Borel subgroups containing a given maximal torus and the set of Weyl chambers in the associated root system (or equivalently the set of positive systems of roots) in the split case, together with the simply transitive action of the combinatorial Weyl group on the set of chambers (and the equality of this Weyl group with the "Weyl group" defined by the reductive group and its chosen maximal torus), we are led to consider the set $\mathscr{P}$ of minimal parabolic k-subgroups P of G that contain the maximal k-split torus S.

There is an evident action of $N(k)$ on $\mathscr{P}$, and we claim that it is transitive. For any $P, P' \in \mathscr{P}$ we know there exists $g \in G(k)$ such that $P' = gPg^{-1}$, so S and gSg^{-1} are maximal k-split tori in P'. But we have shown that in any parabolic k-subgroup of a connected reductive k-group, all maximal k-split tori are k-rationally conjugate. Thus, there exists $p' \in P'(k)$ such that $p'gSg^{-1}p'^{-1} = S$. Hence, $p'g \in N(k)$ and this element conjugates P to P'. This proves the transitivity of the $N(k)$-action on $\mathscr{P}$, and it factors through

$N(k)/Z(k)$ since we know that any parabolic k-subgroup P of G containing S necessarily contains $Z_G(S) = Z$.

Now it remains to define a free action of $W(k)$ on $\mathscr{P}$ that restricts to the above action of $N(k)/Z(k)$ on $\mathscr{P}$. The preceding lemma implies that $W(k) = W(k_s)$, and we know that $W(k_s) = N(k_s)/Z(k_s)$. For any $w \in W(k)$, choose a representative $n \in N(k_s)$ and consider $nP_{k_s}n^{-1}$ for $P \in \mathscr{P}$. This is a parabolic k_s-subgroup of G_{k_s} with the same dimension as P, so if it is defined over k then its k-descent must be a minimal parabolic k-subgroup of G (as we know that the minimal parabolic k-subgroups of G all have the same dimension, due to their $G(k)$-conjugacy). For any $\gamma \in \mathrm{Gal}(k_s/k)$ we have $\gamma(n) = nz$ for some $z \in Z(k_s) \subset P(k_s)$, so

$$\gamma(nP_{k_s}n^{-1}) = \gamma(n)P_{k_s}\gamma(n)^{-1} = nP_{k_s}n^{-1}.$$

Thus, $nP_{k_s}n^{-1}$ descends to a minimal parabolic k-subgroup of G, and we may denote it as $w.P$ since clearly $nP_{k_s}n^{-1}$ depends on n only through its $Z(k_s)$-coset (which in turn depends only on w). Clearly $P \mapsto w.P$ is an action of $W(k)$ on $\mathscr{P}$, and its restriction to an action of $N(k)/Z(k)$ is obviously the action considered above.

To show that this $W(k)$-action on $\mathscr{P}$ is free, we assume $n \in N(k_s)$ satisfies $nP_{k_s}n^{-1} = P_{k_s}$ for some $P \in \mathscr{P}$ then want to deduce that $n \in Z(k_s)$. But the $G(k_s)$-normalizer of P_{k_s} coincides with $P(k_s)$ (Chevalley's theorem on the self-normalizing property of parabolic subgroups in general smooth connected affine groups), so $n \in N(k_s) \cap P(k_s) = Z(k_s)$. ■

When G is quasi-split (i.e., the minimal parabolic k-subgroups are Borel subgroups), there is a refinement of Proposition 4.2 as follows. Let S be a maximal k-split torus in G and B a minimal parabolic k-subgroup containing S, so $Z_G(S)$ is a Levi k-subgroup of B. But a Levi k-subgroup of a Borel k-subgroup is a maximal k-torus (as may be checked over $\overline{k}$), so $T := Z_G(S)$ is a maximal k-torus in G. Any $g \in N_G(S)(k)$ certainly normalizes $Z_G(S) = T$, so we have $N_G(S)(k) \subset N_G(T)(k)$, and this is an equality since any $g \in G(k)$ that normalizes T must also normalize the unique maximal split k-subtorus S in T. By definition we have $Z_G(S) = T$, so by Proposition 4.2 we have the inclusion of groups

$$\begin{aligned}
W(\Phi(G,S)) = W(G,S)(k) = N_G(S)(k)/Z_G(S)(k) &= N_G(T)(k)/T(k) \\
&\subset (N_G(T)/T)(k) \\
&= W(G,T)(k) \\
&= W(G,T)(k_s)^{\mathrm{Gal}(k_s/k)}.
\end{aligned}$$

