Math 249B. Root datum for split reductive groups

The link between (split) connected reductive groups and combinatorial objects called *root data* was first discovered in the theory of compact Lie groups and the structure theory of complex semisimple Lie algebras, where the slightly coarser notion of *root system* was used. Roughly speaking, root systems keep track of group-theoretic information "up to central isogeny" whereas the root datum keeps track of information up to isomorphism. (The root datum viewpoint is also necessary for keeping track of the maximal central torus. But this was not regarded as an important piece of information in the early days of Lie groups, since a central torus is not particularly interesting from a representation-theoretic perspective.)

We begin with a discussion of root groups and then establish the root datum axioms for roots arising from a split connected reductive group. Then we establish some interesting consequences of the link to root systems.

1. Root groups and root data

Throughout this section, G is a connected reductive group over a field k and T is a maximal k-torus that we assume to be k-split. We have seen in the homework for the previous course that in many natural examples, there is no such T (e.g., unit groups of nontrivial central division algebras over k). Those G admitting such a T are called *k-split.* Since every maximal k-torus remains maximal after a ground field extension, and every torus splits over a finite Galois extension, loosely speaking every connected reductive k-group is a "Galois-twisted form" of a split one. Hence, the classification of connected reductive groups comes in two parts: the combinatorial classification in terms of root data in the split case, which we will begin to discuss below, and a Galois cohomological part to keep track of how "twisted" a given group is from a split one (involving the structure of *automorphism groups* of split connected reductive groups, which is best understood with the aid of root data, along with Galois cohomological methods).

Remark 1.1. Everything we do below will rest on the choice of T. Now of course it is typically not true (when $k \neq k_s$) that *every* maximal k-torus in k-split; already for GL_n this fails when k has degree-n finite separable extension fields. But it is true that all k-split T are $G(k)$-conjugate. This is by no means obvious, and its proof rests on the structural understanding of the subgroup structure obtained via root data. Hence, one can keep in mind that at the end of the story all such choices of T will turn out to be "created equal", and so in the end we will get results that are intrinsic to G up to $G(k)$-conjugation (which is best possible, in some sense). For our purposes, the choice of T will simply be fixed throughout the discussion.

The following terminology will be convenient:

Definition 1.2. The *roots* of the pair (G,T) are the non-trivial weights for T under its adjoint action on $\mathfrak{g} = \mathrm{Lie}(G)$. In other words, it is the set $\Phi(G,T) \subset \mathrm{X}(T)$.

For each $a \in \Phi(G,T)$ we know that the corresponding weight space $\mathfrak{g}_a$ in $\mathfrak{g}$ is 1-dimensional, and so we have a weight space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\oplus_{a \in \Phi(G,T)} \mathfrak{g}_a\right)$$

with lines $\mathfrak{g}_a$, where $\mathfrak{t} = \mathrm{Lie}(T)$. In particular, $\Phi(G,T) = \emptyset$ if and only if $G = T$, which is to say that G is commutative (or equivalently, by reductivity, solvable). It is the non-solvable case which is the most important one, and we want to T-equivariantly "exponentiate" each $\mathfrak{g}_a$ to a copy of $\mathbf{G}_{\mathrm{a}}$ in G. Ultimately this rests on a concrete calculation with SL_2. First we prove the general result, and then we see what it says for SL_n.

Proposition 1.3. *For each root a of (G,T), there is a unique smooth connected k-subgroup $U_a \subseteq G$ normalized by T such that the subspace $\mathrm{Lie}(U_a)$ equipped with its T-action is $\mathfrak{g}_a$. Moreover, $U_a \simeq \mathbf{G}_{\mathrm{a}}$ as k-groups.*

The k-group U_a is called the *root group* in G attached to $a \in \Phi(G,T)$. Beware that it is crucial (in positive characteristic) to assume that U_a is T-normalized, not merely that its Lie algebra is T-stable under the adjoint action. Otherwise one can make counterexamples using the graph of Frobenius in $\mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{a}}$ (viewed inside the unipotent radical of a Borel subgroup of SL_3, for example).

Proof. Consider the unique codimension-1 torus $T_a = (\ker a)^0_{\mathrm{red}}$ in T killed by the nontrivial character a of T. The first task is to control all possibilities for U_a by proving that if $H \subseteq G$ is a T-normalized smooth connected k-subgroup for which $\mathrm{Lie}(H) = \mathfrak{g}_a$ then H is contained in the k-group $\mathscr{D}(Z_G(T_a))$ that we know to be k-isomorphic to SL_2 or PGL_2. This is a geometric problem, so we may temporarily assume $k = \overline{k}$.

The Lie algebra condition forces H to be 1-dimensional, so H is either $\mathbf{G}_{\mathrm{a}}$ or GL_1 (since $k = \overline{k}$). The latter case is impossible, since then H would be a torus normalized by T, yet the T-action on H would then be trivial (since T is connected and $\underline{\mathrm{Aut}}(\mathrm{GL}_1) = \mathbf{Z}/2\mathbf{Z}$), contradicting the nontriviality of the T-action on $\mathrm{Lie}(H) = \mathfrak{g}_a$. Hence, H is unipotent.

Next we claim that the T_a-action on H must be trivial, so $H \subseteq G_a := Z_G(T_a)$. Since $H = \mathbf{G}_{\mathrm{a}}$, for any $t \in T(k)$ the conjugation action of t on H is given by an algebraic group automorphism of $\mathbf{G}_{\mathrm{a}}$, and the only such automorphisms are the nonzero constant scalings. In other words, t acts by some $\chi(t) \in k^{\times}$. But then the induced action on $\mathrm{Lie}(H) = \mathrm{Lie}(\mathbf{G}_{\mathrm{a}})$ is easily seen to also be scaling by the same $\chi(t)$ on this line, yet $\mathrm{Lie}(H) = \mathfrak{g}_a$ inside $\mathfrak{g}$ by hypothesis, so $\chi(t) = a(t)$. In particular, if $t \in T_a(k)$ then its action on H is trivial. Since H is unipotent and $G_a/\mathscr{D}(G_a)$ is a torus (being connected reductive and commutative), the containment of H in G_a forces $H \subseteq \mathscr{D}(G_a)$ as desired.

Now we return to the situation over a general field k, knowing that the only possibilities for U_a, if any is to exist at all, are to be found inside the k-subgroup $\mathscr{D}(G_a)$ that we know to be k-isomorphic to SL_2 or PGL_2. In fact, the *proof* of existence of such a k-isomorphism arranged it so that *any* desired 1-dimensional k-split torus in $\mathscr{D}(G_a)$ is carried to the diagonal torus in SL_2 or PGL_2. There is a natural such k-torus: $S_a := T \cap \mathscr{D}(G_a)$! Indeed, since $T_a \times \mathscr{D}(G_a) \to G_a$ is a central isogeny, and the scheme-theoretic preimage of T under this map is $T_a \times (T \cap \mathscr{D}(G_a))$. This preimage is a torus (as for any central isogeny between connected reductive groups), so its direct factor $T \cap \mathscr{D}(G_a)$ is a torus, necessarily 1-dimensional and k-split due to the k-isogeny to the k-split T.

Pick an isomorphism ϕ from $\mathscr{D}(G_a)$ onto SL_2 or PGL_2 such that S_a goes over to the diagonal torus D. Since $T = S_a \cdot T_a$ and T_a centralizes $\mathscr{D}(G_a)$, a k-subgroup of $\mathscr{D}(G_a)$ is T-normalized if and only if it is S_a-normalized, and then the action of T on its Lie algebra is uniquely determined by the action of S_a on the Lie algebra (as T_a must act trivially there). Hence, we have reduced everything to a very special case: G is either SL_2 or PGL_2 and T is the diagonal torus D!! This is so concrete that the rest will be a pleasant calculation.

By direct calculation with $\mathfrak{sl}_2$ and $\mathfrak{pgl}_2$, the non-trivial weights for the adjoint D-action are easily seen (check!) to be the characters

$$a_+ : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^2, \;\; a_- : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{-2}$$

in $\mathrm{X}(D)$ in the SL_2-case, and the characters

$$a_+ : \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mapsto t, \;\; a_- : \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mapsto t^{-1}$$

in the PGL_2-case, with respective weight spaces given respectively by the Lie algebras of the "upper triangular" unipotent subgroup U_+ and the "lower triangular" unipotent subgroup U_-. In both cases, by inspection we see that $U_\pm$ are in fact normalized by D, and $U_\pm \simeq \mathbf{G}_{\mathrm{a}}$ as k-groups. Thus, the existence part of the problem is settled, and it remains to prove *uniqueness*. In particular, now we may and do assume that $k = \overline{k}$, so any possibility which exists must be a copy of $\mathbf{G}_{\mathrm{a}}$ inside our group.

