Math 249B. Examples of relative roots

1. Introduction

In §6 of the handout on the relative Bruhat decomposition, it is shown that if k'/k is a finite separable extension of fields and G' is a connected reductive k'-group then

$$G := \mathrm{R}_{k'/k}(G')$$

is a connected reductive k-group and the functor $\mathrm{R}_{k'/k}$ defines bijections from the set of maximal k'-tori (resp. parabolic, resp. Borel k'-subgroups) of G' to the set of maximal k-tori (resp. parabolic, resp. Borel k-subgroups) of G, with the bijection for parabolic subgroups inclusion-preserving in both directions. In particular, if G' is split then G is quasi-split and the Borel k-subgroups of G have the unique form $\mathrm{R}_{k'/k}(B')$ for Borel k'-subgroups $B' \subset G'$.

The effect of $\mathrm{R}_{k'/k}$ on tori corresponds to induction for Galois lattices, so $\mathrm{R}_{k'/k}(\mathbf{G}_m)$ has as its maximal split k-subtorus the evident copy of $\mathbf{G}_m$. Hence, if S' is a split k'-torus of dimension d then the k-torus $\mathrm{R}_{k'/k}(S')$ of dimension $[k' : k]d$ has maximal split k-subtorus S of dimension d. In §6 of the handout on the relative Bruhat decomposition we also showed that assigning to each maximal split k'-torus $S' \subset G'$ the maximal split k-subtorus $S \subset \mathrm{R}_{k'/k}(S')$ is a bijection between the sets of maximal split tori in G and G'. For example, if G' is split with maximal k'-tori of dimension d then G has maximal k-tori of dimension $[k' : k]d$ whereas its maximal split k-tori have dimension d.

One reason for interest in separable Weil restriction is that the simply connected central cover of any connected semisimple k-group has the form $\prod_i \mathrm{R}_{k_i/k}(G_i)$ for a canonically associated finite étale k-algebra $k' = \prod k_i$ (for fields k_i) and connected semisimple k_i-groups G_i that are absolutely simple and simply connected for each i. Hence, the core cases to understand for the structure of connected semisimple groups are the effect of finite separable Weil restriction and the possibilities in the absolutely simple case.

In this handout, we first explore the effect of finite separable Weil restriction on root systems and root spaces, and then turn our attention to two classes of *absolutely simple* connected semisimple k-groups that are Galois-twisted forms of type A: the units with reduced-norm 1 in central simple k-algebras, and special unitary groups for certain non-degenerate hermitian spaces relative to quadratic Galois extensions of k.

2. Weil restriction

Let S' be a split k'-torus, and S the maximal split k-subtorus of the k-torus $\mathrm{R}_{k'/k}(S')$ (so if S' is a maximal split k'-subtorus of G' for $(k'/k, G', G)$ as in §1 then S is a maximal split k-torus in G, and we want to relate the sets $\Phi(G, S) \subset \mathrm{X}(S)$ and $\Phi(G', S') \subset \mathrm{X}(S')$).

Consider the natural map

$$\theta : \mathrm{X}(S') \to \mathrm{X}(S)$$

defined by $a' \mapsto a := \mathrm{R}_{k'/k}(a')|_S$; this makes sense because $\mathrm{R}_{k'/k}(a') : \mathrm{R}_{k'/k}(S') \to \mathrm{R}_{k'/k}(\mathbf{G}_m)$ must carry S into the maximal split subtorus $\mathbf{G}_m \subset \mathrm{R}_{k'/k}(\mathbf{G}_m)$. The map θ is bijective because compatibility with direct sums in S' reduces this to $S' = \mathbf{G}_m$ and a' the identity k'-endomorphism of $\mathbf{G}_m$, in which case a is clearly the identity k-endomorphism of $\mathbf{G}_m$.

Proposition 2.1. *Assume S' is a maximal split k'-torus in G' for $(k'/k, G', G)$ as in §1. The bijection θ carries $\Phi(G', S')$ onto $\Phi(G, S)$, and for each $a' \in \Phi(G', S')$ and the corresponding $a \in \Phi(G, S)$ the identification of k-vector spaces*

$$\mathfrak{g} = \ker(G(k[\varepsilon]) \to G(k)) = \ker(G'(k'[\varepsilon]) \to G'(k')) = \mathfrak{g}'$$

carries $\mathfrak{g}_a$ onto $\mathfrak{g}'_{a'}$. In particular, all root spaces in $\mathfrak{g}$ have dimension $[k' : k]$.