The refinement of Proposition 4.2 is this:

Proposition 4.3. *Let G be a quasi-split connected reductive group over a field k. For a maximal split k-torus S and the associated maximal k-torus $T = Z_G(S)$, let ${}_kW = N_G(S)(k)/Z_G(S)(k) = W(\Phi(G,S))$ be the relative Weyl group and $W = N_G(T)/T$ the finite étale "absolute" Weyl group. The natural inclusion ${}_kW \hookrightarrow W(k)$ as defined above is an equality.*

Proof. Such equality is clear whenever $\mathrm{H}^1(k,T) = 1$, and that always holds when G is semisimple and either simply connected or of adjoint type. Indeed, in such cases we claim that $T = Z_G(S)$ is an "induced" torus; i.e., a direct product $\prod \mathrm{R}_{k_i/k}(\mathrm{GL}_1)$ of Weil restrictions of GL_1 from finite separable extensions k_i of k. Once such a description is in hand, we have

$\mathrm{H}^1(k,T) = \prod \mathrm{H}^1(k_i, \mathrm{GL}_1)$ by Shapiro's Lemma (functoriality of group cohomology with respect to induction), and this vanishes by Hilbert 90.

To obtain the "induced" description, pick a Borel k-subgroup $B \supset S$, so T is contained in B and $\Phi(B_{k_s}, T_{k_s})$ is a *Galois-stable* positive system of roots in $\Phi(G_{k_s}, T_{k_s})$. The basis Δ of simple positive roots is therefore also Galois-stable, and in the adjoint type case Δ is a $\mathbf{Z}$-basis of the character lattice $\mathrm{X}(T_{k_s})$ while in the simply connected case $\Delta^\vee$ is a Galois-stable basis of the dual lattice $\mathrm{X}_*(T_{k_s})$. Thus, in both cases the character lattice is a permutation representation of $\mathrm{Gal}(k_s/k)$ (i.e., there is a $\mathbf{Z}$-basis on which $\mathrm{Gal}(k_s/k)$ acts through permutations), so T is induced (the factor fields k_i correspond to the open stabilizers in $\mathrm{Gal}(k_s/k)$ for an element in each orbit in the $\mathbf{Z}$-basis).

We shall reduce the general quasi-split case to the semisimple case of adjoint type via the insensitivity of relative and absolute root systems to the formation of *central* quotients. The central quotient $\overline{G} = G/Z_G$ is semisimple of adjoint type. The respective images $\overline{S}$ and $\overline{T}$ of S and T in $\overline{G}$ are a maximal split k-torus and maximal k-torus containing it (the maximality of $\overline{S}$ in $\overline{G}$ rests crucially on $\overline{G}$ being a central quotient of G), with $\overline{T} = Z_{\overline{G}}(\overline{S})$ because torus centralizers behave well under images. The formation of the relative and absolute root systems is insensitive to passing to a central quotient, so the natural maps

$$N_G(T)(k)/T(k) \to N_{\overline{G}}(\overline{T})(k)/\overline{T}(k), \;\; W(G,T)(k_s) \to W(\overline{G},\overline{T})(k_s)$$

are isomorphisms because these maps are respectively identified with the natural equalities

$$W(\Phi(G,S)) = W(\Phi(\overline{G},\overline{S})), \;\; W(\Phi(G_{k_s},T_{k_s})) = W(\Phi(\overline{G}_{k_s},\overline{T}_{k_s}))$$

(using that the $\mathbf{Q}$-vector spaces spanned by the relative and absolute roots depend only on the derived group, with $\mathscr{D}(G) \to \mathscr{D}(\overline{G})$ a central *isogeny*).

