

1. MOTIVATION

This handout aims to establish in the general case two key features of the split case: the applicability of BN-pair formalism (a.k.a. “Tits systems”) and a theory of relative root groups which permits a refined formulation of the relative Bruhat decomposition (corresponding geometrically to the description of each locally closed Bruhat cell as an explicit product variety of a Borel subgroup against an affine space). We also use root groups to relate *k-isotropic* “*k*-simple factors” to irreducible components of ${}_k\Phi$. We saw with both Weil restriction and special unitary groups in class (see §11.2) that relative root groups can have dimension larger than 1, and that this can occur even in the absolutely simple case.

In the split case, much of this was explained in the handout on the geometric Bruhat decomposition. Let’s recall the main argument concerning the refined Bruhat decomposition in the split case. Let G be a split connected reductive group over a field k . Choose a split maximal torus T and a Borel k -subgroup B . Let $W = N_G(T)(k)/T(k) = W(\Phi(G, T))$. The Bruhat decomposition gives

$$G(k) = \coprod_{w \in W} B(k)n_w B(k)$$

where $n_w \in N_G(T)(k)$ is a choice of representative of w . There is usually tremendous non-uniqueness in expressions $bn_w b'$ for a given $g \in G(k)$, so for computations and other purposes it is convenient to refine our description of these double cosets.

For any $w \in W$, let $\Phi_w^+ = \{a \in \Phi^+ \mid w^{-1}.a \in \Phi^+\}$ and let $\Phi'_w = \Phi^+ - \Phi_w^+$ be its complement. Choose an enumeration of Φ^+ for which Φ'_w appears before Φ_w^+ . Thus,

$$Un_w = \prod_{a \in \Phi'_w} U_a \times \prod_{a \in \Phi_w^+} U_a n_w = \left(\prod_{a \in \Phi'_w} U_a \right) n_w \cdot \prod_{a \in \Phi_w^+} (n_w^{-1} U_a n_w).$$

But $n_w^{-1} U_a n_w = U_{w^{-1}.a} \subset U$ for $a \in \Phi_w^+$, so

$$Un_w U = \left(\prod_{a \in \Phi'_w} U_a \right) n_w \times U = \prod_{a \in \Phi'_w} U_a \times n_w \prod_{a \in \Phi^+} U_a.$$

The same computation works on k -points, and if we include a factor of T (which normalizes U and is normalized by $N_G(T)(k)$) then we arrive at the description

$$B(k)n_w B(k) = \prod_{a \in \Phi'_w} U_a(k) \times T(k)n_w \times \prod_{a \in \Phi^+} U_a(k)$$

via multiplication inside $G(k)$. This really is a direct product decomposition, since we can move n_w all the way to the left to arrive at the equivalent description

$$n_w \cdot \prod_{a \in \Phi'_w} U_{w^{-1}.a}(k) \times T(k) \times \prod_{a \in \Phi^+} U_a(k) \subset n_w(U^- \times T \times U)(k)$$

inside a translate of the open cell associated to the pair (T, B) .

In this description, the order of enumeration Φ'_w might seem to matter in terms of the image of $\prod_{a \in \Phi'_w} U_a(k)$ inside $B(k)$. To explain why it is independent of the enumeration, we shall use a dynamic technique from [CGP] whose proof rests on elementary properties of

root systems: any closed set of roots Ψ in a (possibly non-reduced) root system Φ_0 is the intersection $\Phi_0 \cap A$ of Φ_0 with a subsemigroup A of the ambient \mathbf{Q} -vector space V generated by Φ_0 , so any such Ψ is the set of roots in the subsemigroup of V generated by Ψ . This latter condition on a subset Ψ of Φ_0 is called being *saturated* (so any closed set of roots is saturated).

For any saturated set of roots Ψ in the reduced root system ${}_k\Phi$, we will see in Proposition 3.3 that for general G (without a split hypothesis) the associated root groups U_a ($a \in \Psi$) directly span *in any order* a common unipotent smooth connected k -subgroup $U_\Psi \subset G$ (normalized by T). This rests on a “direct spanning” construction from [CGP, §3.3] that has absolutely *nothing* to do with the theory of reductive groups.

We can summarize the refined Bruhat decomposition in the split case as

$$B(k)n_w B(k) = U_{\Phi'_w}(k) \times T(k) \times U_{\Phi^+}(k),$$

where $U_\Psi := \prod_{a \in \Psi} U_a$ (multiplication *in any order*) for any saturated subset Ψ of $\Phi(G, T)$. As was noted in Remark 1.1 in the handout on the geometric Bruhat decomposition, $\#\Phi'_w$ is the length $\ell(w)$ of w relative to the set of reflections $\{r_a\}_{a \in \Delta}$ in $W(G, T)$ giving a Coxeter presentation, where Δ is the basis of Φ^+ . This gives geometric meaning to the Bruhat cells $Bn_w B = \mathbf{A}_k^{\ell(w)} \times n_w B$ in the split case. A version of this in the general case is given on k -points in Corollary 3.8 using a direct spanning property on k -points, and it is crucial in many applications (such as the development of Bruhat–Tits theory).

2. ROOT GROUPS AND TITS SYSTEMS

Now let G be an arbitrary connected reductive group over a field k , and let S be a maximal k -split torus. Let S_0 be the maximal k -split torus and $S' = (S \cap G')_{\text{red}}^0$ (where $G' := \mathcal{D}(G)$), so $S_0 \times S' \rightarrow S$ is an isogeny and under the decomposition $X(S)_{\mathbf{Q}} = X(S_0)_{\mathbf{Q}} \oplus X(S')_{\mathbf{Q}}$. ${}_k\Phi = \Phi(G, S)$ spans $X(S')_{\mathbf{Q}}$ making $(X(S')_{\mathbf{Q}}, {}_k\Phi)$ a root system. For the rest of the handout we may as well assume ${}_k\Phi \neq \emptyset$, which is to say S is not central in G , as otherwise there is no content to anything that follows.

We begin by defining the *root group* $U_a \subset G$ associated to each $a \in {}_k\Phi$. Beware that in general (unlike in the split case) this may have rather large dimension, and can even be non-commutative; we will discuss the precise structure of such U_a later. The crux of the matter is the following general dynamical construction that has *nothing* to do with reductive groups and was recorded earlier in the course when we prepared for the discussion of subgroups directly spanned by root groups in the split case.

Proposition 2.1. *Let G be a smooth connected affine k -group equipped with an action by a split k -torus S , and let $A \subseteq X(S)$ be a subsemigroup not containing 0. There exists a unique S -stable smooth connected k -subgroup $U_A(G)$ such that $\text{Lie}(U_A(G))$ is the span of the a -weight spaces for all $a \in A \cap \Phi(G, S)$. This k -subgroup is unipotent and contains any S -stable smooth connected k -subgroup $H \subseteq G$ such that all S -weights on $\text{Lie}(H)$ occur in A .*

Additional properties of $U_A(G)$ will be reviewed as the need arises below.