Any possibility U for U_a yields a k-subgroup $D \ltimes U$ that is 2-dimensional, smooth, connected, and solvable, so by dimension reasons it must be a Borel subgroup that contains D. But the elementary Bruhat decomposition for $\mathrm{SL}_2(k)$ and $\mathrm{PGL}_2(k)$ with $k = \overline{k}$ yields that the two Borel subgroups $B_\pm = D \ltimes U_\pm$ are the *only* ones containing D. This forces $U \subseteq B_\pm$, so $U = U_\pm$ for dimension reasons. Then correspondingly $\mathfrak{g}_a = \mathrm{Lie}(U) = \mathfrak{g}_{a_\pm}$, so the Lie algebra condition on U inside $\mathfrak{g}$ picks out exactly one of the two possibilities as the only one which can work, and we have seen that this possibility really does work. ■

Example 1.4. Let $G = \mathrm{SL}_n$ and $T = D$ the diagonal torus. Then for each $1 \le i \ne j \le n$ let U_{ij} be the k-subgroup $u_{ij} : \mathbf{G}_{\mathrm{a}} \hookrightarrow G$ defined by setting $u_{ij}(x)$ to be the matrix whose diagonal entries are 1 and all other entries vanish except for the ij-entry which is x. This is easily seen to be a k-subgroup of G that is normalized by D, with $t = \mathrm{diag}(t_1, \dots, t_n)$ acting by $t \cdot u_{ij}(x) \cdot t^{-1} = u_{ij}((t_i/t_j)x)$. Thus, the space $\mathrm{Lie}(U_{ij}) \subset \mathfrak{sl}_n$ is a T-weight space for the nontrivial weight $a_{ij}(t) = t_i/t_j$. (Note that for $n = 2$ and $(i, j) = (1, 2)$, we get $t_1/t_2 = t_1^2$ since $t_2 = 1/t_1$ due to being in SL_2.) This already gives us a collection of weight spaces filling up the entire dimension of $\mathfrak{sl}_n$ away from the diagonal part $\mathfrak{t}$, so we have found all of the roots, as well as the root groups.

Another fun example is $G = \mathrm{Sp}_{2n}$ for a suitable "diagonal" T. This is worked out from scratch in §9.6.5 of the 2nd edition of the book *Pseudo-reductive groups*.

Having assembled the set of roots $\Phi(G, T)$ and the T-normalized root group $U_a \simeq \mathbf{G}_{\mathrm{a}}$ inside G for each root a, we next introduce the *coroots*. This will be a collection of nontrivial cocharacters $a^\vee : \mathrm{GL}_1 \to T$ which again arise from special arguments with SL_2 and PGL_2:

Proposition 1.5. *For each $a \in \Phi(G, T)$, there is a unique k-homomorphism $a^\vee : \mathrm{GL}_1 \to S_a := T \cap \mathscr{D}(Z_G(T_a))$ such that $a \circ a^\vee \in \mathrm{End}(\mathrm{GL}_1) = \mathbf{Z}$ is 2; i.e., $a(a^\vee(t)) = t^2$. That is, relative to any k-isomorphism $u_a : \mathbf{G}_{\mathrm{a}} \simeq U_a$, we have*

$$a^\vee(t) u_a(x) a^\vee(t)^{-1} = u_a(t^2 x).$$

In the PGL_2-case the map $a^\vee$ is a degree-2 isogeny, and in the SL_2-case it is an isomorphism.

Note that the choice of u_a really does not matter, since any two are related by composition with $\mathrm{Aut}_k(\mathbf{G}_{\mathrm{a}}) = k^\times$, which clearly preserves the proposed condition.

Proof. The problem is intrinsic to the k-split pair $(\mathscr{D}(Z_G(T_a)), S_a)$ that we have seen is k-isomorphic to (SL_2, D) or (PGL_2, D), and by composing such an isomorphism with a representative of the nontrivial class in the Weyl group of D if necessary we may arrange that the a-root group U_a goes over to the upper-triangular unipotent subgroup U_+. So now the problem is an entirely concrete one about U_+ and D inside SL_2 and PGL_2. In particular, we may and do use the choice $u_a(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. The existence of $a^\vee$ is now by inspection: $a^\vee(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ in the SL_2-case and $a^\vee(t) = \left(\begin{smallmatrix} t^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in the PGL_2-case. For uniqueness it suffices to check on $\overline{k}$-points, and that is safely left to the reader. ■

Definition 1.6. The set of *coroots* of (G, T) is the subset $\Phi^\vee(G, T) \subset \mathrm{X}_*(T)$ consisting of the cocharacters $a^\vee$ for all $a \in \Phi(G, T)$.

By construction, $(-a)^\vee = -a^\vee$. We will see soon that $a^\vee$ determines a, but if you think about it briefly this is not immediately obvious from the definitions. Before we take up a more detailed study of coroots, we wish to give another description of them. Let $D \subset \mathrm{SL}_2$ and $\overline{D} \subset \mathrm{PGL}_2$ denote the diagonal split maximal k-tori (of dimension 1). Note that $\mathrm{PGL}_2 = \mathrm{GL}_2/\mathbf{G}_\mathrm{m}$ naturally acts on SL_2, and more specifically $\mathrm{PGL}_2(k) = \mathrm{GL}_2(k)/k^\times$ acts on the k-group SL_2 via conjugation. The resulting homomorphism

$$\mathrm{PGL}_2(k) \to \mathrm{Aut}_{k\text{-gp}}(\mathrm{SL}_2)$$

is injective. Indeed, if $g \in \mathrm{GL}_2(k)$ centralizes SL_2 then computation with its action on D shows that g is diagonal, so we may scale g to be $\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The conjugation action of g on $u(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in U^+$ satisfies $gu(x)g^{-1} = u(cx)$, so triviality of this action forces $c = 1$ (so the original g lies in $k^\times$). This action plays a key role in:

Proposition 1.7. *Let (G, T) be a split reductive pair over a field k, and choose $a \in \Phi(G, T)$. There exists a central k-isogeny $f : \mathrm{SL}_2 \to \mathscr{D}(G_a)$ satisfying $f(D) = S_a = T \bigcap \mathscr{D}(G_a)$ and $f : U^+ \simeq U_a$, and the $\overline{D}(k)$-action on SL_2 is simply transitive on the set of such f. In particular, all such f induce the same k-isogeny $D \to S_a$, and under the isomorphism $\mathbf{G}_\mathrm{m} \simeq D$ defined by $c \mapsto \left(\begin{smallmatrix} c & 0 \\ 0 & 1/c \end{smallmatrix}\right)$ the resulting k-homomorphism $\mathbf{G}_\mathrm{m} \to S_a \hookrightarrow T$ is the coroot $a^\vee$.*

Moreover, $f \mapsto \mathrm{Lie}(f)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{g}_a - \{0\}$ is a bijection from the set of such f onto the set of bases of $\mathfrak{g}_a$.

Note that the requirement $f(U^+) = U_a$ (as opposed to $f(U^+) = U_{-a}$) is what distinguishes a and $-a$. Also, this result shows that to specify f without any ambiguity is exactly the same as to choose a basis of $\mathfrak{g}_a$; such a choice is a special case of an important general structure called a *pinning* that will be studied later (to eliminate conjugation ambiguities when passing between split connected reductive groups and root data).

Proof. The construction of $a^\vee$ used such an f, so we fix one such f_0 and need to consider an arbitrary f satisfying the desired conditions. Since f is a central isogeny, so $\ker f \subset Z_{\mathrm{SL}_2} = \mu_2$, we have either $\ker f = 1$ or $\ker f = \mu_2$. But $\mathrm{SL}_2/\mu_2 = \mathrm{PGL}_2 \not\simeq \mathrm{SL}_2$ (e.g., compare scheme-theoretic centers), so necessarily f has degree 1 when $\mathscr{D}(G_a) \simeq \mathrm{SL}_2$ and f has degree 2 when $\mathscr{D}(G_a) \simeq \mathrm{PGL}_2$. Referring to these two cases as the "SL_2 case" and "PGL_2 case" respectively, we have that $\ker f = 1$ in the SL_2 case and $\ker f = \mu_2$ in the PGL_2 case. Thus, in these respective cases the isogeny $f : D \to S_a$ is identified with an endomorphism of $\mathbf{G}_\mathrm{m}$ with respective degrees 1 and 2 (using *fixed* k-isomorphisms of D and S_a with $\mathbf{G}_\mathrm{m}$).

For $d > 0$ the only degree-d endomorphisms of $\mathbf{G}_\mathrm{m}$ are $t \mapsto t^{\pm d}$. Thus, there are at most *two* possibilities for $f : D \to S_a$, related through inversion (on the source or target). Hence, to show that this map between tori is uniquely determined as we vary f, we just have to rule out the possibility that f and f_0 are off by inversion on D.