Putting this all together, we have a commutative diagram

$$\begin{tikzcd} N_G(T)(k)/T(k) \arrow[r, "?"] \arrow[d, "\simeq"'] & W(G,T)(k) \arrow[d, "\simeq"] \\ N_{\overline{G}}(\overline{T})(k)/\overline{T}(k) \arrow[r] & W(\overline{G},\overline{T}) \end{tikzcd}$$

in which the vertical maps are isomorphisms and the top map is what we want to be an isomorphism. This commutativity reduces our task to the isomorphism property for the bottom map, which is an isomorphism by the settled case of quasi-split semisimple groups of adjoint type. ■

5. An interesting Galois cohomology example

The remainder of this handout is devoted to giving two classes of quasi-split examples for which $\mathrm{H}^1(k,Z) \neq 0$. The point of the quasi-split condition is to ensure that the Levi factor $Z_G(S)$ of a minimal parabolic k-subgroup is commutative (equivalently, a torus), so our discussion does not involve non-abelian Galois cohomology.

Remark 5.1. One place where *not* to look are the simply connected and adjoint type cases, because in such cases the maximal k-torus $T = Z_G(S)$ is "induced" and hence has vanishing degree-1 cohomology, as we saw in the proof of Proposition 4.3.

We shall work with $G = \mathrm{R}_{k'/k}(\mathrm{SL}_n)/\mu_n$ for a finite separable extension k'/k and an integer $n > 1$, subject to some conditions with Brauer groups that we will arrive at near the end of the calculation. In accordance with the lesson from Remark 5.1, note that if $k' \neq k$ (as we will want) then this is neither simply connected nor adjoint type since G_{k_s} is the quotient of a direct product of $[k' : k]$ copies of SL_n modulo the diagonally embedded central μ_n.

To get started, we need to identify a maximal *split* torus in G. It is easy to make a guess: if D is the diagonal split k-torus in the k-group SL_n then $D \hookrightarrow \mathrm{R}_{k'/k}(D_{k'})$ should be maximal as a split k-torus of $\mathrm{R}_{k'/k}(\mathrm{SL}_n)$ and D/μ_n should be a maximal split k-torus in G. We wish to justify this by addressing more generally how maximal split tori interact with isogenies (central or not!) and with Weil restriction.

Recall that for any central quotient map $f : G' \to G$ between connected reductive groups there is a bijection between sets of maximal split k-tori in G and G' via $S' \mapsto S := f(S')$ and $S \mapsto f^{-1}(S)^0_{\mathrm{red}}$. This can fail when we drop the centrality condition on the kernel of the isogeny. (For example, let k be a local function field of characteristic p, and Δ is a central division algebra of rank p^2 over k. Consider $G' = \mathrm{SL}_1(\Delta)$, which is k-anisotropic. The Frobenius base change $G := G'^{(p)}$ along the degree-p Frobenius endomorphism $k \to k$ is SL_p since a degree-p extension field splits a Brauer class of degree p. Hence, the Frobenius isogeny $G' \to G$ from an anisotropic k-group to a split connected semisimple k-group is a counterexample.)

Hence, to identify a maximal k-split torus in $G = \mathrm{R}_{k'/k}(\mathrm{SL}_n)/\mu_n$ it suffices to identify one in the central isogenous cover $\mathrm{R}_{k'/k}(\mathrm{SL}_n)$. Of course, if D is the diagonal split k-torus in the k-group SL_n then it is natural to guess that $D \hookrightarrow \mathrm{R}_{k'/k}(D_{k'})$ is maximal as a split k-torus of $\mathrm{R}_{k'/k}(\mathrm{SL}_n)$. Since it will be useful quite generally, let's briefly digress to discuss how Weil restriction interacts with maximal tori, maximal split tori, and parabolic subgroups in a wider framework. Then we will focus on our specific example of interest.