The bijection $\mathfrak{g} \simeq \mathfrak{g}'$ of k-vector spaces is also compatible with Lie brackets; for a proof, see A.7.5–A.7.6 in [CGP].

Proof. It is harmless to make a ground field extension K/k such that $K \otimes_k k'$ is a field. Taking $K = k(x)$ if k is finite, we may assume k is infinite. Thus, $S(k)$ is Zariski-dense in S, and likewise for $S'(k') \subset S'$. Consequently, we can keep track of weight spaces for S and S' by studying the actions of $S(k)$ and $S'(k')$. In particular, $\mathfrak{g}_a$ is the set of $X \in \mathfrak{g}$ such that $\mathrm{Ad}_G(s)(X) = a(s)X$ for all $s \in S(k)$. Likewise, $\mathfrak{g}'_{a'}$ is the set of $X' \in \mathfrak{g}'$ such that $\mathrm{Ad}_{G'}(s')(X') = a'(s')X'$ for all $s' \in S'(k')$.

By definition of the adjoint representation in terms of conjugation of dual-number points and definition of the identification of $\mathfrak{g}$ and $\mathfrak{g}'$, via the inclusion $S(k) \hookrightarrow \mathrm{R}_{k'/k}(S')(k) = S'(k')$ we have $\mathrm{Ad}_{G'}(s) = \mathrm{Ad}_G(s)$. Hence, $\mathfrak{g}_a$ is the set of $X' \in \mathfrak{g}'$ such that $\mathrm{Ad}_{G'}(s)(X') = a(s)X'$.

Writing X' uniquely as a sum of its components along the S'-weight spaces in $\mathfrak{g}'$, which is also an $S(k)$-equivariant decomposition (via the inclusion of $S(k)$ into $S'(k')$), it suffices to check that $a'|_{S(k)} = a$ (valued in $k^\times \subset \mathrm{R}_{k'/k}(\mathbf{G}_m)(k) = k'^\times$). But the very definition of a in terms of a' makes this equality obvious (since applying the functor of k-points after applying $\mathrm{R}_{k'/k}$ is naturally identified with applying the functor of k'-points). ■

3. Central simple algebras

Let A be a central simple algebra over k of dimension N^2, so $A \simeq \mathrm{Mat}_d(D)$ for some $d|N$ and a central division algebra D over k of dimension m^2 for $m = N/d$. Let G be the algebraic group of units in A with reduced norm equal to 1; i.e., for any k-algebra R the group $G(R)$ consists of the units $u \in (R \otimes_k A)^\times$ such that $\mathrm{Nrd}(u) = 1$ in $R^\times$, where $\mathrm{Nrd} : \underline{A} \to \mathbf{A}^1_k$ is the multiplication morphism given by the reduced norm. (See Exercise 5 in HW4 of the previous course.)

One usually writes $\mathrm{SL}_d(D)$ to denote G. This is a Galois-twisted form of $\mathrm{SL}_{dm} = \mathrm{SL}_N$, and its Lie algebra is naturally identified with the kernel of the reduced trace in A (using commutator in the associative algebra A for the Lie bracket); see Exercise 1(ii) in HW7 of the previous course.

The central subring inclusion $k \hookrightarrow D$ defines a k-algebra inclusion $\mathrm{Mat}_d(k) \hookrightarrow \mathrm{Mat}_d(D)$ along which the restriction of Nrd is $\det^m$. In particular, this yields a k-subgroup inclusion $\iota : \mathrm{SL}_d \hookrightarrow G$.

Proposition 3.1. *Let $S \subset \mathrm{SL}_d$ be the split diagonal k-torus. Via ι this is a maximal split k-torus in G, and $\Phi(G, S) = \Phi(\mathrm{SL}_d, S)$. For $a \in \Phi(G, S) = \Phi(\mathrm{SL}_d, S)$ corresponding to the ij-entry for $i \neq j$, $\mathfrak{g}_a$ is identified with the ij-entry space D inside $\mathfrak{g} \subset A = \mathrm{Mat}_d(D)$.*

Moreover, $Z_G(S) \subset (\underline{D}^\times)^m$ is the group of points $(\xi_1, \ldots, \xi_m)$ such that $\prod \mathrm{Nrd}(\xi_j) = 1$.

Informally, the large dimensions for the root spaces and $Z_G(S)$ are related to the additive and unit groups of D, but it doesn't make sense to speak of Weil restriction to k of $\mathbf{G}_{\rm a}$ or $\mathbf{G}_m$ from $\mathrm{Spec}(D)$ since D is non-commutative (away from the uninteresting case $D = k$).