Proof. This is [CGP, Prop. 3.3.6], where the notation $H_A(G)$ is used (as it is not assumed there that $0 \notin A$, so $H_A(G)$ might not be unipotent). ■

For any $a \in X(S) - \{0\}$, let $\langle a \rangle$ be the semigroup consisting of positive integral multiples of a . For $a \in {}_k\Phi$, $U_a := U_{\langle a \rangle}(G)$ is the smooth connected unipotent k -subgroup in G that is normalized by S and for which $\text{Lie}(U_a)$ is the span of the weight spaces in $\text{Lie}(G)$ for the weights in $\Phi(G, S)$ that are *positive integral* multiples of a . Moreover, U_a contains every smooth connected k -subgroup $H \subseteq G$ normalized by S such that S acts on $\text{Lie}(H)$ only with weights that are positive integral multiples of a . In particular, $U_{2a} \subseteq U_a$ if $2a \in {}_k\Phi$.

Definition 2.2. The k -group U_a is called the *root group of G associated to $a \in {}_k\Phi$* .

We may sometimes refer to such U_a are *relative root groups* for emphasis (to distinguish from the split case). Here is a more concrete description of U_a . Let $\mathcal{S}_a = \ker a$ (which might not be smooth if a is divisible in $X(S)$, and can happen when ${}_k\Phi$ has irreducible factors of type C_n or type BC_n with $n > 1$) and let $G_a = Z_G(\mathcal{S}_a)^0$. By the handout ‘‘Reductive centralizer’’, G_a is a connected reductive k -subgroup of G (visibly containing S), and $\Phi(G_a, S)$ has rank 1: it consists of the roots that are integer multiples of a (since we used the centralizer of the entire kernel of a , not just $(\ker a)^0$).

Lemma 2.3. *As k -subgroups of G , we have $U_a \subset G_a$.*

Proof. The smooth connected root group U_a is normalized by S . We have to show that under the resulting S -action, the closed k -subgroup $\ker a$ acts trivially on U_a . Its effect on the span $\text{Lie}(U_a)$ of the weight spaces inside \mathfrak{g} for the positive integral multiples of a in ${}_k\Phi$ is certainly trivial.

But rather generally, the action on a smooth connected affine group by a multiplicative type group is trivial when the action on the Lie algebra is trivial! This follows from a calculation on infinitesimal neighborhoods of the identity, using the complete reducibility of linear representations of multiplicative type groups. A proof is given in the self-contained second paragraph of the proof of [CGP, Cor. A.8.11]. ■

For $\lambda_a \in X_*(S)$ such that $\langle a, \lambda_a \rangle > 0$, we have $U_{G_a}(\lambda_a) \subseteq U_{U_a}(\lambda_a) = U_a$ inside G_a . This inclusion is between smooth connected k -groups, and it is an equality on Lie algebras (as $\Phi(G_a, S)$ consists of the integer multiples of a inside ${}_k\Phi$, and $\text{Lie}(U_a)$ is the span of the positive integral multiples of a in ${}_k\Phi$). Thus, we get an equality as k -subgroups: $U_a = U_{G_a}(\lambda_a)$.

Now we obtain two nice consequences.

Proposition 2.4. *If $H \subset G$ is a closed k -subgroup scheme normalized by S then $U_a \cap H = U_H(\lambda_a)$. In particular, if H is smooth and connected then so is $U_a \cap H$.*

Since $P_{G_a}(\pm\lambda_a)$ are the two proper parabolic k -subgroups of G_a , we recover an alternative description of U_a :

Proposition 2.5. *For the unique proper parabolic k -subgroup $P_a \subset G_a = Z_G(\ker a)^0$ that contains S and has a as an S -weight on its Lie algebra, $U_a = \mathcal{R}_{u,k}(P_a)$.*

Since $P_{\pm a}$ are the only two proper parabolic k -subgroups of G_a containing S , so each is minimal, they each contain $Z_{G_a}(S) = Z_G(S)$ as a Levi factor. In particular, U_a is normalized by $Z_G(S)$ for any $a \in {}_k\Phi$. The dynamic description $U_a = U_{G_a}(\lambda_a)$ also implies that U_a is a k -split smooth connected unipotent group (a general property of the $U(\lambda)$ construction

[CGP, Prop. 2.1.10]). When G is split, so ${}_k\Phi$ is a reduced root system, we recover the notion of root group in the split theory.

The commutativity and vector group properties of relative root groups will be explored later. For now, we use root groups to establish a link with the notion of Tits system. Recall from the previous course:

Definition 2.6. A *Tits system* (or BN-pair) is a 4-tuple $(\mathcal{G}, B, N, \Sigma)$ where \mathcal{G} is an abstract group, B and N are subgroups, and $\Sigma \subseteq N/(B \cap N)$ is a subset such that the following four axioms are satisfied:

- (T1) $B \cup N$ generates \mathcal{G} and $B \cap N$ is normal in N ,
- (T2) the elements of Σ have order 2 in the quotient $W := N/(B \cap N)$ and generate W ,
- (T3) for all $\sigma \in \Sigma$ and $w \in W$, $\sigma B w \subseteq B w B \cup B \sigma w B$ (using any representatives for σ and w in N , the choices of which do not matter),
- (T4) $\sigma B \sigma \not\subseteq B$ for all $\sigma \in \Sigma$ (which is equivalent to $\sigma B \sigma \neq B$, since $\sigma^2 = 1$ in W).

We refer the reader to §2 in Chapter IV of Bourbaki for the basic properties of Tits systems. By Remark (1) in §2.5 of Chapter IV of Bourbaki, Σ is uniquely determined by (\mathcal{G}, B, N) (thereby explaining why such data is often called a BN-pair for \mathcal{G}).

Theorem 2.7 (Borel–Tits). *Let $N = N_G(S)$, $Z = Z_G(S)$, and P a minimal parabolic k -subgroup of G containing S . Let ${}_k\Delta$ be the basis of the positive system of roots ${}_k\Phi^+ = \Phi(P, S)$, and let $R = \{r_a \mid a \in {}_k\Delta\}$ be the associated set of simple positive reflections. The 4-tuple $(G(k), P(k), N(k), R)$ is a Tits system with Weyl group ${}_kW (= N(k)/Z(k))$.*

This is the *standard Tits system* associated to (G, S, P) . Since $P(k) \cap N(k) = Z(k)$, we have $N(k)/(P(k) \cap N(k)) =: {}_kW = W({}_k\Phi)$. That is, the Weyl group of the standard Tits system naturally coincides with $W(G, S)$.

Proof. The relative Bruhat decomposition $G(k) = \coprod_{w \in {}_kW} P(k)n_w P(k)$ implies axiom (T1). Moreover, the quotient group $N(k)/Z(k) = {}_kW = W({}_k\Phi)$ is generated by R . This is axiom (T2). To prove axiom (T4), for $r = r_a \in R$ (with $a \in {}_k\Delta \subset {}_k\Phi$) it suffices to prove that $rP(k)r \neq P(k)$. Clearly rPr contains $rU_a r = U_{r(a)} = U_{-a}$, so it suffices to prove that $U_{-a}(k)$ is not contained in $P(k)$. We will prove $U_{-a} \cap P = 1$, so $U_{-a}(k) \cap P(k) = 1$, which does the job since $U_{-a}(k) \neq 1$ (as the unipotent smooth connected k -group U_{-a} is nontrivial and k -split).

By Proposition 2.4, $U_{-a} \cap P$ is *smooth and connected*. But its Lie algebra is trivial since $\Phi(P, S) = {}_k\Phi^+$ does not contain any S -weights that occur on U_{-a} , so $U_{-a} \cap P = 1$ as desired.