6. Weil restriction, maximal tori, and parabolic subgroups

Let k be a field, k' a finite étale k-algebra, and X' an affine finite type k'-scheme. (We consider the generality in which k' may not be a field because for field extensions L/k the finite étale L-algebra $k' \otimes_k L$ is usually not a field even when k' is a field, and we will be especially interested in $L = k_s$.) Concretely, if $\prod k'_i$ is the decomposition of k' into factor fields k'_i then $X' = \coprod X'_i$ over $\mathrm{Spec}(k') = \coprod \mathrm{Spec}(k'_i)$ for an affine finite type k'_i-scheme X'_i for each i. Then $X := \mathrm{R}_{k'/k}(X') = \prod \mathrm{R}_{k'_i/k}(X'_i)$, as may be readily checked via functorial considerations: if A is a k-algebra then

$$X(A) = X'(k' \otimes_k A) = \prod_i X'_i(\prod k'_i \otimes_k A) = \prod_i \mathrm{R}_{k'_i/k}(X'_i)(A).$$

In case X' is a k'-group scheme, each X'_i is a k'_i-group scheme, and likewise for smoothness.

Clearly X is naturally a k-group scheme when X' is, and if $X' \to \mathrm{Spec}(k')$ has geometrically connected (resp. smooth) fibers then X is geometrically connected (resp. smooth) over k. Indeed, since we set things up in the generality with k' a finite étale k-algebra that isn't necessarily a field, it is harmless to first apply a scalar extension to k_s and thereby replace k with k_s and replace k' with $k' \otimes_k k_s$ (a finite étale k_s-algebra that is typically not a field even if k' is a field!) so that $k = k_s$. Hence, $k' = k^n$ is a product of copies of k, and $X' = \coprod X'_i$ is a disjoint union of n affine k-schemes of finite type. Thus, $X = \mathrm{R}_{k^n/k}(\coprod X'_i) = \prod X'_i$ over k. By inspection, the right side is geometrically connected (resp. smooth) over k when each X'_i is so.

Let $G' = \coprod G'_i$ be a smooth affine k'-group with connected fibers. Then $G := \mathrm{R}_{k'/k}(G')$ is a smooth connected affine k-group. If G' is reductive (resp. a torus) then the same holds for G, as we see by inspecting G_{k_s} as a product in accordance with the above calculation. The same holds for solvability, and if H' is a smooth closed k'-subgroup of G' that is fiberwise parabolic then $\mathrm{R}_{k'/k}(H')$ is a parabolic k-subgroup of G since it suffices to check this over k_s (where the product decompositions make it evident). Thus, the same holds for the Borel property. This brings us to:

Lemma 6.1. *Let G' be a smooth affine k'-group with connected fibers. The maps $T' \mapsto \mathrm{R}_{k'/k}(T')$ and $P' \mapsto \mathrm{R}_{k'/k}(P')$ are bijections between the sets of fiberwise maximal k'-tori and fiberwise parabolic k'-subgroups of G' and the sets of maximal k-tori and parabolic k-subgroups of G. The same holds for "Borel" in place of "parabolic". Moreover, $P' \subseteq Q'$ inside G' if and only if $\mathrm{R}_{k'/k}(P') \subseteq \mathrm{R}_{k'/k}(Q')$ inside G, and if H' is a smooth closed k'-subgroup of G' with $H := \mathrm{R}_{k'/k}(H') \subset G$ then $\mathrm{R}_{k'/k}(Z_{G'}(H')) = Z_G(H)$.*

Proof. In view of the generality of the bijectivity assertions, *Galois descent* reduces the verifications to the case over k_s (replacing k' with $k' \otimes_k k_s$). Make sure you see why this is correct!

As a consequence, we may now assume $k = k_s$, so k' is a product of copies of k and the Weil restrictions are compatible direct products. Thus, the centralizer assertion at the end is obvious and the other assertions come down to the fact that in a direct product of smooth affine groups, maximal tori and parabolic subgroups are built via direct products along the factors. To verify this it suffices to check over $\overline{k}$ (why?), so then geometric conjugacy considerations reduce the torus case to the obvious fact (check!) that a direct product of maximal tori is a maximal torus.