Equivalently, if we define the k-isomorphism $\lambda : \mathbf{G}_\mathrm{m} \simeq D$ by $\lambda(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}\right)$ then for $\mu := f_0 \circ \lambda \in \mathrm{X}_*(S_a)$ we need to rule out the possibility $f \circ \lambda = -\mu$. Suppose to the contrary that $f \circ \lambda = -\mu$. In GL_2 we have

$$\lambda(t) \begin{pmatrix} x & y \\ z & w \end{pmatrix} \lambda(t)^{-1} = \begin{pmatrix} x & t^2 y \\ t^{-2} z & w \end{pmatrix},$$

so $U_{\mathrm{SL}_2}(\lambda) = U^+$. The good behavior of the "$U(\lambda)$" construction under quotient maps gives that $f(U^+) = f(U_{\mathrm{SL}_2}(\lambda)) = U_{\mathscr{D}(G_a)}(f \circ \lambda)$, so

$$U_{\mathscr{D}(G_a)}(-\mu) = f(U^+) = U_a = f_0(U^+) = U_{\mathscr{D}(G_a)}(\mu).$$

But in general the dynamic method for any 1-parameter subgroup μ of a linear algebraic group H gives that the multiplication map

$$U_H(-\mu) \times Z_H(\mu) \times U_H(\mu) \to H$$

is an open immersion, so $U_H(-\mu) \bigcap U_H(\mu) = 1$. Hence, the above equality relating μ and $-\mu$ is impossible (since $U_a \neq 1$), so the uniqueness of $f|_D$ is proved.

Now that f is uniquely determined on D, we take into consideration the effect of precomposition with the action of $\overline{D}(k)$ on SL_2. Upon fixing isomorphisms $U^+ \simeq \mathbf{G}_\mathrm{a}$ and $U_a \simeq \mathbf{G}_\mathrm{a}$, $f : U^+ \simeq U_a$ is identified with an automorphism of the k-group $\mathbf{G}_\mathrm{a}$. Such an automorphism is a degree-1 additive polynomial, which is to say $x \mapsto bx$ for $b \in k^\times$. The induced map on Lie algebras is multiplication by b, so the natural map of sets

$$\mathrm{Isom}_{k\text{-gp}}(U^+, U_a) \to \mathrm{Isom}_k(\mathrm{Lie}(U^+), \mathrm{Lie}(U_a)) = \mathfrak{g}_a - \{0\}$$

is bijective (the last step being evaluation on $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$). Precomposition with conjugation by $\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \overline{D}(k)$ has the effect of replacing b with bc, so this $\overline{D}(k)$-action is simply transitive on the set of all k-isomorphisms $U^+ \simeq U_a$. By using this action, our problem is now reduced to showing that a k-homomorphism $f : \mathrm{SL}_2 \to H$ to any linear algebraic group H is uniquely determined by its restriction to the Borel subgroup $B^+ = D \ltimes U^+$.

More generally, if $f : H' \to H$ is a homomorphism between linear algebraic groups over k and H' is connected, then f is uniquely determined by its restriction to a parabolic k-subgroup P. Indeed, if $F : H' \to H$ is a second homomorphism such that $F|_P = f|_P$ then the map of k-schemes $H' \to H$ defined by $h' \mapsto f(h')F(h')^{-1}$ is right P-invariant, so it factors through a k-scheme map $q : H'/P \to H$. But H'/P is a smooth connected proper k-scheme with a k-point and H is affine of finite type over k, so q must be a constant map to $q(1) = 1$. That is, $F = f$. ∎

An interesting consequence of the *method* of proof of Proposition 1.7 is:

Corollary 1.8. *For any field k, the injective maps*

$$\mathrm{PGL}_2(k) \hookrightarrow \mathrm{Aut}_{k\text{-gp}}(\mathrm{SL}_2), \ \ \mathrm{PGL}_2(k) \hookrightarrow \mathrm{Aut}_{k\text{-gp}}(\mathrm{PGL}_2)$$

are bijective. In particular, all k-automorphisms of PGL_2 are inner, and the same holds for SL_2 if and only if $k^\times = (k^\times)^2$.

Proof. Since the maps in question are injective, by Galois descent it suffices to check the equality over k_s (as then passing to $\mathrm{Gal}(k_s/k)$-invariants on both sides over k_s gives the result over k). Hence, we may and do assume $k = k_s$, so all maximal k-tori in a connected linear algebraic k-group H are in the same $H(k)$-conjugacy class (see Proposition 3.6 in the handout on Lang's theorem and dynamic methods). It follows that for any k-automorphism ϕ of H and fixed maximal k-torus $T \subset H$, the composition of ϕ with a suitable $H(k)$-conjugation carries T into itself. Thus, in our situations with SL_2 and PGL_2 equipped with their respective diagonal maximal k-tori D and $\overline{D}$, we can focus our attention on those automorphisms ϕ which carry D or $\overline{D}$ into itself.

Consider the resulting automorphism of D or $\overline{D}$ induced by ϕ. This is the identity or inversion, as we may check by identifying this torus with $\mathbf{G}_\mathrm{m}$. Composing ϕ with conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ if necessary, we may arrange that ϕ restricts to the *identity* on the diagonal torus. Hence, its action on the character group is the identity, thereby preserving the two roots, so by the unique characterization of the root groups it follows that ϕ must carry each root group into itself. In particular, ϕ restricts to an automorphism of the upper triangular root group U^+ or $\overline{U}^+$ (depending on whether we are in the SL_2 case or PGL_2 case).

This root group U is k-isomorphic to $\mathbf{G}_{\mathrm{a}}$, so $\mathrm{Aut}(U) \to \mathrm{GL}(\mathrm{Lie}(U)) = k^{\times}$ is bijective. But the natural action by $\overline{D}(k) \subset \mathrm{PGL}_2(k)$ is trivial on the diagonal torus (of SL_2 and PGL_2) and preserves U, making $\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$ act on $\mathrm{Lie}(U)$ as multiplication by c. We conclude that the composition of ϕ with the action of a unique element of $\overline{D}(k)$ brings us to an automorphism that is the identity on both the diagonal torus *and* on U, hence on the Borel k-subgroup B that they generate. But we saw at the end of the preceding proof that the only such automorphism is the identity. ∎

The first step towards understanding the injectivity of $a \mapsto a^{\vee}$ is to give an alternative way to think about coroots in terms of the finite *Weyl group* $W(G,T) = (N_G(T)/T)(k) = N_G(T)(k)/T(k)$ (latter equality by Hilbert 90, since T is k-split!). For each root a, the pair $(\mathscr{D}(G_a), S_a)$ is k-isomorphic to (SL_2, D) or (PGL_2, D), and in particular has a Weyl group of order 2. All elements of $\mathscr{D}(G_a)$ centralize the codimension-1 torus T_a, so since $T = T_a \cdot S_a$ we see that any representative $n_a \in \mathscr{D}(G_a)$ of the non-trivial class in $W(\mathscr{D}(G_a), S_a)$ actually normalizes *all* of T and does not centralize it! That is, we have an *injective* homomorphism

$$W(\mathscr{D}(G_a), S_a) \hookrightarrow W(G,T).$$

We let $w_a \in W(G,T)$ denote the image of the nontrivial element in this order-2 subgroup. Under the natural faithful action of $W(G,T)$ on T, this element acts trivially on T_a and acts via inversion on S_a since it is represented by an element of $N_{\mathscr{D}(G_a)}(S_a)$ not centralizing S_a. Thus, on $\mathrm{X}(T)_{\mathbf{Q}}$ it acts trivially on a hyperplane and via negation on a complementary line, so it is a *reflection*.

We define $s_a \in \mathrm{End}(\mathrm{X}(T))$ to be the endomorphism induced by w_a; note that $w_{-a} = w_a$, $s_{-a} = s_a$.

Proposition 1.9. *Let* $\langle \cdot, \cdot \rangle : \mathrm{X}(T) \times \mathrm{X}_*(T) \to \mathrm{End}(\mathrm{GL}_1) = \mathbf{Z}$ *be the natural perfect pairing* $\langle \chi, \lambda \rangle = \chi \circ \lambda$ *between finite free* $\mathbf{Z}$*-modules. Then*

$$s_a(x) = x - \langle x, a^{\vee} \rangle a.$$

There will be more work to do in order to show that $a^{\vee}$ determines a.