It remains to prove axiom (T3). That is, for $r := r_a$ (with $a \in {}_k\Delta$) and $w \in {}_kW$ we claim that

$$(*) \quad rP(k)\{w, rw\}P(k) \subseteq P(k)\{w, rw\}P(k).$$

This is given in the proof of 21.15 in the 2nd edition of Borel’s textbook on linear algebraic groups, and here we give a mild reformulation of that calculation.

Let $\lambda : \mathrm{GL}_1 \rightarrow S$ be a 1-parameter k -subgroup such that $P = P_G(\lambda) = Z_G(\lambda) \rtimes U_G(\lambda)$, so the minimality of P implies that $Z_G(\lambda) = Z_G(S) = Z$ and $U_G(\lambda) = \mathcal{R}_{u,k}(P)$. Thus, $P = Z \rtimes \mathcal{R}_{u,k}(P)$ and ${}_k\Phi^+ = \Phi(P, S) = ({}_k\Phi)_{\lambda \geq 0} = ({}_k\Phi)_{\lambda > 0}$, so $\langle a, \lambda \rangle > 0$. More generally, a

k -root $b \in {}_k\Phi$ is *positive* (i.e., it lies in ${}_k\Phi^+$) if and only if $\langle b, \lambda \rangle > 0$. If necessary, replacing w with rw , we may (and we will) assume that $w^{-1}(a)$ is positive.

Now let $U = \mathcal{R}_{u,k}(P) = U_G(\lambda)$, so for $G_a := Z_G(\ker a)^0$ the root group U_a is equal to $U_{G_a}(\lambda) = G_a \cap U$. Let $S_a = (\ker a)_{\text{red}}^0$. As a is nondivisible in ${}_k\Phi$ (since $a \in {}_k\Delta!$), an element of ${}_k\Phi$ is a rational multiple of a if and only if it is an integral multiple of a . Thus, the containment $G_a \subseteq Z_G(S_a)$ between smooth connected k -groups is an equality via comparison of their Lie algebras.

Choose a 1-parameter k -subgroup $\lambda_a : \text{GL}_1 \rightarrow S_a$ such that $\langle b, \lambda_a \rangle > 0$ for all $b \in {}_k\Delta - \{a\}$. Note that $\langle a, \lambda_a \rangle = 0$, so $G_a = Z_G(\lambda_a)$. Let $V = U_G(\lambda_a)$, so $P_G(\lambda_a) = Z_G(\lambda_a) \times U_G(\lambda_a) = G_a \times V$ and hence $U = U_a \times V$. We also noted above that $P = Z \times U$. Since G_a contains Z as well as a representative of $r = r_a$ (by definition of r_a), it follows that V is normalized by r and hence,

$$rP(k)\{w, rw\}P(k) = Z(k)rU(k)\{w, rw\}P(k) = Z(k)V(k)rU_a(k)\{w, rw\}P(k).$$

Thus,

$$rP(k)\{w, rw\}P(k) \subseteq P(k)G_a(k)wP(k).$$

For the parabolic k -subgroup $P_a := P_{G_a}(\lambda) = G_a \cap P = Z \times U_a$ of the *connected reductive* G_a , the Bruhat decomposition for $G_a(k)$ with respect to $P_a(k)$ implies that

$$G_a(k) = Z(k)U_a(k)\{1, r\}U_a(k),$$

so

$$rP(k)\{w, rw\}P(k) \subseteq P(k)\{1, r\}U_a(k)wP(k).$$

But the positivity of $w^{-1}(a)$ implies that $w^{-1}U_a w \subset P$, so

$$rP(k)\{w, rw\}P(k) \subseteq P(k)\{w, rw\}P(k).$$

■

Remark 2.8. Since ${}_kW = W({}_k\Phi)$ acts simply transitively on the set of minimal parabolic k -subgroups of G containing S (denote this action as $(w, P) \mapsto w \circ P$), if P is such a k -subgroup corresponding to a positive system of roots ${}_k\Phi^+$ then for a suitable $w_0 \in {}_kW$ the k -subgroup $w_0 \circ P$ corresponds to the positive system of roots $-{}_k\Phi^+$. Writing $P = P_G(\lambda) = Z_G(\lambda) \times U_G(\lambda)$ for some $\lambda \in X_*(S)$, we have $w_0 \circ P = P_G(-\lambda)$. Thus, $P \cap w_0 \circ P = Z_G(\lambda) = Z_G(S) =: Z$, so $Z(k) \subseteq \bigcap_{w \in {}_kW} wP(k)w^{-1} \subseteq P(k) \cap w_0P(k)w_0^{-1} = Z(k)$; i.e., $Z(k) = \bigcap_{w \in {}_kW} wP(k)w^{-1}$.

For $N := N_G(S)$ we have $Z(k) = P(k) \cap N(k)$ due to the minimality of P , so

$$\bigcap_{n \in N(k)} nP(k)n^{-1} = P(k) \cap N(k).$$

This says that the Tits system in Theorem 2.7 is *saturated* in the sense of Exercise 5(a) in §2 of Chapter IV in Bourbaki. The established property that $P(k)$ is a semidirect product of its subgroup $Z(k) = P(k) \cap N(k)$ against a nilpotent subgroup $U_G(\lambda)(k)$ is expressed by saying that that the Tits system is *split*.

In §2 of the handout on Tits systems, we saw that the connection to Tits systems in the split case yields striking simplicity results for $G(k)/Z_G(k)$ in the split simply connected case. That simplicity proof relied on an equality “ $G(k) = G(k)^+$ ” which says that $G(k)$ is

generated by the rational points of the split k -unipotent radicals of parabolic k -subgroups of G (or equivalently, just for the minimal parabolic k -subgroups, since if $Q \subset P$ is a containment of parabolic k -subgroups then $\mathcal{R}_{u,k}(P) \subset \mathcal{R}_{u,k}(Q)$, as we may check over \bar{k}). This property in the split case is not true in general, and its validity over global fields lies quite deep (see §3 in the handout on Tits systems). Consequently, we do not discuss simplicity results for $G(k)/Z_G(k)$ beyond the split case, and instead demonstrate the power of the link with Tits systems via the following striking result.

Theorem 2.9. *With notation as in Theorem 2.7, the following hold:*

- (i) *There are $2^{\#R}$ subgroups of $G(k)$ containing $P(k)$, and any such subgroup equals $Q(k)$ for a unique parabolic k -subgroup Q containing P .*
- (ii) *For parabolic k -subgroups Q and Q' of G , $Q \subset Q'$ if and only if $Q(k) \subset Q'(k)$.*

Informally, this theorem says that “parabolic subgroups” of $G(k)$ relative to its Tits system structure (see §2.6 in Chapter IV of Bourbaki for this terminology) are in inclusion-preserving bijective correspondence with the parabolic k -subgroups of G via the formation of rational points. Note that we allow the case that k is finite, and in (ii) we don’t assume Q and Q' share a common split maximal k -torus.

Proof. Since $(G(k), P(k), N(k), R)$ is a Tits system (with Weyl group ${}_k W$), by Theorem 3(b) in §2.5 of Chapter IV in Bourbaki there is a natural bijection $X \mapsto G(k)_X$ from the set of subsets $X \subseteq R$ to the set of subgroups of $G(k)$ containing $P(k)$, so there are exactly $2^{\#R}$ subgroups of $G(k)$ containing $P(k)$.