The same argument applies to Borel subgroups (via the characterization of Borel subgroups as solvable with complete coset space), and for the case of parabolicity (over $\overline{k}$) we just have to check that if $\{G_i\}$ are smooth connected affine groups over $k = \overline{k}$ and P is a parabolic subgroup of $G = \prod G_i$ then $P = \prod P_i$ for parabolic subgroups P_i of G_i. We can certainly pass to the quotient by $\mathscr{R}_u(G) = \prod \mathscr{R}_u(G_i)$ so that each G_i is reductive. Then we have the dynamical description $P = P_G(\lambda)$ for some cocharacter $\lambda : \mathrm{GL}_1 \to G = \prod G_i$. Letting λ_i be the ith component of λ, clearly $P_G(\lambda) = \prod P_{G_i}(\lambda_i)$. ■

Somewhat more delicate is the interaction of maximal split tori with Weil restriction, as the trick of scalar extension to k_s cannot be applied without ruining our situation.

Proposition 6.2. *Let G' be a smooth affine k'-group with connected reductive fibers G'_i, where $k' = \prod k'_i$ for fields k'_i finite separable over k. Let $S' = \coprod S'_i$ be a fiberwise maximal split k'-torus in G'. Then the maximal split k-subtorus S inside $\mathrm{R}_{k'/k}(S')$ is a maximal split k-torus in $G = \mathrm{R}_{k'/k}(G')$, and conversely if $\mathcal{S}$ is a maximal split k-torus in G then the image of $\mathcal{S}_{k'} \hookrightarrow G_{k'} \twoheadrightarrow G'$ is a fiberwise maximal split k'-torus in G'. These operations are inverse bijections between the sets of maximal split tori in G and G'.*

In this result, the map $\mathrm{R}_{k'/k}(X')_{k'} \to X'$ over k' (applied for $X' = G'$) "corresponds" to the identity map on $\mathrm{R}_{k'/k}(X')$; on points valued in a k'-algebra A' it is the map $X'(k' \otimes_k A') \to X'$ induced by the k'-*algebra* map $k' \otimes_k A' \to A'$ carrying $c' \otimes a'$ to $c'a'$.

Proof. Since we have a good rational conjugacy theorem over fields for maximal split tori, we first note that it does suffice to treat the factor fields of k' over k separately (check!), so we may and do assume k' is a field. If we begin with S' and make S in accordance with the given procedure (which we do not yet know to be maximal as a split k-torus in G) then we claim that the image of $S_{k'} \hookrightarrow G_{k'} \twoheadrightarrow G'$ is S'. This latter composite map is the same as $S_{k'} \to S' \hookrightarrow G'$ (check!), so for the purpose of identifying the image we can replace G' with S' and then even reduce to the case $G' = \mathrm{GL}_1$. In this case, S is the evident copy of GL_1 inside $\mathrm{R}_{k'/k}(\mathrm{GL}_1)$ (informally corresponding to $k^\times \subset k'^\times$). Indeed, this can be seen via rationalized Galois lattice considerations since the induction of the 1-dimensional trivial character through an inclusion of finite groups contains a single copy of the trivial representation (by Frobenius reciprocity over $\mathbf{Q}$). Thus, the assertion comes down to the claim that the natural map

$$T_{k'} \hookrightarrow \mathrm{R}_{k'/k}(T_{k'})_{k'} \twoheadrightarrow T_{k'}$$

is surjective for $T = \mathrm{GL}_1$, and computing on functorial points shows that this is the identity map (with T allowed to be any affine k-scheme of finite type whatsoever).