Proof. By definition s_a is the action of w_a induced on $\mathrm{X}(T)$. But w_a acts trivially on $T_a \subset \ker a$, and it acts by inversion on the subtorus $S_a = a^{\vee}(\mathbf{G}_{\mathrm{m}})$ that is an isogeny complement to T_a. Thus, the isomorphism $\mathrm{X}(T)_{\mathbf{Q}} \simeq \mathrm{X}(S_a)_{\mathbf{Q}} \times \mathrm{X}(T_a)_{\mathbf{Q}}$ induced by the isogeny $S_a \times T_a \to T$ implies that $s_a(a) = -a$ and s_a fixes a hyperplane pointwise, so it is a reflection on $\mathrm{X}(T)_{\mathbf{Q}}$. Since it negates $a \neq 0$, necessarily $s_a(x) = x - \ell_a(x)a$ for a unique nonzero linear form ℓ_a on $\mathrm{X}(T)_{\mathbf{Q}}$. Writing $\ell_a = \langle \cdot, x'_a \rangle$ for $x'_a \in \mathrm{X}_*(T)_{\mathbf{Q}}$, our problem is to prove that $x'_a = a^{\vee}$.

Under the perfect duality pairing, the dual automorphism $s_a^{\vee}$ on the dual lattice $\mathrm{X}(T)^{\vee} = \mathrm{X}_*(T)$ is also induced by n_a-conjugation on T (check!), so it negates the line $\mathrm{X}_*(S_a)_{\mathbf{Q}}$ through $a^{\vee}$ (since $S_a = a^{\vee}(\mathbf{G}_{\mathrm{m}})$). But it is easy to directly compute the dual of $x \mapsto x - \langle x, x'_a \rangle a$, namely $\lambda \mapsto \lambda - \langle a, \lambda \rangle x'_a$, and evaluating it on $\lambda = a^{\vee}$ gives

$$-(a^{\vee}) = s_a^{\vee}(a^{\vee}) = a^{\vee} - \langle a, a^{\vee} \rangle x'_a = a^{\vee} - 2x'_a,$$

so $x'_a = a^{\vee}$. ∎

Proposition 1.10. *The surjective map of sets* $\Phi(G,T) \to \Phi^{\vee}(G,T)$ *defined by* $a \mapsto a^{\vee}$ *is bijective.*

Proof. Consider roots a and b such that $a^{\vee} = b^{\vee}$ in $\mathrm{X}(T)$. Consider the element $w_a w_b \in W(G,T) \subset \mathrm{GL}(\mathrm{X}(T))$. This is the product $s_a s_b$, and from the explicit formulas

$$s_a(x) = x - \langle x, a^{\vee} \rangle a, \quad s_b(x) = x - \langle x, b^{\vee} \rangle b = x - \langle x, a^{\vee} \rangle b$$

it is easy to compute

$$s_a s_b(x) = x + \langle x, a^{\vee} \rangle (a - b).$$

Working in $\mathrm{X}(T)_{\overline{\mathbf{Q}}}$, consider an eigenvector v of $s_a s_b$, so $s_a s_b(v) = cv$. Thus, $cv = v - \langle v, a^\vee\rangle(a-b)$. If $c \neq 1$ then v is a multiple of $a-b$, yet $a-b$ is fixed by $s_a s_b$ because

$$\langle a-b, a^\vee\rangle = \langle a, a^\vee\rangle - \langle b, a^\vee\rangle = \langle a, a^\vee\rangle - \langle b, b^\vee\rangle = 2-2 = 0.$$

This would force v to also be fixed by $s_a s_b$, contradicting that $c \neq 1$. In other words, $c = 1$ after all. That is, the only eigenvalue of $s_a s_b$ is 1, which is to say that $s_a s_b$ is unipotent. But $s_a s_b$ lies in the *finite* subgroup $W(G,T)$ on the automorphism group of $\mathrm{X}(T)$, so unipotence forces this operator to be the identity.

We conclude that s_a and s_b are inverse to each other. Yet each is a reflection, hence of order 2, so in fact $s_a = s_b$. But inspection of the above explicit formulas for these reflections (using the assumption of equality of the associated coroots) gives that $\ell(x)a = \ell(x)b$ for all $x \in \mathrm{X}(T)$ and the nonzero linear form $\ell = \langle \cdot, a^\vee\rangle$. Hence, $a = b$. ■

Proposition 1.11. *For each root a, the reflection $s_a : x \mapsto x - \langle x, a^\vee\rangle a$ on $\mathrm{X}(T)$ preserves the finite set of roots $\Phi(G,T)$. Also the dual reflection*

$$s_a^\vee : \lambda \mapsto \lambda - \langle a, \lambda\rangle a^\vee$$

on the dual lattice $\mathrm{X}_*(T)$ preserves the finite set of coroots $\Phi^\vee(G,T)$.

Proof. By our preceding calculations, the actions of s_a and its dual are exactly the natural actions induced by the action of w_a on T. Thus, the first assertion is a consequence of the obvious fact that the action of $N_G(T)$ on T permutes the set $\Phi(G,T)$ of nontrivial T-weights on $\mathrm{Lie}(G)$. For the second assertion, it is likewise suffices to prove that the $N_G(T)$-action on T permutes the set of coroots. For any root a and any $n \in N_G(T)$ representing $w \in W(G,T)$, $w.a^\vee$ is a cocharacter of $nS_a n^{-1} = S_{w.a}$ (equality since $S_a := T \cap \mathscr{D}(Z_G(T_a))$ and $T_a := (\ker a)^0_{\mathrm{red}}$). It is easy to check that it satisfies the property in Proposition 1.5 for the root $w.a$ (verify!), so it must be $(w.a)^\vee$. ■

Definition 1.12. A *root datum* is a 4-tuple $(X, R, X^\vee, R^\vee)$ consisting of a pair of finite free $\mathbf{Z}$-modules X and $X^\vee$ equipped with a perfect duality pairing $\langle \cdot, \cdot\rangle : X \times X^\vee \to \mathbf{Z}$ and a pair of finite subsets $R \subset X$ and $R^\vee \subset X^\vee$ such that there exists a bijection $a \mapsto a^\vee$ satisfying the axioms:

(1) For all $a \in R$, $\langle a, a^\vee\rangle = 2$.
(2) For all $a \in R$, the dual endomorphisms $s_{a,a^\vee}$ of X and $s_{a^\vee,a}$ of $X^\vee$ defined by

$$s_{a,a^\vee}(x) = x - \langle x, a^\vee\rangle a, \;\; s_{a^\vee,a}(x^*) = x^* - \langle a, x^*\rangle a^\vee$$

satisfy $s_{a,a^\vee}(R) = R$ and $s_{a^\vee,a}(R^\vee) = R^\vee$.

Note that we allow $R = \emptyset$ with $X \neq 0$ (tori!), and the first axiom forces $a, a^\vee \neq 0$ as well as the fact that $s_{a,a^\vee}$ and $s_{a^\vee,a}$ respectively negate the lines through a and $a^\vee$ and pointwise fix the hyperplanes orthogonal to $a^\vee$ and a (working over $\mathbf{Q}$, say); i.e., pointwise fix the kernel hyperplanes to each in the dual $\mathbf{Q}$-vector space. Hence, each is a *reflection*. In particular, since $s_{a,a^\vee}(a) = -a$, we see that R is stable under negation. Also, the axioms are entirely symmetric: the 4-tuple $(X^\vee, R^\vee, X, R)$ is also a root datum (using the inverse bijection $a^\vee \mapsto a$).

There is a subtlety lurking here: we did not impose the specification of the bijection $a \mapsto a^\vee$ as part of the definition. Rather, this was simply assumed to exist in some way. Most textbooks impose the bijection as part of the structure of a root datum, and the entire basic theory can be developed in this way. But it is more elegant to not impose this, which we can do thanks to:

Proposition 1.13. *In a root datum, the bijection $a \mapsto a^\vee$ is uniquely determined. Writing $s_a := s_{a,a^\vee}$ and $s_{a^\vee} = s_{a^\vee,a} = s_a^\vee$, we also have $s_a(b)^\vee = s_{a^\vee}(b^\vee)$ for all $a, b \in R$.*

Proof. This is Lemma 3.2.4 in "Pseudo-reductive groups" (2nd ed.); the proof is an elementary argument in linear algebra, relying on a small calculation via the axioms given in [SGA3, XXI, 1.1.4]. More specifically, the argument in "Pseudo-reductive groups" shows that whatever bijection $a \mapsto a^\vee$ can exist between R and $R^\vee$ satisfying the axioms for a root datum, the functional $\langle \cdot, a^\vee \rangle$ on the $\mathbf{Q}$-span of R is uniquely determined independently of that bijection. If $a \mapsto a^*$ is a second such bijection, then for any $a \in R$ the functionals $\langle \cdot, a^\vee \rangle$ and $\langle \cdot, a^* \rangle$ on the $\mathbf{Q}$-span of R coincide. We have $a^* = b^\vee$ for some $b \in R$, and want to know that $b = a$. But the equality $\langle \cdot, a^\vee \rangle = \langle \cdot, b^\vee \rangle$ on the $\mathbf{Q}$-span of R forces $a = b$ by [SGA3, XXI, 1.1.4]. ■

The entire preceding analysis shows that to any split pair (G, T) we have associated a root datum

$$R(G,T) = (\mathrm{X}(T), \Phi(G,T), \mathrm{X}_*(T), \Phi^\vee(G,T)),$$

under which the reflections $s_a \in \mathrm{End}(\mathrm{X}(T))$ are induced by the elements $w_a \in W(G,T)$. Thus, the subgroup of $W(G,T)$ generated by the reflections s_a is intrinsic to the root datum, and it is denoted $W(R(G,T))$. (In Corollary 2.11 we will show that these groups coincide.)