Explicitly, in terms of the relative Bruhat decomposition $G(k) = \coprod_{w \in {}_k W} P(k)n_w P(k)$ for a choice of representative $n_w \in N(k)$ of each $w \in {}_k W$, $G(k)_X$ is the union of the $P(k)$ -double cosets indexed by the elements of the subgroup ${}_k W_X$ of ${}_k W$ generated by X . Note that the subgroup $G(k)_X$ is the unique subgroup of $G(k)$ containing $P(k)$ such that the image of $N(k) \cap G(k)_X$ in ${}_k W$ is ${}_k W_X$. (In particular, ${}_k W_X$ determines X .) In general ${}_k W_X \cap {}_k W_{X'} = {}_k W_{X \cap X'}$, so $G(k)_X \cap G(k)_{X'} = G(k)_{X \cap X'}$. It follows that if $G(k)_X \subseteq G(k)_{X'}$ then $G(k)_X = G(k)_{X \cap X'}$ and hence $X = X \cap X' \subseteq X'$.

Any smooth closed k -subgroup Q of G containing P is parabolic, and there are exactly $2^{\#R}$ such k -subgroups (corresponding to subsets I of ${}_k \Delta$). We shall now display them all. For each subset $I \subseteq {}_k \Delta$, we fix a cocharacter $\lambda_I \in X_*(S)$ such that $\langle a, \lambda_I \rangle \geq 0$ for all $a \in {}_k \Delta$, with equality if and only if $a \in I$. Define $P_I = P_G(\lambda_I) = Z_G(\lambda_I)U_G(\lambda_I)$. In view of how ${}_k \Delta$ is defined in terms of P , by weight space computations in \mathfrak{g} we see that $P = P_\emptyset$ and $P_J \subseteq P_{J'}$ when $J \subseteq J'$; in particular, $P \subseteq P_I$. We will prove that if $J \neq J'$ then $P_J(k) \neq P_{J'}(k)$ (so $P_J \neq P_{J'}$). It would then follow that the collection $\{P_I \mid I \subseteq {}_k \Delta\}$ consists of $2^{\#{}_k \Delta} = 2^{\#R}$ parabolic k -subgroups of G containing P , and the collection $\{P_I(k) \mid I \subseteq {}_k \Delta\}$ consists of $2^{\#R}$ subgroups of $G(k)$ containing $P(k)$. Hence, the former would be the collection of all parabolic k -subgroups of G containing P and the latter would be the collection of all subgroups of $G(k)$ containing $P(k)$, so (i) would be proved.

Fix an I , and let $N_I = N \cap P_I = N_{P_I}(S)$. We have noted above that I is determined by the subgroup ${}_k W_I \subseteq {}_k W$ generated by $\{r_a\}_{a \in I} \subseteq R$, so it suffices to show that the image of $N(k) \cap P_I(k) = N_I(k)$ in ${}_k W$ is ${}_k W_I$ (and then $P_I = G(k)_I$, due to the unique

characterization of $G(k)_X$ noted above). This image $N_I(k)/Z(k)$ is the “relative Weyl group” $N_{P_I}(S)(k)/Z_{P_I}(S)(k)$ of (P_I, S) , though P_I isn’t reductive for $I \neq {}_k\Delta$.

Let’s first show that the subgroup $W(Z_G(\lambda_I), S)(k) \subseteq {}_k W$ is generated by $\{r_a\}_{a \in I}$ (i.e., it coincides with ${}_k W_I$). Due to how λ_I was defined (with $I \subset {}_k\Delta$), the root system ${}_k\Phi_I := \Phi(Z_G(\lambda_I), S)$ is the span of I inside ${}_k\Phi$, with I moreover as a basis. Thus, $W(Z_G(\lambda_I), S)(k)$ is generated by reflections $\{r'_{a,I}\}_{a \in I}$. The natural inclusion $W(Z_G(\lambda_I), S)(k) \hookrightarrow {}_k W$ inside $\mathrm{GL}(X(S)_{\mathbf{Q}})$ carries $r'_{a,I}$ to r_a for each $a \in I$, due to the well-known fact that for any nonzero finite-dimensional vector space V over a field of characteristic 0 and any *finite* subgroup $F \subset \mathrm{GL}(V)$, a reflection $r \in F$ is uniquely determined by the line that it negates.

We have shown that ${}_k W_I \subset N_I(k)/Z(k)$ via the inclusion of $Z_G(\lambda_I)$ into $P_G(\lambda_I) = P_I$, and to prove equality it suffices to show that these finite groups have the same size. More specifically, using the natural identification of $Z_G(\lambda_I)$ with the maximal reductive quotient $\overline{P}_I = P_G(\lambda_I)/U_G(\lambda_I)$ of P_I and the isomorphism of S onto its maximal k -split torus image \overline{S} in \overline{P}_I , to complete the proof of (i) it suffices to show that the natural map $N_I(k)/Z(k) \rightarrow W(\overline{P}_I, \overline{S})(k)$ is an isomorphism. The key point is that the inclusion $N_I(k)/Z(k) \rightarrow W(P_I, S)(k) := (N_{P_I}(S)/Z_{P_I}(S))(k)$ is an equality. This is an analogue for P_I of what we know for G , and the proof of that result for connected reductive groups was entirely formal up to the known fact that all maximal k -split tori of P_I are $P_I(k)$ -conjugate (as P_I is a parabolic k -subgroup of the connected reductive G) and the following two facts:

Lemma 2.10. *For any minimal parabolic k -subgroup P of P_I containing S , $N_{P_I}(S) \cap P = Z_{P_I}(S)$. Moreover, the minimal parabolic k -subgroups of P_I are $P_I(k)$ -conjugate.*

Proof. The first assertion is inherited from the same for G in place of P_I (as a parabolic k -subgroup of P_I is precisely a parabolic k -subgroup of G contained in P_I , due to the parabolicity of P_I). For the second assertion, note that any parabolic k -subgroup of P_I must contain the *normal* split unipotent smooth connected k -subgroup $U_G(\lambda_I)$. Thus, we get an inclusion-preserving bijection between the sets of parabolic k -subgroups of P_I containing S and those of the connected reductive $\overline{P}_I = P_I/U_G(\lambda_I)$ containing the maximal k -split torus image of S . Hence, the known k -rational conjugacy result for the reductive \overline{P}_I implies the same for P_I because $P_I(k) \rightarrow \overline{P}_I(k)$ is surjective (as $U_G(\lambda_I)$ is split unipotent). ■

We may now recast our problem in terms of finite étale k -groups rather than groups of k -points: we want $N_I/Z \rightarrow W(\overline{P}_I, \overline{S})$ to be an isomorphism. This is a special case of a general fact having nothing to do with reductive groups:

Lemma 2.11. *Let H be a smooth connected affine group over a field k , U a normal unipotent smooth connected k -subgroup, and $H' = H/U$. Let T be a k -torus in H and let T' be its image in H' . The natural map $N_H(T)/Z_H(T) \rightarrow N_{H'}(T')/Z_{H'}(T')$ between finite étale k -groups is an isomorphism.*

Proof. This is [CGP, Lemma 3.2.1] (which permits $U = \ker(H \twoheadrightarrow H')$ to be a bit more general, but whose proof is self-contained relative to our situation). ■