Proof. The effect of S-conjugation on the k-algebra $\mathrm{Mat}_d(D) = \mathrm{Mat}_d(k) \otimes_k D$ has no effect on the D-tensor factor, being concentrated on $\mathrm{Mat}_d(k)$ (where it is the standard action). The gives the expected weight-space decomposition for S acting through conjugation on $\underline{A}^\times$, so likewise on its derived group that is G (as we may check over k_s, where D splits).

Since $\underline{D}^\times/\mathbf{G}_m$ is k-anisotropic, due to the k-anisotropicity of the centrally isogenous $\mathrm{SL}_1(D)$ (as D is a central division algebra; see Exercise 1(i) in HW8 of the previous course), once we establish the asserted description of $Z_G(S)$ it will follow that S is maximal split inside $Z_G(S)$. Hence, it remains to prove that $Z_G(S)$ as as described. But the proposed description is a smooth connected group (as we may check over k_s, where D splits) that is certainly contained inside the smooth connected k-group $Z_G(S)$, so to prove equality it suffices to compare Lie algebras.

Dropping the "SL"-condition at the cost of an extra 1-dimensional central $\mathbf{G}_m$, it suffices to check that the S-centralizer in the Lie algebra A of $\underline{A}^\times$ coincides with the "diagonal" subalgebra D^m. But this is clear since we have identified weight spaces for S in A for the weights in $\Phi(\mathrm{SL}_d, S)$, and those together with D^m already span the entire Lie algebra (so there is no room for a larger subspace of S-invariants). ■

4. Special unitary groups

Let k'/k be a quadratic Galois extension. Denote the nontrivial Galois automorphism as $z \mapsto \overline{z}$. We will build special unitary groups over k relative to k'/k, and see that typically these have relative root system that is non-reduced (and even occurs for $k = \mathbf{R}$, so this phenomenon impacts the structure theory of semisimple Lie groups and Lie algebras over $\mathbf{R}$).

Let $V' = k'^n$ with $n \geq 2$, and choose a positive integer q such that $n \geq 2q$. Let $h : V' \times V' \to k'$ be the sesquilinear map defined by

$$h(\vec{x}, \vec{y}) = \sum_{i=1}^{q} (x_i \overline{y}_{q+i} + x_{q+i} \overline{y}_i) + h_0((x_{2q+1}, \dots, x_n), (y_{2q+1}, \dots, y_n))$$

where

$$h_0 = \sum_{i=2q+1}^{n} c_i x_i \overline{y}_i$$

with $c_{2q+1}, \dots, c_n \in k^\times$. Assume the quadratic form $q_{h_0} = \sum_{i=2q+1}^{n} c_i x_i \overline{x}_i$ on k'^{n-2q} that is valued in k is k-anisotropic. Regarding (k'^{n-2q}, q_{h_0}) as a quadratic space over k, it has *even* dimension $2(n-2q)$ and is automatically non-degenerate.

Clearly

$$h(\vec{y}, \vec{x}) = \overline{h(\overline{x}, \overline{y})};$$

we say h is *hermitian*. In terms of the language of matrices for sesquilinear forms, h has the matrix

$$[h] = \begin{pmatrix} & 1_q & \\ 1_q & & \\ & & C \end{pmatrix}$$

where

$$C = \mathrm{diag}(c_{2q+1}, \dots, c_n).$$

As in Exercise 3 of HW7 of the previous course, the k-group $G = \mathrm{SU}(h) \subset \mathrm{R}_{k'/k}(\mathrm{SL}(V'))$ is a Galois-twisted form of SL_n, so the absolute root system for G is A_{n-1}.

Note that the subgroup of G "supported" in the lower-right $(n-2q) \times (n-2q)$ corner is a copy of the connected reductive k-group $\mathrm{SU}(h_0) \subset \mathrm{SO}(q_{h_0})$ (containment shown in Exercise 3(iii) of HW7 of the previous course). Thus, the assumed k-anisotropicity of the quadratic space q_{h_0} over k implies that the connected reductive k-group $\mathrm{SU}(h_0)$ is k-anisotropic (as we know $\mathrm{SO}(q_{h_0})$ must be k-anisotropic).