Let (G,T) be a split connected reductive group over a field k. Inside the finite group $W(G,T)$, we built a subgroup $W(R(G,T))$ that is intrinsic to the root datum. The finiteness of this subgroup is a general combinatorial fact unrelated to algebraic groups. This is the final part of:

Proposition 1.14. *Let $(X, R, X^\vee, R^\vee)$ be a root datum, and $Q \subset X$ the $\mathbf{Z}$-span of R. Also define the linear map $f : X \to X^\vee$ by $f(x) = \sum_{a \in R} \langle x, a^\vee \rangle a^\vee$, and $X_0 = \ker f$.*

1. *For each $a \in R$, $\langle a, f(a) \rangle \neq 0$ and $f(a) = (1/2)\langle a, f(a) \rangle a^\vee$.*
2. *The kernel X_0 coincides with the common annihilator $\bigcap_{a \in R} \ker \langle \cdot, a^\vee \rangle \subset X$, and the natural map $Q \oplus X_0 \to X$ is injective with image of finite index.*
3. *The subgroup $W(R) \subset \mathrm{GL}(X)$ generated by the reflections $\{s_a\}_{a \in R}$ is finite.*

The group $W(R)$ in (3) is the *Weyl group* of the root datum; it is trivial precisely when R is empty. The "isogeny decomposition" $Q \oplus X_0$ of X in (2) is analogous to the central isogeny $G \to (G/\mathscr{D}(G)) \times (G/Z)$ for a connected reductive group G and its maximal central torus Z; see Example 2.1.

Proof. For any $b \in R$, the reflection s_b permutes the finite set R, so we can apply a "change of variables" $a \mapsto s_b(a)$ and use the formula $s_b(a)^\vee = s_{b^\vee}(a^\vee)$ from Proposition 1.13 to get

$$f(b) = \sum_{a \in R} \langle b, a^\vee \rangle a^\vee = \sum_{a \in R} \langle b, s_{b^\vee}(a^\vee) \rangle s_{b^\vee}(a^\vee).$$

Since $s_{b^\vee}(a^\vee) = a^\vee - \langle b, a^\vee \rangle b^\vee$, so

$$\langle b, s_{b^\vee}(a^\vee) \rangle = \langle b, a^\vee \rangle - \langle b, a^\vee \rangle \langle b, b^\vee \rangle = -\langle b, a^\vee \rangle,$$

we find that

$$f(b) = -f(b) + (\sum_{a \in R} \langle b, a^\vee \rangle^2) b^\vee,$$

so $f(b) = c_b b^\vee$ with $c_b \in (1/2)\mathbf{Z}_{\geq 4}$ since $\langle a, a^\vee \rangle^2 = 4$. (Once we prove later that $(-a)^\vee = -a^\vee$ for any root datum, by combining the equal terms for each $\{a, -a\}$ in the sum over R defining f, it will follow that $c_b \in \mathbf{Z}_{\geq 2}$.) Applying $\langle b, \cdot \rangle$ to the identity $f(b) = c_b b^\vee$, we get $c_b = (1/2)\langle b, f(b) \rangle$. This proves (1).

The image of f clearly lies in the $\mathbf{Q}$-span $V' \subset X^\vee_{\mathbf{Q}}$ of the elements of $R^\vee$, and the formula in (1) shows that the image of $f_{\mathbf{Q}}$ exhausts V'. More specifically, if $V = Q_{\mathbf{Q}} \subset X_{\mathbf{Q}}$ denotes the $\mathbf{Q}$-span of R then (1) shows that $f_{\mathbf{Q}}(V) = V'$. Thus, the resulting isomorphism $(X/X_0)_{\mathbf{Q}} \simeq V'_{\mathbf{Q}} = f(Q)_{\mathbf{Q}}$ implies that the map $Q \oplus X_0 \to X$ becomes surjective after extending scalars to $\mathbf{Q}$. Hence, to prove (2) the problem is entirely one of comparison of $\mathbf{Q}$-dimensions. But the isomorphism $(X/X_0)_{\mathbf{Q}} \simeq V'_{\mathbf{Q}} = f(Q)_{\mathbf{Q}}$ gives that $\dim V' \leq \dim Q_{\mathbf{Q}} = \dim V$, with equality if and only if the result in (2) holds. We can run through the entire argument with the dual root datum (using the map $f' : X^\vee \to X$ analogous to f) to get the reverse inequality $\dim V \leq \dim V'$, so (2) is proved.

Finally, to prove (3) we consider the decomposition $X_{\mathbf{Q}} = V \oplus (X_0)_{\mathbf{Q}}$ provided by (2). Since $(X_0)_{\mathbf{Q}}$ is annihilated by the coroots, the reflections s_a on $X_{\mathbf{Q}}$ restrict to the identity on $(X_0)_{\mathbf{Q}}$ and also preserve the $\mathbf{Q}$-span of the roots. Thus, $W(R)$ maps isomorphically to its image in $\mathrm{GL}(V)$, which is a subgroup of V that acts through permutations on a *finite* $\mathbf{Q}$-spanning set R of V. This injects $W(R)$ into the permutation group of the finite set R, so $W(R)$ is finite. ■

Corollary 1.15. *For each $a \in R$ and $q \in \mathbf{Q}$ such that $qa \in R$ inside $X_{\mathbf{Q}}$, we have $(qa)^\vee = (1/q)a^\vee$. In particular, $(-a)^\vee = -a^\vee$ for all $a \in R$. In general, the only possibilities for q are $\{\pm 1, \pm 2, \pm 1/2\}$.*

A root datum is *reduced* if the only elements of R linearly dependent with each $a \in R$ are $\{\pm a\}$. For example, we know that the root datum arising from a split connected reductive group is always reduced.

Proof. Consider the linear map $f : X \to X^\vee$ from Proposition 1.14. For $a \in R$ and $q \in \mathbf{Q}$ such that $qa \in R$, let's compute both sides of the identity $f(qa) = qf(a)$. The left side is $(1/2)\langle qa, f(qa)\rangle(qa)^\vee = (q^2/2)\langle a, f(a)\rangle(qa)^\vee$, whereas the right side is $(q/2)\langle a, f(a)\rangle a^\vee$. But we know that $\langle a, f(a)\rangle \neq 0$, so we get $(qa)^\vee = (1/q)a^\vee$ as claimed. Taking $q = -1$ gives $(-a)^\vee = -a^\vee$ for all $a \in R$.

Since $\langle a, (qa)^\vee\rangle \in \mathbf{Z}$ and $(qa)^\vee = (1/q)a^\vee$, we see that $2/q \in \mathbf{Z}$. Likewise, since $\langle qa, a^\vee\rangle \in \mathbf{Z}$, we have $2q \in \mathbf{Z}$. It follows that the numerator and denominator of q (in reduced form) must divide 2, so this leaves only the possibilities $q \in \{\pm 1, \pm 2, \pm 1/2\}$. ■

Let Q be the span of R in X, Q' the span of $R^\vee$ in $X^\vee$, and define the annihilators

$$X_0 = \bigcap_{a' \in R^\vee} \ker\langle \cdot, a'\rangle, \;\; X'_0 = \bigcap_{a \in R} \ker\langle a, \cdot\rangle,$$

so $Q \oplus X_0$ is a finite-index subgroup of X and (by consideration of the dual root datum) $Q' \oplus X'_0$ is a finite-index subgroup of $X^\vee$. The lattices Q and Q' contain the most interesting information (the roots and coroots), and the duality pairing between them is perfect over $\mathbf{Q}$ (since $X_{\mathbf{Q}}$ and $X^\vee_{\mathbf{Q}}$ are in duality, with $(X_0)_{\mathbf{Q}}$ the annihilator of $V' := Q'_{\mathbf{Q}}$ and $(X'_0)_{\mathbf{Q}}$ the annihilator of $V := Q_{\mathbf{Q}}$). However, Q and Q' generally are *not* in perfect duality over $\mathbf{Z}$. For example, if $G = \mathrm{SL}_2$ and T is the diagonal maximal torus then inside $X = \mathrm{X}(T) \simeq \mathbf{Z}$ we have $Q = 2X$ but $Q' = X^\vee$.