To prove assertion (ii) in Theorem 2.9, consider arbitrary parabolic k -subgroups Q and Q' of G such that $Q(k) \subseteq Q'(k)$. Fix a minimal parabolic k -subgroup P contained in Q . By the conjugacy of minimal parabolic k -subgroups of G , there exists $g \in G(k)$ such that

$gPg^{-1} \subseteq Q'$. Hence, $Q'(k)$ contains a $G(k)$ -conjugate of $P(k)$, so it is a “parabolic subgroup” of $G(k)$ relative to the Tits system structure. But $P(k)$ and $gP(k)g^{-1}$ are contained in $Q'(k)$, so $g \in Q'(k)$ by Theorem 4(i) in §2.6 of Chapter IV of Bourbaki. Thus, $P \subseteq Q'$, so Q and Q' are parabolic k -subgroups containing the common minimal parabolic k -subgroup P . But then $Q \cap Q'$ is another such k -subgroup yet its k -points coincide with those of $Q(k)$, so (i) implies that $Q = Q \cap Q' \subseteq Q'$. ■

3. STRUCTURE OF ROOT GROUPS AND DIRECT SPANNING

To begin a deeper study of root groups, we first establish that U_a equipped with its S -action is a vector group with S acting through a , except for possibly when $2a \in {}_k\Phi$. Even in this multipliable cases, we can say something useful.

Throughout this section, we appeal to the largely self-contained [CGP, §3.3] for general results concerning the action of a split torus on a smooth connected affine group (having *nothing* to do with the reductive case).

Proposition 3.1. *Let G be a connected reductive k -group, S a maximal k -split torus in G , $N = N_G(S)$, and $Z = Z_G(S)$. Let ${}_k\Phi = \Phi(G, S)$. Choose $a \in {}_k\Phi$ and let $U_{\pm a}$ be the root groups associated to $\pm a$.*

If $2a \notin {}_k\Phi$ then U_a is a vector group admitting a unique linear structure relative to which the S -action is linear. If $2a \in {}_k\Phi$ then $(U_a, U_a) \subseteq U_{2a}$, $U_{2a} \subset Z_{U_a}$, and U_a/U_{2a} is a vector group admitting a unique linear structure relative to which the S -action is linear. In such multipliable cases, the underlying k -scheme of U_a is isomorphic to an affine space.

Proof. The centralizer $Z_G(\ker a)$ has reductive identity component and contains $U_{\pm a}$. Since $Z_G(S) \subseteq Z_G(\ker a)^0$, so we may (and do) replace G with $Z_G(\ker a)^0$. Now ${}_k\Delta = \{a\}$, the Weyl group ${}_kW$ is a group of order 2 generated by r_a , and $P := Z \rtimes U_{-a}$ is a minimal parabolic k -subgroup of G .

Since ${}_k\Phi$ is a root system, the only positive integral multiple of a , other than a , which can be in ${}_k\Phi$ is $2a$. By construction, $U_a = U_{\langle a \rangle}(G)$ and $U_{2a} = U_{\langle 2a \rangle}(G)$, so general dynamic principles (see [CGP, Prop. 3.3.5]) ensure that if $2a \in {}_k\Phi$ then $(U_a, U_a) \subseteq U_{2a} \subseteq Z_{U_a}$; likewise, if $2a \notin {}_k\Phi$ then we similarly conclude that U_a is commutative. It then follows from [CGP, Lemma 3.3.8] (a general result on smooth connected commutative unipotent groups equipped with a sufficiently nontrivial torus action, having nothing to do with reductive groups) that U_a is a vector group in the non-multipliable case and that U_a/U_{2a} is a vector group in the multipliable case, with these vector groups admitting a unique S -equivariant linear structure.

In particular, in the multipliable case U_a is a U_{2a} -torsor over the affine space U_a/U_{2a} . Since U_a is a vector group it follows (from the triviality of étale \mathbf{G}_a -torsors over affine schemes) that $U_a \rightarrow U_a/U_{2a}$ admits a section and hence the k -scheme U_a is isomorphic to an affine space. ■

Remark 3.2. Let G be a smooth connected affine k -group, S a maximal k -split torus of G , and P a minimal parabolic k -subgroup of G containing S . Let $N = N_G(S)$ and $Z = Z_G(S)$. By minimality of P we have $P = P_G(\lambda) = U_G(\lambda) \rtimes Z_G(\lambda)$ with $Z_G(\lambda) = Z$ and $\mathcal{R}_{u,k}(P) = U_G(\lambda)$ for some $\lambda \in X_*(S)$.

Let $G(k)^+$ be the normal subgroup of $G(k)$ generated by the k -rational points of the k -split unipotent radicals of the parabolic k -subgroups of G . It is immediate from the preceding

proposition that $G(k)^+ \cap N(k)$ maps onto the k -Weyl group ${}_k W = N(k)/Z(k)$. Now let \mathcal{G} be a subgroup of $G(k)$ that contains $G(k)^+$, $\mathcal{N} := N(k) \cap \mathcal{G}$, and $\mathcal{B} := P(k) \cap \mathcal{G}$. Then \mathcal{N} maps onto ${}_k W$. As $U_G(\lambda)(k) \subset \mathcal{B}$, we see that $P(k) = U_G(\lambda)(k) \rtimes Z(k) = \mathcal{B}Z(k)$. Moreover, for all $n \in \mathcal{N}$ we have $P(k)nP(k) = \mathcal{B}n\mathcal{B}Z(k)$ and $\mathcal{G} \cap (P(k)nP(k)) = \mathcal{B}n\mathcal{B}$. It is seen from this, using the relative Bruhat decomposition of $G(k)$, that $G(k) = \mathcal{G}Z(k)$. Now it is easily deduced from Theorem 2.7 and Remark 2.8 that the 4-tuple $(\mathcal{G}, \mathcal{B}, \mathcal{N}, R)$, with R as in Theorem 2.7, is a Tits system with Weyl group ${}_k W$. We shall call this Tits system a *standard* Tits system in \mathcal{G} . Its rank (i.e., the cardinality of R) is equal to the k -rank of $\mathcal{D}(G)$.

The equality $G(k) = \mathcal{G} \cdot Z(k)$ implies that all maximal k -split tori (resp., all minimal parabolic k -subgroups) of G are conjugate to each other under \mathcal{G} .

Recall that a subset Ψ of $\Phi(G, S)$ is *saturated* if the subsemigroup A of $X(S)$ spanned by Ψ does not contain 0 and $\Psi = A \cap {}_k \Phi$. By [CGP, Prop. 2.2.7], for any positive system of roots ${}_k \Phi^+$ in ${}_k \Phi$, any closed set of roots $\Psi \subseteq {}_k \Phi^+$ is saturated. The two most interesting examples of saturated (even closed) subsets of ${}_k \Phi$ for our purposes are obtained as follows. for any 1-parameter k -subgroup $\lambda : \mathrm{GL}_1 \rightarrow S$ we can take Ψ to be ${}_k \Phi_{\lambda > 0}$, and for any linearly independent $a, b \in {}_k \Phi$ we can take Ψ to be the set (a, b) of elements in ${}_k \Phi$ of the form $ma + nb$ for positive integers m and n .