Consider the q-dimensional split k-torus $S \subset G$ consisting of points

$$\lambda(t) = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1_{n-2q} \end{pmatrix}$$

for $t = (t_1, \dots, t_q) \in \mathbf{G}_m^q$ viewed as a diagonal matrix in the evident manner. A basis of $\mathrm{X}(S)$ is given by the projections $a_i : \lambda(t) \mapsto t_i$. As a Lie subalgebra of $\mathrm{Lie}(\mathrm{R}_{k'/k}(\mathrm{SL}(V'))) = \mathfrak{sl}(V') = \mathfrak{sl}_n(k')$ over k, the Lie algebra $\mathfrak{g}$ of G consists of those $M' \in \mathfrak{sl}_n(k')$ satisfying $M'^{\mathrm{t}}[h] + [h]\overline{M'} = 0$. Using that $\overline{C} = \varepsilon C$, we find that such M' are the block matrices over k' given by

$$M' = \begin{pmatrix} Y & X & CU \\ X' & -\overline{Y}^{\mathrm{t}} & CV \\ -C^{-1}\overline{V}^{\mathrm{t}}C & -C^{-1}\overline{U}^{\mathrm{t}}C & W \end{pmatrix}$$

where $X, X', Y \in \mathrm{Mat}_q(k')$, $W \in \mathrm{Mat}_{n-2q}(k')$, and $U, V \in \mathrm{Mat}_{q \times (n-2q)}(k')$ satisfy

$$\overline{X}^{\mathrm{t}} = -X,\ \ \overline{X'}^{\mathrm{t}} = -X',\ \ \overline{W}^{\mathrm{t}} = -CWC^{-1},\ \ \mathrm{Tr}(W) = \overline{\mathrm{Tr}(Y)} - \mathrm{Tr}(Y).$$

(The fourth condition says exactly that $M' \in \mathfrak{sl}_n(k')$, and note that applying trace to the third condition gives $\mathrm{Tr}(W)$ is negative of its own conjugate, as forced by the fourth condition.) It follows that the strictly lower triangular parts of X and X' are determined by their respective strictly upper triangular parts, and their diagonals are valued in k. In particular, the diagonal entries for X and X' are 1-dimensional over k whereas the off-diagonal entries are 1-dimensional over k' (so 2-dimensional over k).

The structure of the S-weight spaces works out as follows (allowing $i = j$ below!):

- the ij-entry for Y has weight $a_i - a_j$,
- the ij-entry for X has weight $a_i + a_j$,
- the ij-entry for X' has weight $-a_i - a_j$,
- the ij-entry for $-\overline{Y}^{\mathrm{t}}$ has weight $-a_i + a_j$,
- the ith row of CU and ith column of $-C^{-1}\overline{U}^{\mathrm{t}}C$ has weight a_i,
- the ith row of CV and ith column of $-C^{-1}\overline{V}^{\mathrm{t}}C$ has weight $-a_i$

- all entries for W have S-weight 0.

Note that for the entries of X and X' we may restrict attention to the upper triangular part (including the diagonal). In particular, within the upper-left $2q \times 2q$ block all S-weight spaces for nontrivial weights are k'-lines (so 2-dimensional over k') *except* for the weights $\pm 2a_i$ that are 1-dimensional over k.

Let $T_0 = \ker(\mathrm{N}_{k'/k} : \mathrm{R}_{k'/k}(\mathbf{G}_m) \to \mathbf{G}_m)$ be the anisotropic norm-1 subtorus of $\mathrm{R}_{k'/k}(\mathbf{G}_m)$, so the formula for h gives that T_0^q embeds into $\mathrm{U}(h)$ within the diagonal of the upper-left $2q \times 2q$ part by a formula as for S except that we do not involve any inversion. To lie inside $G \subset \mathrm{R}_{k'/k}(\mathrm{SL}_n)$ we need to work with the subtorus

$$S_0 = \{(t_1, \dots, t_q) \in T_0^q \mid \prod t_j = 1\}.$$

This subtorus commutes with S and meets S in $S_0[2] \simeq \mu_2^{q-1}$. It follows from Lie algebra considerations with smooth connected k-groups that

$$Z_G(S) = (S_0 \cdot S) \times \mathrm{SU}(h_0),$$

so $Z_G(S)/S = (S_0/S_0[2]) \times \mathrm{SU}(h_0)$ is k-anisotropic, so S is a *maximal* split k-torus in G. Note that the S-weights $\pm 2a_i$ are divisible (by 2) in the set of nontrivial S-weights on $\mathfrak{g}$ precisely when $\pm a_i$ occurs as an S-weight, which is precisely when $n > 2q$. More specifically, the upper-left $2q \times 2q$ part of G contributes the set of S-weights constituting a C_q root system in the q-dimensional space $\mathrm{X}(S)_{\mathbf{Q}}$ (with long roots $\pm 2a_i$).