Having shown that S' is the image of $S_{k'}$ in the quotient G' of $G_{k'}$, consider a split k-torus $\mathcal{S}$ containing S. Thus, $\mathcal{S}_{k'}$ contains $S_{k'}$, so the image of $\mathcal{S}_{k'}$ contains S'. But $\mathcal{S}_{k'}$ is a fiberwise split k'-torus, so by the maximality of S' it follows that $\mathcal{S}_{k'}$ has image S' inside G' too. This quotient map $\mathcal{S}_{k'} \twoheadrightarrow S'$ corresponds by adjointness of Weil restriction and base change to a k-homomorphism $\mathcal{S} \to \mathrm{R}_{k'/k}(S')$. By construction, this is compatible with how each sit inside $G = \mathrm{R}_{k'/k}(G')$ (check!), so we see that $\mathcal{S}$ is a split k-subtorus of $\mathrm{R}_{k'/k}(S')$. By S was chosen to be the maximal split k-subtorus of $\mathrm{R}_{k'/k}(S')$, so the inclusion $S \subseteq \mathcal{S}$ is an equality. Hence, S is maximal inside G. This proves that the proposed recipe $S' \mapsto S$ *makes sense* as a map from the set of fiberwise maximal split k'-tori in G' to the set of maximal split k-tori in G. But all maximal split k-tori in G are $G(k)$-conjugate to each other, and via the visible equality $G(k) = G'(k')$ it follows that every maximal split k-torus in G arises from some such S'. Moreover, since we have shown that S' is the image of $S_{k'}$ under $G_{k'} \twoheadrightarrow G'$, such an S' is unique. ■

7. The calculation

Now we focus on the example $G = \mathrm{R}_{k'/k}(\mathrm{SL}_n)/\mu_n$. Let D be the diagonal split maximal k-torus in SL_n as a k-group, so the preceding section ensures that $T := \mathrm{R}_{k'/k}(D_{k'})/\mu_n$ is a maximal k-torus in G with $S := D/\mu_n$ as a maximal split k-torus in G. We claim that the inclusion $T \subset Z_G(S)$ is an equality. This can be done by bare hands, or alternatively we note that since G is quasi-split (a Borel k-subgroup is $\mathrm{R}_{k'/k}(B')/\mu_n$ for a Borel k'-subgroup of SL_n) we know that $Z_G(S)$ has to be a k-torus (so maximality of T forces $T = Z_G(S)$).

Our aim is to compute $\mathrm{H}^1(k, Z_G(S))$. The reader who is unfamiliar with fppf cohomology should now assume $\mathrm{char}(k)$ doesn't divide n, so μ_n is smooth over k. Hence, we have an exact sequence

$$1 \to \mu_n \to \mathrm{R}_{k'/k}(D_{k'}) \to Z_G(S) \to 1$$

of commutative affine k-groups of finite type. We have $\mathrm{H}^1(k, \mathrm{R}_{k'/k}(D_{k'})) = \mathrm{H}^1(k', D_{k'})$ by Shapiro's Lemma, and this vanishes by Hilbert 90 since $D_{k'}$ is a split k'-torus. Hence,

$$\mathrm{H}^1(k, Z_G(S)) = \ker(\mathrm{H}^2(k, \mu_n) \to \mathrm{H}^2(k, \mathrm{R}_{k'/k}(D_{k'}))).$$

Again using Shapiro's Lemma, the final H^2 is identified with

$$\mathrm{H}^2(k', D_{k'}) = \{(c_i) \in \mathrm{Br}(k')^n \mid \sum c_j = 0\},$$

and by Hilbert 90 we know $\mathrm{H}^2(k, \mu_n) = \mathrm{Br}(k)[n]$. In this way, we have

$$\mathrm{H}^1(k, Z_G(S)) = \ker(\mathrm{Br}(k)[n] \to \mathrm{Br}(k')^n)$$

using the diagonal mapping of $\mathrm{Br}(k)$ into $\mathrm{Br}(k')^n$. (Make sure you see why this identification of the kernel is correct.) Thus, $\mathrm{H}^1(k, Z_G(S))$ is identified with the kernel of the inclusion of $\mathrm{Br}(k)[n]$ into $\mathrm{Br}(k')$, which is the n-torsion in the kernel of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(k')$.

For example, if k is a non-archimedean local field and k'/k is a finite extension whose degree is divisible by n then this kernel is nontrivial, and in the global field setting we similarly get nontrivial kernel in many cases.