We seek to construct a smooth connected unipotent k -subgroup U_Ψ attached to any saturated Ψ , and to describe it as a direct span (in any order!) of suitable root groups. However, if ${}_k \Phi$ is not reduced and $c, 2c \in \Psi$ then we have to be careful not to use both U_c and U_{2c} in such a direct spanning (as $U_{2c} \subset U_c$).

Proposition 3.3. *Let $(G, S, {}_k \Phi)$ be as above.*

- (1) *For any saturated subset Ψ of ${}_k \Phi$ there is a unique S -stable smooth connected unipotent k -subgroup U_Ψ such that $\mathrm{Lie}(U_\Psi)$ is the span of the subspaces $\mathrm{Lie}(U_c)$ for $c \in \Psi$. It is normalized by $Z_G(S)$, and directly spanned in any order by the subgroups U_c for $c \in \Psi$ that are not divisible in Ψ .*
- (2) *For any parabolic k -subgroup P of G containing S , the subset Ψ of ${}_k \Phi$ consisting of $a \in {}_k \Phi$ such that $-a \notin \Phi(P, S)$ is saturated, and $\mathcal{R}_{u,k}(P) = U_\Psi$.*

We will make the description of Ψ and U_Ψ in (2) more explicit in Remark 3.4

Proof. To prove assertion (1), let A be the subsemigroup of $X(S)$ generated by Ψ . Then as Ψ is saturated, $0 \notin A$ and $A \cap \Phi(G, S) = \Psi$. Let $U_\Psi = U_A(G)$ as in Proposition 2.1, which ensures (since $0 \notin A$) that U_Ψ is a smooth connected unipotent k -subgroup. To prove that $Z_G(S)$ normalizes U_Ψ , we note that the description of $U_A(G)$ makes sense without reference to the maximality of S as a k -split torus in G , and is compatible with any extension of the ground field. Thus, using scalar extension to k_s implies that $(U_\Psi)_{k_s}$ is normalized by $Z_G(S)(k_s)$, and hence U_Ψ is normalized by $Z_G(S)$. The rest of assertion (1) follows easily from [CGP, Theorem 3.3.11] applied to the smooth connected solvable k -group $U_\Psi = U_A(G)$ equipped with its natural action by S (using the disjoint union decomposition $\Psi = \coprod \Psi_i$ where the Ψ_i are the non-empty intersections of Ψ with half-lines in $X(S)_{\mathbf{Q}}$).

To prove assertion (2), we fix a $\lambda \in X_*(S)$ such that $P = P_G(\lambda)$. Thus, $\mathcal{R}_{u,k}(P) = U_G(\lambda)$. But $U_\Psi = U_G(\lambda)$ for $\Psi = {}_k \Phi_{\lambda > 0}$ by uniqueness in (1), so (2) is proved. \blacksquare

Remark 3.4. Fix a positive system of roots ${}_k\Phi^+$, with ${}_k\Delta$ its basis, so we get the notion of *standard* parabolic k -subgroup of G as those parabolic k -subgroups Q containing S such that ${}_k\Phi^+ \subset \Phi(Q, S)$, or equivalently Q contains the minimal parabolic k -subgroup $P_0 \supset S$ such that $\Phi(P_0, S) = {}_k\Phi^+$. We know that the standard parabolic k -subgroups are in inclusion-preserving bijection with the subsets $I \subset {}_k\Delta$ via the condition $\Phi({}_kP_I, S) = {}_k\Phi^+ \cup [I]$. As a refinement of Proposition 3.3(2), we shall now describe $\mathcal{R}_{u,k}({}_kP_I)$ and a Levi factor directly in terms of I .

The closed set Ψ_I of roots occurring as S -weights on $\text{Lie}(\mathcal{R}_{u,k}({}_kP_I))$ are precisely the positive relative roots not in $[I]$, which is to say those whose ${}_k\Delta$ -expansion is not supported inside I . Thus, $\mathcal{R}_{u,k}({}_kP_I)$ is directly spanned in any order by the relative root groups U_c for non-divisible $c \in {}_k\Phi^+$ not contained in $[I]$. A Levi factor L_I of ${}_kP_I$ containing S must contain $Z_G(S)$ and must contain the root groups U_c for $c \in [I]$, so L_I is generated by those k -subgroups for Lie algebra reasons (in view of our description of $\mathcal{R}_{u,k}({}_kP_I)$); in particular, such an L_I containing S is uniquely determined.

But we can describe L_I in the form $Z_G(S_I)$ for an explicit subtorus $S_I \subset S$ as follows. Let

$$S_I = \left(\bigcap_{a \in I} \ker a \right)_{\text{red}}^0 \subset S.$$

Note that for every $c \in [I]$ we have $c|_{S_I} = 1$, so the S_I -centralizer in U_c has full Lie algebra. But such a torus centralizer in U_c is smooth and connected, so S_I centralizes U_c . (This argument permits the possibility that c is multipliable, in which case U_c is generally non-commutative!) In other words, the smooth connected k -subgroup $Z_G(S_I) \subset G$ contains L_I . We claim that the inclusion $L_I \subset Z_G(S_I)$ is an equality, thereby completing our explicit description

$$P_I = Z_G(S_I) \ltimes U_{\Psi_I}$$

in terms of I .

Since $Z_G(S_I)$ is smooth and connected, it remains to check that the only nontrivial S -weights occurring on its Lie algebra are the ones in $[I]$. That is, we claim that if $a \in {}_k\Phi$ and $a|_{S_I} = 1$ then $a \in [I]$ (the converse being obvious). Since I is a linearly independent subset of $X(S)_{\mathbf{Q}}$, the characters of S that are trivial on S_I are exactly those that are a \mathbf{Q} -linear combination of elements of I . But ${}_k\Delta$ is a basis of ${}_k\Phi$ containing I , so a relative root is in the \mathbf{Q} -span of I if and only if it is in the \mathbf{Z} -span of I (i.e., lies in $[I]$).

For independent $a, b \in {}_k\Phi$, if we apply Proposition 3.3(1) to the saturated subset $\Psi = (a, b)$ then the construction of U_{Ψ} and [CGP, Prop. 3.3.5] yield:

Corollary 3.5. *We have*

$$(U_a, U_b) \subseteq U_{(a,b)} = \prod_i U_{c_i}$$

where the direct spanning is taken with respect to an arbitrary enumeration $\{c_i\}$ of the set of non-divisible elements of (a, b) .

Remark 3.6. For $a \in {}_k\Phi$, let $M_a = m(u)Z_G(S)$, where u is any nontrivial element of $U_a(k)$ and the element $m(u) \in N_G(S)(k)$ is as in Lemma 4.2 of the handout on Cartan's connectedness theorem. It is easy to see using Proposition 3.1, Corollary 3.5, and Lemma 4.2

from the handout on Cartan's connectedness theorem that $(Z_G(S)(k), (U_a(k), M_a(k))_{a \in \Phi})$ is a generating root datum (*donnée radicielle génératrice*) in $G(k)$ in the sense of Bruhat and Tits (see (6.1.1) in their initial big IHES paper on what came to be called Bruhat–Tits theory). This is very useful in the calculations that underlie Bruhat–Tits theory.