2. Applications of root data

We already have enough to begin seeing how the combinatorial root datum is convenient for analyzing some basic properties of split reductive groups.

Example 2.1. Consider the root datum $R(G,T) = (X, R, X^\vee, R^\vee)$ associated to a split connected reductive group (G,T) over a field k, so $X = \mathrm{X}(T)$. The group $X'_0 \subset X^\vee = \mathrm{X}_*(T)$ consists of the cocharacters $\lambda : \mathbf{G}_{\mathrm{m}} \to T$ such that $a \circ \lambda$ is trivial for all roots a, which is to say that $\lambda(\mathbf{G}_{\mathrm{m}})$ acts trivially on $\mathfrak{g}$. (Equivalently, X'_0 is the annihilator of Q in $X^\vee$.) This in turns says exactly that the smooth connected subgroup $Z_G(\lambda(\mathbf{G}_{\mathrm{m}}))$ in G has full Lie algebra, which is to say that λ is valued in Z_G. Hence, $X'_0 = \mathrm{X}_*(Z)$ where $Z \subset T$ is the maximal central k-torus in G. In particular, *G is semisimple if and only if Q has finite index in X*. Meanwhile, the coroots $a^\vee : \mathbf{G}_{\mathrm{m}} \to T$ are constructed to be valued in $\mathscr{D}(G_a) \subset \mathscr{D}(G)$, so $Q' \subset \mathrm{X}_*(\mathscr{T})$ where $\mathscr{T} := T \bigcap \mathscr{D}(G)$ is a maximal k-torus of $\mathscr{D}(G)$ (as proved in class early in the course).

Since surjections carry maximal tori onto maximal tori, the isogeny $Z \times \mathscr{D}(G) \to G$ with Z a torus shows that the maximal tori of $\mathscr{D}(G)$ have codimension $\dim Z$ in the maximal tori of G. But $X'_0 = \mathrm{X}_*(Z)$ has rank $\dim Z$, so it follows that the common $\mathbf{Z}$-ranks of Q and Q' coincide with the dimensions of the maximal tori of $\mathscr{D}(G)$. Hence, the inclusion $Q' \subset \mathrm{X}_*(\mathscr{T})$ must have finite index and so the coroots span a finite-index subgroup of the cocharacter group of $\mathscr{T}$. This shows that $\mathrm{X}_*(\mathscr{T})$ is the saturation of Q' in $\mathrm{X}_*(T) = X^\vee$ and the coroot groups $a^\vee(\mathbf{G}_\mathrm{m})$ *generate* $\mathscr{T}$. Lie algebra considerations show that the group generated by the maximal torus $\mathscr{T}$ and the root groups U_a exhausts $\mathscr{D}(G)$, and inspection of SL_2 and its quotient PGL_2 shows that each $\mathscr{D}(G_a)$ is generated by the roots groups $U_{\pm a}$. But $a^\vee(\mathbf{G}_\mathrm{m}) \subset \mathscr{D}(G_a)$, so we conclude that $\mathscr{D}(G)$ *is generated by the root groups*.

The natural map $Q \to \mathrm{X}(\mathscr{T})$ defined by restriction of characters is injective and hence (by $\mathbf{Z}$-rank reasons) a finite-index inclusion because the composite map

$$Q \to \mathrm{X}(\mathscr{T}) = \mathrm{Hom}(\mathrm{X}_*(\mathscr{T}), \mathbf{Z}) \to \mathrm{Hom}(Q', \mathbf{Z})$$

is exactly the natural lattice pairing between Q and Q' that is perfect over $\mathbf{Q}$. This proves that the roots of (G,T) restrict to a spanning set of a finite-index subgroup of $\mathrm{X}(\mathscr{T})$, and that $\mathrm{X}_*(\mathscr{T})$ is the *saturation* Q'_{sat} of Q' in $X^\vee$ (so $\mathrm{X}(\mathscr{T})$ is the quotient of X modulo the annihilator of $R^\vee$). In particular, we can compute the root datum $R(\mathscr{D}(G), \mathscr{T})$ purely combinatorially in terms of $R(G,T)$. Namely, since X_0 is identified with the annihilator $(Q')^\perp$ of the $\mathbf{Z}$-span Q' of the coroots,

$$R(\mathscr{D}(G), T \bigcap \mathscr{D}(G)) = (X/(Q')^\perp, R, Q'_{\mathrm{sat}}, R^\vee).$$

Example 2.2. Let's work out the case $G = \mathrm{GL}_n$ with $T = \mathbf{G}_\mathrm{m}^n$ the split diagonal torus. In this case $\mathscr{D}(G) = \mathrm{SL}_n$ and naturally $X = \mathrm{X}(T) = \mathbf{Z}^n$ with $\Phi = \Phi(G,T)$ the set of differences $e_i - e_j$ of distinct standard basis vectors (corresponding to the characters $a_{ij} : \mathrm{diag}(t_1, \dots, t_n) \mapsto t_i/t_j$). The set $\Phi^\vee \subset X^\vee = \mathbf{Z}^n$ of coroots consists of the differences $a_{ij}^\vee = e_i^* - e_j^*$ in the dual lattice, so the root lattice $Q = \sum_{i \neq j} \mathbf{Z}(e_i - e_j) \subset X$ consists of the vectors in $\mathbf{Z}^n$ whose coordinates sum to 0 and the common annihilator $X_0 \subset X$ of the coroots is the diagonal $\mathbf{Z} \subset \mathbf{Z}^n$.

The character groups for the corresponding diagonal maximal tori $T \cap \mathscr{D}(G) \subset \mathrm{SL}_n$ and $T/Z_G \subset \mathrm{PGL}_n$ are $X/X_0 = \mathbf{Z}^n/\Delta(\mathbf{Z})$ and Q respectively, and the kernel μ_n of $\mathrm{SL}_n \to \mathrm{PGL}_n$ is the center. Adding up coordinates modulo n identifies $\mathrm{coker}(Q \to X/X_0)$ with $\mathbf{Z}/n\mathbf{Z}$, whose $\mathbf{G}_\mathrm{m}$-dual is $Z_{\mathrm{SL}_n} = \mu_n$ as we know must be the case.

Under the *Isogeny Theorem* that functorially relates split reductive groups to root data (see Remark 2.12), it follows from the computations in Example 2.1 that the finite-index inclusion $X \to (X/(Q')^\perp) \oplus (Q^\perp)^*$ corresponds to the central isogeny $\mathscr{D}(G) \times Z \to G$. Likewise, the inclusion $Q \oplus (Q')^\perp \to X$ corresponds via the Isogeny Theorem to the central isogeny $G \to (G/Z) \times (G/\mathscr{D}(G))$. The possible failure of Q and Q' to be in perfect duality over $\mathbf{Z}$ (i.e., $Q \to Q'^*$ may not be an isomorphism) thereby corresponds to the fact that the central isogeny of semisimple groups $\mathscr{D}(G) \to G/Z$ is generally not an isomorphism (e.g., for $G = \mathrm{GL}_n$ with $n > 1$, the associated central isogeny $\mathrm{SL}_n \to \mathrm{PGL}_n$ is not an isomorphism).

Example 2.1 illustrates the interest in a combinatorial datum that is weaker than a root datum: the $\mathbf{Q}$-vector space V spanned in $X_\mathbf{Q}$ by the finite set R of nonzero elements and its dual V' spanned in $X_\mathbf{Q}^\vee$ by the coroots. These satisfy the axioms in the following definition.

Definition 2.3. A *root system* is a finite-dimensional $\mathbf{Q}$-vector space V equipped with a finite spanning set R of nonzero elements such that for each $a \in R$ there exists $a^\vee \in V^*$ satisfying:

(1) $a^\vee(R) \subset \mathbf{Z}$ and $a^\vee(a) = 2$,
(2) $s_{a,a^\vee}(R) \subset R$, where $s_{a,a^\vee}(v) = v - \langle v, a^\vee \rangle a$.

The dimension $\dim V$ is the *rank* of the root system.

Remark 2.4. It is a matter of convention as to whether one should allow $(V, R) = (0, \emptyset)$ to be a root system. Bourbaki allows this, but some references do not. We allow it so that we can associate a root system (and root datum) to the trivial algebraic group. Obviously this is not important.