Since $Z_G(S)/S \neq 1$ in all cases away from $(n, q) = (2, 1)$, G is always non-split except when $(n, q) = (2, 1)$ (the quasi-split $\mathrm{SU}(2)$ is split). Moreover, G is quasi-split if and only if $Z_G(S)$ is a torus, or equivalently $Z_G(S)/S$ is a torus, which is to say that the connected semisimple group $\mathrm{SU}(h_0)$ is a torus; this happens precisely when $n = 2q, 2q + 1$.

Example 4.1. In the special case $n = 2q$, $Z_G(S)$ is a torus containing S with codimension $q - 1$. Hence, in such cases G is quasi-split but not split when $q > 1$, and the absolute root system $\mathrm{A}_{n-1} = \mathrm{A}_{2q-1}$ whereas $\Phi(G, S)$ is the root system C_q of rank q.

If instead $n = 2q + 1$ then in addition to the C_q from the upper-left $2q \times 2q$ part, we get additional (short, multipliable) roots $\pm a_i$ whose weight spaces are each a copy of k'.

Example 4.2. To get a grip on the non-reducedness of the root system, let's focus on BC_1-cases to see what is going on: we consider $n = 3$ and $q = 1$. This makes h a hermitian form in 3 variables,

$$h(\vec{x}, \vec{y}) = x_1\overline{y}_2 + x_2\overline{y}_1 + cx_3\overline{y}_3,$$

where $c \in k^\times$. Inside $\mathrm{R}_{k'/k}(\mathrm{SL}_3)$, the k-torus $S = \mathbf{G}_m$ consists of points $\mathrm{diag}(t, 1/t, 1)$, with $\mathrm{X}(S) = \mathbf{Z}a$ for $a : \mathrm{diag}(t, 1/t, 1) \mapsto t$.

The smooth connected subvariety U consisting of points

$$\begin{pmatrix} 1 & w & -c\overline{v} \\ & 1 & \\ & v & 1 \end{pmatrix}$$

with $v, w \in \mathrm{R}_{k'/k}(\mathbf{G}_{\mathrm{a}})$ satisfying

$$w + \overline{w} + cv\overline{v} = 0$$

is a 3-dimensional k-subgroup normalized by S (easy) on which the group law is given by

$$(v_1, w_1) \cdot (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + c\overline{v}_1 v_2),$$

so U is non-commutative. The Lie algebra $\mathrm{Lie}(U)$ is the span of the 1-dimensional weight space for $2a$ and the 2-dimensional weight space for a. In terms of the notion of "relative root group" to be defined later, U is the a-root group (its Lie algebra also supports the $2a$-weight space).

Setting v to be 0 gives a 1-dimensional k-subgroup $U_{2a} \simeq \mathbf{G}_{\mathrm{a}}$ whose Lie algebra is the $2a$-weight space, and $U/U_{2a} \simeq \mathrm{R}_{k'/k}(\mathbf{G}_{\mathrm{a}})$ is commutative for which the S-action on the Lie algebra has a as its only weight.

There is no S-equivariant homomorphic section to $U \to U/U_{2a}$, so this is a seriously non-commutative situation (i.e., far from a semi-direct product) For example, one might try imposing the condition $w = -cv\overline{v}/2$ (as then $w + \overline{w} + cv\overline{v} = 0$), but this does not give a *homomorphic* section.

In general, when $n > 2q$, we see that $\Phi(G, S) = \mathrm{BC}_q$ for which the weight spaces for the shortest roots $\pm a_i$ have dimension $2(n - 2q)$. For $n > 2q + 1$ these large dimensional spaces are *not* secretly lines over extension fields of k. That is, such large dimensions are not explained by the intervention of a Weil restriction from a finite separable extension field (in contrast with the case $n = 2q + 1$, or in general with the non-multipliable weights, for which the weight spaces are k'-lines arising from the intervention of $\mathrm{R}_{k'/k}$ via the inclusion of $\mathrm{SU}(h)$ inside $\mathrm{R}_{k'/k}(\mathrm{SL}(V')))$.

Note that by inspection, $(\mathrm{X}(S)_{\mathbf{Q}}, \Phi(G, S))$ is always a rank-q root system, whereas the absolute root system has rank $n - 1$ with $n \geq 2q$. Hence, both the absolute and relative root systems are irreducible (though the latter is typically non-reduced) and there is typically a huge gulf between their ranks. As $n - 2q$ grows, it is the weight spaces for multipliable roots that account for ever more of the dimension.