Remark 3.7. It is natural to wonder if the reduced root system ${}_k\Phi'$ consisting of non-multipliable roots in the k -root system ${}_k\Phi$ coincides with the root system of a split connected reductive k -subgroup of G containing S as a maximal torus. This issue arose in the handout on Cartan's connectedness theorem, where we settled the case of k -rank equal to 1. The affirmative answer in general is give by Theorem 7.2 in the Borel–Tits IHES 27 paper on reductive groups, and a much simpler proof is given by [CGP, Thm. C.2.30] (specialized to the reductive case, where some technicalities simplify and the rank-1 calculations were already carried out in the handout on Cartan's connectedness theorem).

As an application of the direct spanning of the group U_Ψ for saturated Ψ as in Proposition 3.3, we can prove:

Corollary 3.8. *Let P be a minimal parabolic k -subgroup of G containing S , and ${}_k\Phi^+ = \Phi(P, S)$. For $w \in {}_kW = W({}_k\Phi)$, define*

$$\Phi_w^+ = \{a \in {}_k\Phi^+ \mid w^{-1}.a \in {}_k\Phi^+\}, \quad \Phi'_w = {}_k\Phi^+ - \Phi_w^+.$$

For a representative $n_w \in N_G(S)(k)$ of $w \in {}_kW$, we have

$$P(k)n_wP(k) = U_{\Phi'_w}(k) \times Z_G(S)(k) \times n_wU_{{}_k\Phi^+}(k).$$

In view of the direct spanning property of the groups $U_{\Phi'_w}$ and $U_{{}_k\Phi^+}$, this is proved by the exact same computation as reviewed in the split case in §1. Of course, unlike in the split case, in the general case there is no combinatorial recipe for the dimensions of the affine spaces $U_{\Phi'_w}$ or $U_{{}_k\Phi^+}$ since the dimensions of the root groups U_a are unknown!

4. SIMPLE FACTORS

We proved in class (see Theorem 8.2.5) that if G is a connected semisimple k -group and $\{G_i\}$ is its set of minimal nontrivial smooth connected normal k -subgroups then: $\{G_i\}$ is finite, the G_i 's pairwise commute, each G_i is k -simple (i.e., has no nontrivial smooth connected proper normal k -subgroup), and the multiplication map

$$\prod G_i \rightarrow G$$

is a central isogeny. The essential case in the proof was the k -split case (such as over k_s), from which the general case was descent by Galois descent from G_{k_s} .

In the split case we saw an additional property: if $T \subset G$ is a split maximal k -torus and $T_i = T \cap G_i$ (a split maximal k -torus of G_i ; see Lemma 4.1 below) then via the decomposition $X(T)_{\mathbf{Q}} = \prod X(T_i)_{\mathbf{Q}}$, the root systems $\Phi(G_i, T_i)$ are the irreducible components of $\Phi(G, T)$. In particular, G is absolutely simple if and only if $\Phi(G, T)$ is irreducible. We shall establish an analogue in general, but must be attentive to the fact that some G_i might be k -anisotropic (and so do not contribute to ${}_k\Phi$; they will lie in $Z_G(S)$!). As a preliminary step, we record two lemmas.

Lemma 4.1. *Let G be a connected reductive k -group and N a smooth connected normal k -subgroup of G (so N is connected reductive). Let S be a maximal k -split torus of G , and T a maximal k -torus of G . Then $S_N := (S \cap N)_{\text{red}}^0$ is a maximal k -split torus of N , and $T \cap N$ is a maximal torus of N .*

Proof. By the $G(k)$ -conjugacy of all such S and the normality of N in G , the dimension of S_N is independent of S . Thus, for the assertion concerning S it suffices to find *one* S for which S_N is maximal as a k -split torus in N . Pick a maximal k -split torus S' in N , so it lies in a maximal k -split torus S'' of G . Clearly $S''_N = S'$ by maximality of S' in N , so we are done.

Next, we show that $T \cap N$ is a maximal k -torus of N . For this purpose we may extend scalars so that $k = \bar{k}$, and then we can apply the preceding case to see that $(T \cap N)_{\text{red}}^0$ is a maximal torus of N . It then remains to check that $T \cap N$ is smooth and connected. Since $T = Z_G(T)$, clearly $T \times (T \cap N)$ is the centralizer of T in $T \times N$. But a torus centralizer in a smooth connected affine group is always smooth and connected, so we are done. ■

Lemma 4.2. *For any smooth connected normal k -subgroup M in a connected reductive k -group H and for any maximal k -torus T in M , there is an almost direct product decomposition $T = T' \cdot Z$ where $T' = T \cap \mathcal{D}(M)$ is a maximal k -torus in $\mathcal{D}(M)$ and Z is the maximal subtorus of T that centralizes M .*

Proof. By Galois descent we may assume $k = k_s$, so by the split theory such M is generated by some of the (commuting) simple factors of H and a subtorus of the maximal central k -torus of H . Since T compatibly decomposes as an almost direct product of the maximal central torus and maximal tori in each of the simple factors, the assertion is evident. ■

Here is the main result:

Proposition 4.3. *Let G be a connected reductive k -group, S a maximal k -split torus, and ${}_k\Phi = \Phi(G, S)$. Let S' be the maximal k -split torus $(S \cap \mathcal{D}(G))_{\text{red}}^0$ of $\mathcal{D}(G)$. Assume $S' \neq 1$, or equivalently S is non-central in G . For each irreducible component Ψ of ${}_k\Phi$, let N_Ψ be the nontrivial smooth connected k -subgroup generated by $\{U_a\}_{a \in \Psi}$.*

- (1) *The k -group G is generated by $Z_G(S)$ and the N_Ψ 's.*
- (2) *Each N_Ψ is semisimple and normal in G .*
- (3) *The maximal k -split torus $S_\Psi := (S \cap N_\Psi)_{\text{red}}^0$ of N_Ψ is nontrivial, $\prod_\Psi S_\Psi \rightarrow S'$ is an isogeny, and the k -root system of N_Ψ with respect to S_Ψ is naturally identified with Ψ via the natural isomorphism $X(S')_{\mathbf{Q}} \simeq \prod_\Psi X(S_\Psi)_{\mathbf{Q}}$.*
- (4) *Each k -isotropic normal connected semisimple k -subgroup of G contains some N_Ψ .*
- (5) *No N_Ψ contains a nontrivial smooth connected proper normal k -subgroup.*

Proof. Each k -subgroup N_Ψ is clearly normalized by $Z_G(S)$, and Lie algebra considerations imply that G is generated by $Z_G(S)$ and the N_Ψ 's. For a choice of Ψ and roots $a \in \Psi$ and $b \in \Phi - \Psi$, U_a commutes with U_b by Proposition 3.5 since $ma + nb \notin {}_k\Phi$ for any positive integers m and n . Since G is generated by $Z_G(S)$ and the root groups $\{U_b\}_{b \in \Phi}$, we see that N_Ψ is a (smooth connected nontrivial) normal k -subgroup of G , so N_Ψ is reductive.

Applying Lemma 4.2 to G and its normal subgroup N_Ψ and a maximal k -torus T of G containing S , the almost direct product decomposition of T passes to one on maximal split

subtori (with the help of Lemma 4.1) we see that S is an almost direct product of the maximal k -split torus $(S_\Psi \cap \mathcal{D}(N_\Psi))_{\text{red}}^0$ of $\mathcal{D}(N_\Psi)$ and the maximal k -subtorus of S that centralizes N_Ψ . Thus, $(S, U_a) = (S_\Psi, U_a) \subseteq \mathcal{D}(N_\Psi)$ for all $a \in \Psi$, yet $(S, U_a) = U_a$ by the known S -equivariant structure of U_a in the non-multipliable case and of U_{2a} and U_a/U_{2a} in the multipliable case (see Proposition 3.1), so N_Ψ is perfect (thus semisimple) and $S_\Psi \neq 1$. We have established (1), (2), and the first assertion in (3).