As we saw for root data, for any root system it is automatic that each $s_{a,a^\vee}$ is a reflection that negates a (so R is stable under negation). Also, for any root system, the assignment $a \mapsto a^\vee$ is an injection of R into $V^* - \{0\}$. To see this, we first note that the group W generated by the reflections $s_{a,a^\vee}$ is finite because it lies in $\mathrm{GL}(V)$ and permutes the finite *spanning set* R of V (so W injects into the permutation group of the finite set R). Consequently, the injectivity argument in the proof of Proposition 1.10 can be applied in the root system setting. The finite group W is called the *Weyl group* of the root system. In the proof of Proposition 1.14 we saw that the Weyl group of any root datum maps isomorphically onto the Weyl group of the associated root system. The link between root systems and root data can be run backwards in a limited sense:

Proposition 2.5. *Let (V, R) be a root system, and define X to be the $\mathbf{Z}$-span Q of R and $X^\vee$ to be its dual lattice in V^* (so $X^\vee$ contains the set $R^\vee = \{a^\vee\}_{a \in R}$). The 4-tuple $(X, R, X^\vee, R^\vee)$ is a root datum.*

Proof. The only non-obvious requirement for the axioms of a root datum to be satisfied is that $s_{a^\vee,a}$ preserves $R^\vee$. Unsurprisingly, we claim that necessarily $s_{a^\vee,a}(b^\vee) = s_{a,a^\vee}(b)^\vee$ for all $a, b \in R$. To prove this, fix a $\mathbf{Q}$-valued positive-definite symmetric bilinear form $(\cdot, \cdot)$ on V that is invariant under the action of the finite Weyl group W of the root system. This identifies V with V^*, and by design the elements $s_{a,a^\vee} \in W$ leave the bilinear form invariant. Thus, $s_{a,a^\vee}$ must preserve the hyperplane H_a orthogonal to the line $\mathbf{Q}a$, yet it negates this line and hence must have all eigenvalues equal to 1 on H_a. But $s_{a,a^\vee}$ has *finite* order, so its effect on H_a must be the identity. That is, $s_{a,a^\vee}$ can be recovered from the geometry: it must be the orthogonal reflection

$$x \mapsto x - (x, 2a/(a, a))a$$

in the line $\mathbf{Q}a$. This says that $a^\vee = 2a/(a, a)$ under the identification of V^* with V defined by $(\cdot, \cdot)$.

It may seem that we are almost going in a circle (and not making any progress), since the bilinear form $(\cdot, \cdot)$ was chosen to be W-invariant, and W was defined in terms of the reflections $s_{a,a^\vee}$ that depend on the specification of the coroots $a^\vee$. Nonetheless, we can in fact now prove that $s_{a,a^\vee}(b)^\vee = s_{a^\vee,a}(b^\vee)$. The linear form $s_{a,a^\vee}(b)^\vee \in V^*$ goes over to

$$\frac{2s_{a,a^\vee}(b)}{(s_{a,a^\vee}(b), s_{a,a^\vee}(b))} = \frac{2s_{a,a^\vee}(b)}{(b, b)} = s_{a,a^\vee}(2b/(b, b)),$$

so the problem is to check that the linear identification of V^* with V carries $s_{a^\vee,a}(b^\vee)$ over to $s_{a,a^\vee}(2b/(b, b))$. By definition,

$$s_{a^\vee,a}(b^\vee) = b^\vee - \langle a, b^\vee \rangle a^\vee \mapsto \frac{2b}{(b, b)} - \langle a, b^\vee \rangle \cdot \frac{2a}{(a, a)} \in V,$$

and $s_{a,a^\vee}(2b/(b, b)) = 2b/(b, b) - \langle 2b/(b, b), a^\vee \rangle a$, so we are reduced to checking that

$$\frac{\langle a, b^\vee \rangle}{(a, a)} \stackrel{?}{=} \frac{\langle b, a^\vee \rangle}{(b, b)}.$$

But we have already seen that $\langle x, c^\vee \rangle = (x, 2c/(c, c))$ for all $c \in R$ and $x \in V$, so the desired identity is clear. ■

An immediate consequence, via Proposition 1.13, is the following basic fact, for which we emphasize that no Euclidean structure has been specified (and hence it goes beyond the uniqueness in many expositions of the theory of of root systems).

Corollary 2.6. *For any root system (V, R), the map $R \to V^*$ defined by $a \mapsto a^\vee$ is uniquely determined by the properties in the axioms (so we may write s_a rather than $s_{a,a^\vee}$).*

In Bourbaki LIE VI, it is root systems and not root data which are studied. This is akin to the dichotomy between central isogeny classes of split connected semisimple groups and isomorphism classes of split connected reductive groups: all of the real work is at the level of the root system, but the root datum is necessary to keep track of things at a level finer than isogenies. Loosely speaking, passage from a root datum to its associated root system is akin to replacing the study of a connected reductive group with the study of the central isogeny class of its derived group. Likewise, in the theory of Lie algebras over $\mathbf{C}$ one has $\mathfrak{gl}_n = \mathbf{C} \oplus \mathfrak{sl}_n$ (so it hardly seems worthwhile to consider $\mathfrak{gl}_n$ in the structure theory) but at the level of algebraic groups GL_n is *not* the direct product of its central $\mathbf{G}_{\mathrm{m}}$ and SL_n. Isogenies also matter in representation theory; e.g., in characteristic 0 some irreducible representations of SL_n do not factor through PGL_n, even though their Lie algebras coincide. Hence, it is in the study of Lie groups and algebraic groups, rather than their Lie algebras, that the usefulness of the root datum (as opposed to the root system) becomes apparent.

By Proposition 2.5, we can view Corollary 1.15 as really being a statement about root systems: in any root system (V, R), if $a \in R$ and $c \in \mathbf{Q}$ satisfies $ca \in R$ then $c \in \{\pm 1/2, \pm 1, \pm 2\}$. This fact is proved in virtually every exposition of the theory of root systems, generally via Euclidean geometry by extending scalars to $\mathbf{R}$ and using a W-invariant inner product; e.g., see Proposition 8(i) in §1.3 of Bourbaki LIE VI. Many expositions of root systems *require* the specification of an inner product $(\cdot, \cdot)$ as part of the framework for the definitions (taking V to be an $\mathbf{R}$-vector space) and work directly with orthogonal reflections relative to the inner product (so $a^\vee$ is *defined* to be $2a/(a, a)$, subject to integrality conditions in the axioms). It seems more elegant (as in Bourbaki) to axiomatize these concepts without reference to a Euclidean structure, even though the choice of such an auxiliary structure is certainly convenient in proofs (as we saw in the proof of Proposition 2.5).

Remark 2.7. Remarkably, under a mild restriction, the W-invariant inner product on a root system (viewed over $\mathbf{R}$, say) is *unique* up to $\mathbf{R}^\times$-scaling (from which the analogous uniqueness holds over $\mathbf{Q}$ or any field of characteristic 0). To make this precise, we need a new concept. Observe that if (V, R) and (V', R') are two root systems, then $(V \oplus V', R \coprod R')$ is a root system, called the *direct sum* of the two given root systems. Note that the Weyl group of such a direct sum is identified with $W(R) \times W(R')$.

A root system (V, R) *reducible* if it is isomorphic to a direct sum of two nonzero root systems, and is *irreducible* if $V \neq 0$ and it is not reducible. In §1.2 of Bourbaki LIE VI the notion of irreducibility is studied, and it is proved there (allowing V to be over any field of characteristic 0) that (i) (V, R) is irreducible as a root system if and only if V is absolutely irreducible as a representation of the finite group $W(R)$, (ii) every root system is uniquely (up to relabeling) a direct sum of irreducible root systems. (The logically-inclined reader will verify that this is consistent with considering $(0, \emptyset)$ as a root system.)

Clearly if a root system is reducible then Weyl-invariant inner products can be chosen independently from each other on the different irreducible components. But in the irreducible case it makes sense to ask about the uniqueness up to scaling. Such uniqueness is immediate from Schur's Lemma, in view of the absolute irreducibility of the $W(R)$-action in the irreducible case (as this ensures that any two $W(R)$-equivariant isomorphisms $V \simeq V^*$ are scalar multiples of each other).

Curiously, in view of Corollary 2.6, for any (V, R) there is a *canonical* positive-definite symmetric $W(R)$-invariant bilinear form on V:

$$B_R(v, v') = \sum_{a \in R} \langle v, a^\vee \rangle \langle v', a^\vee \rangle.$$

This is visibly symmetric, it is $W(R)$-invariant due to the identity $s_b(a)^\vee = s_{b^\vee}(a^\vee)$, and it is positive-definite because $R^\vee$ spans V^* in any root system. (See Proposition 3 in §1.1 of Bourbaki LIE VI for another argument.) See §1.12 of Bourbaki LIE VI for further discussion of this bilinear form.