Recall that a smooth connected normal k -subgroup of a smooth connected normal k -subgroup of G is normal in G (as follows from the general structure of smooth connected normal subgroups of connected reductive groups over \bar{k} , via irreducible components of root systems). Thus, a minimal nontrivial smooth connected normal k -subgroup N of G cannot contain a smooth connected proper k -subgroup that is normal in N .

Let N be a non-commutative smooth connected normal k -subgroup of G , so the k -subgroup $\mathcal{D}(N)$ of N is nontrivial and perfect and normal in G . Thus, if such an N is minimal among the non-commutative smooth connected normal k -subgroups of G then it is perfect and hence is contained in $\mathcal{D}(G)$ (which is itself perfect). But S is an almost direct product of S' and the maximal k -split central torus of G . Each N_Ψ lies in $\mathcal{D}(G)$ by perfectness, so the k -root system of G with respect to S is identified with that of $\mathcal{D}(G)$ with respect to S' . We may therefore replace G with $\mathcal{D}(G)$ (and S with $S' \neq 1$), so G is semisimple and $S' = S$. In particular, by the split theory over k_s , every nontrivial normal smooth connected k -subgroup of G is semisimple.

Let $\{N_i\}_{i \in I}$ be the set of minimal nontrivial smooth connected normal k -subgroups of $G = \mathcal{D}(G)$, so every N_i is semisimple. By the split theory over k_s , such N_i are almost direct products of $\text{Gal}(k_s/k)$ -orbits among the minimal normal smooth connected k_s -subgroups of G_{k_s} . This description over k_s implies the following properties if the N_i 's: they pairwise commute, the product homomorphism

$$\pi : N = \prod_{i \in I} N_i \longrightarrow G$$

is a central isogeny, and every normal smooth connected k -subgroup of G is generated by $\{N_j\}_{j \in J}$ for a unique subset $J \subseteq I$. Note also that no N_i contains a nontrivial smooth connected proper normal k -subgroup, due to the minimality of N_i in G . We shall eventually see that the k -isotropic N_i 's are precisely the N_Ψ 's.

For $i \in I$, let $S_i = (S \cap N_i)_{\text{red}}^0$, a maximal k -split torus of N_i . Let I^\sharp be the set of $i \in I$ such that N_i is k -anisotropic. Thus, $S_i = 1$ precisely for $i \in I^\sharp$, and the pair (N_i, S_i) has a non-empty k -root system for $i \in I - I^\sharp$ (since otherwise the nontrivial split torus S_i is central in the semisimple N_i , an absurdity). We claim that S is an isogenous quotient of the maximal k -split torus $\prod_{i \in I - I^\sharp} S_i$ of N . Choose a maximal k -torus T of G containing S , so by Lemma 4.1 we see that $T_i := T \cap N_i$ is a maximal k -torus of N_i that contains S_i . The product $\prod T_i$ is a maximal k -torus of N , so its image in $G = \pi(N)$ is a maximal k -torus of G . This forces $\prod T_i \rightarrow T$ to be surjective, so the maximal split k -subtorus $\prod_{i \in I} S_i = \prod_{i \in I - I^\sharp} S_i$ of $\prod T_i$ maps onto the maximal split k -subtorus S of T .

The isogeny $\prod_{i \in I - I^\sharp} S_i \rightarrow S$ implies that: $N_i \subseteq Z_G(S)$ for every $i \in I^\sharp$, π induces an injective homomorphism of $X_*(\prod_{i \in I - I^\sharp} S_i) = \prod_{i \in I - I^\sharp} X_*(S_i)$ into $X_*(S)$ whose image is a

subgroup of finite index, and $I - I^\sharp$ is non-empty. Thus, for any subset $J \subseteq I^\sharp$, the k -subgroup $N_J := \langle N_j \rangle_{j \in J}$ is k -anisotropic (because we can apply the preceding considerations to N_{I^\sharp} in the role of G to deduce that N_{I^\sharp} is k -anisotropic).

For any $\lambda \in \prod_{i \in I - I^\sharp} X_*(S_i)$, the restriction of π to $U_N(\lambda)$ is an isomorphism onto $U_G(\pi \circ \lambda)$ since the kernel of π is central (forcing $\ker \pi \subseteq Z_N(\lambda)$). Hence, the set of root groups of G relative to S is the (disjoint) union of the set of root groups of N_i relative to S_i over all $i \in I - I^\sharp$. Thus, Φ is the direct sum of the non-empty k -root systems $\Phi_i := \Phi(N_i, S_i)$ for $i \in I - I^\sharp$ compatibly with the decomposition $X(S)_{\mathbf{Q}} = \prod_{i \in I - I^\sharp} X(S_i)_{\mathbf{Q}}$.

We claim that for each $i \in I - I^\sharp$, the k -root system Φ_i in $X(S_i)_{\mathbf{Q}} \neq 0$ is irreducible. Granting this, $\{\Phi_i\}_{i \in I - I^\sharp}$ is the set of irreducible components of the root system ${}_k\Phi$ in the \mathbf{Q} -vector space $X(S)_{\mathbf{Q}} = \prod_{i \in I - I^\sharp} X(S_i)_{\mathbf{Q}}$ and (by minimality) the nontrivial perfect normal k -subgroup N_{Φ_i} of N_i must exhaust N_i for each $i \in I - I^\sharp$. This would establish (3) and (4), as well as (5). Thus, we can replace (G, S) with the k -isotropic (N_{i_0}, S_{i_0}) for each $i_0 \in I - I^\sharp$ separately. Now the k -isotropic connected semisimple G has no nontrivial smooth connected normal k -subgroup apart from itself. In such cases it suffices to prove that the non-empty root system ${}_k\Phi$ is irreducible.

Let Ψ be an irreducible component of ${}_k\Phi$; our aim is to prove that $\Psi = {}_k\Phi$. Since N_Ψ is a nontrivial smooth connected normal k -subgroup of G , $N_\Psi = G$. Suppose there exists $b \in {}_k\Phi - \Psi$, so the nontrivial smooth connected unipotent k -group U_b centralizes $N_\Psi = G$ and hence is central in G . This contradicts that G is reductive, so $\Psi = {}_k\Phi$. ■

The preceding proof yields immediately:

Corollary 4.4. *Assume G is semisimple. Let $\{\Phi_i\}$ be the set of irreducible components of ${}_k\Phi$, and let $\{N'_j\}$ be the set of k -anisotropic minimal nontrivial normal connected semisimple k -subgroups of G . The multiplication homomorphism*

$$\prod N'_j \times \prod N_{\Phi_i} \rightarrow G$$

is a central isogeny, and every normal connected semisimple k -subgroup N of G is an almost direct product among the N'_j 's and the N_{Φ_i} 's.

The main point of this corollary is that the isotropic minimal nontrivial smooth connected normal k -subgroups of $\mathcal{D}(G)$ are constructed from the root groups for the roots in an irreducible component of ${}_k\Phi$.