Example 2.8. Let (G, T) be a split connected *semisimple* k-group, with root datum $(X, R, X^\vee, R^\vee)$, and let Q be the lattice spanned over $\mathbf{Z}$ by R and Q' the $\mathbf{Z}$-span of $R^\vee$ in V^*. We call Q the *root lattice* and $P := (Q')^*$ (the vectors on which the coroots take values in $\mathbf{Z}$) the *weight lattice*. Note that Q and P are completely determined by the root system (V, R) (so they are insensitive to central isogenies in G). The reason that P is called the "weight lattice" is that in the application to complex semisimple Lie algebras, it turns out to be precisely the elements of $V = \mathrm{Lie}(T)^*$ that can arise as "highest weight" vectors in irreducible representations of $\mathfrak{g}$.

The root system (V, R) provides the pair $Q \subset P$ of $\mathbf{Z}$-lattices in V, which in turn are upper and lower bounds on the possibilities for X, since $Q \subset X \subset P$. The Existence, Isomorphism, and Isogeny Theorems (see Remark 2.12) imply that split connected semisimple k-groups G up to isomorphism correspond to isomorphism classes of triples (V, R, X) where (V, R) is a root system and X is an intermediate group between P and Q. Since the scheme-theoretic center Z_G is exactly the common kernel of all roots on T, it follows that the finite étale Cartier dual to Z_G is exactly the constant k-group X/Q. Hence, the case $X = Q$ corresponds to the condition $Z_G = 1$, and is called *adjoint* (because the kernel of $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ turns out to always be Z_G, so the case $Z_G = 1$ is precisely when the adjoint representation faithfully represents G on $\mathfrak{g}$). Likewise, the case $X = P$ corresponds to the center being "as large as possible", though it is by no means obvious for each root system (V, R) that such a case is necessarily realized by some split semisimple G over k. The key to proving the Existence Theorem is to construct, for each irreducible root system, a split connected semisimple (G, T) for which $\mathrm{X}(T) = P$; such G are called *simply connected* (because they turn out to satisfy a mapping property that is similar that of simply connected Lie groups). In general P/X is called the *fundamental group* of G because its Cartier dual turns out to be the automorphism group of the (unique) simply connected central covering $\widetilde{G} \to G$, and P/Q is called the fundamental group of the root system (V, R) (so it coincides with the fundamental group of G in the adjoint case).

Remark 2.9. Although split connected reductive groups only give rise to reduced root data, and so many texts ignore the non-reduced case, the latter are important! First of all, in the study of connected reductive k-groups G which are not necessarily split but do contain a non-trivial k-split torus (perhaps not maximal as a k-torus), one associates a so-called *relative root datum* which is a root datum that can be non-reduced. These already show up in the classification of connected semisimple $\mathbf{R}$-groups which are not split and have non-compact group of $\mathbf{R}$-points. The same happens over all fields that aren't separably closed.

In the classification of root data via root systems, the only "irreducible" cases for which there are roots which are non-trivially divisible in the character lattice X are "simply connected" type C, which correspond to symplectic groups (so in fact non-reducedness is a somewhat rare occurrence, but it cannot be entirely ignored). For example, SL_2 has its roots that are divisible by 2 in the character lattice, but for PGL_2 this does not happen.

In the classification of irreducible root systems (as in Bourbaki LIE VI), for each $n \geq 1$ there is (up to isomorphism) a unique rank-n irreducible root system that is non-reduced. It is called BC_n because it is a union of the root systems B_n and C_n (within the same n-dimensional V). This non-reduced root system arises "in nature" via the exceptional (purely inseparable) non-central isogeny $\mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n}$ in characteristic 2 that carries an orthogonal transformation of a $(2n+1)$-dimensional (V, q) to the induced automorphism of the associated symplectic space $(V/V^{\perp}, B_q)$ of dimension $2n$, where $V^{\perp}$ is the defect line for B_q on V. This exceptional isogeny puts the roots of both groups within the common $\mathbf{Q}$-vector space associated to compatible maximal tori, thereby yielding BC_n.

There is much more, such as relating the root datum to the subgroup structure. We end with the proof (conditional on results in the theory of root systems) that the containment $W(R(G,T)) \subseteq W(G,T)$ is an equality (i.e., $W(G,T)$ is generated by the reflections s_a). This rests on:

Proposition 2.10. *A Borel subgroup B in G containing T is uniquely determined by the set $\Phi(B)$ of roots a such that $\mathfrak{g}_a \subseteq \mathrm{Lie}(B)$: it is generated by T and the root groups U_a for all such a.*

Proof. We may choose a regular cocharacter $\lambda \in \mathrm{X}_*(T)$ such that $B = B(\lambda) := T \ltimes U_G(\lambda)$ (since such a λ exists over $\overline{k}$ and $\mathrm{X}_*(T_{\overline{k}}) = \mathrm{X}_*(T)$ due to T being k-split). Hence,

$$\mathrm{Lie}(B) = \mathrm{Lie}(T) \oplus (\oplus_{\langle a,\lambda\rangle > 0} \mathfrak{g}_a).$$

It follows that if $a \in \Phi(G,T)$ and $\mathfrak{g}_a \subset \mathrm{Lie}(B)$ then $\langle a, \lambda \rangle > 0$. But for any such a we have $U_a \simeq \mathbf{G}_{\mathrm{a}}$ on which any $\lambda(t)$ ($t \in \mathbf{G}_{\mathrm{m}}$) acts as scaling by $a(\lambda(t)) = t^{\langle a,\lambda\rangle}$ with $\langle a, \lambda\rangle > 0$. Thus, the functorial characterization of $U_G(\lambda)$ gives that $U_a \subseteq U_G(\lambda)$. Varying over all such a, the k-subgroups U_a in $U_G(\lambda)$ have Lie algebras that directly span $\mathrm{Lie}(U_G(\lambda))$, so the smooth connected k-subgroup they generate must equal $U_G(\lambda)$ (as $U_G(\lambda)$ is connected). But $B(\lambda) = T \ltimes U_G(\lambda)$, so $B(\lambda)$ is generated by T and the root groups U_a for those roots a whose weight space is contained in $\mathrm{Lie}(B(\lambda))$. ∎

Within the theory of root systems, there is the concept of a *positive system of roots* (see §1.6–§1.7 in Bourbaki LIE VI): these can be defined in several non-obviously equivalent ways, one of which is the sets of roots cut out by a condition $\langle a, \lambda\rangle > 0$ for a linear form λ on $X_{\mathbf{Q}}$ that is non-vanishing on all roots (see Corollary 1 in §1.7 of Bourbaki LIE VI). It is a general fact that the Weyl group of the root system simply transitively permutes the set of such positive systems (see Theorem 2(i) in §1.5 of Bourbaki LIE VI). Such positive systems Φ^+ in $\Phi(G,T)$ are exactly the sets of roots that occur in the Lie algebra of a Borel subgroup containing T.

Corollary 2.11. *Let (G,T) be a split connected reductive group over a field k. The inclusion of groups $W(R(G,T)) \subset W(G,T)$ is an equality.*

Proof. Choose $w \in W(G,T)$. By the definitions, clearly $\Phi(w.B) = w.\Phi(B)$. Since $W(R(G,T))$ acts (simply) transitively on the set of all positive systems of roots in $\Phi(G,T)$, there exists w' in the subgroup $W(R(G,T))$ such that $w.\Phi(B) = w'.\Phi(B)$, so $\Phi(w^{-1}w'.B) = \Phi(B)$. By Proposition 2.10 this forces $w^{-1}w'.B = B$, and hence (by Proposition 1.13 in the handout on basics of reductivity and semisimplicity) $w = w'$! ∎

Not only have we proved that the Weyl group of (G,T) is exactly the Weyl group of the associated root datum (or root system), but we also showed that the set of Borel subgroups containing T is in natural bijective correspondence with the set of positive systems of roots in the root system.

Remark 2.12. The next step is to formulate and prove the Existence, Isomorphism, and Isogeny theorems which characterize isomorphism classes of k-split pairs (G,T) *up to the* $(T/Z_G)(k)$*-action* *on* G in terms of root data, as well as characterize isogenies between two such pairs in terms of the root data. (Beware that typically $T(k)/Z_G(k)$ is smaller than $(T/Z_G)(k)$ when Z_G is not a torus, such as $G = \mathrm{SL}_n$ with $k^\times \neq (k^\times)^n$.) This can also be refined via an additional structure called a *pinning* which removes interference of the $(T/Z_G)(k)$-action on G.

To prove the Existence theorem one "just" has to find a split semisimple group realizing each reduced irreducible root system, with $X = P$. Special linear and symplectic groups are types A and C respectively, and spin groups of split quadratic forms (built via Clifford algebras) give types B and D. For some exceptional root systems there are constructions based on octonion algebras, Jordan algebras, and so on. I believe that there is no known construction for E_8 in all characteristics other than the uniform method which applies to all reduced irreducible root systems.