Math 249B. Simple isogeny factors over a field

1. Introduction

Recall that a connected semisimple group G over a field k is called *k-simple* if G is nontrivial and has no nontrivial smooth connected proper normal k-subgroup. We say that G is *absolutely simple* if $G_{\overline{k}}$ is $\overline{k}$-simple (in which case G_K is K-simple for every extension K/k, as it is sufficient to check for algebraically closed K, and by a "spreading out and specialization" argument if $K \supset \overline{k}$ and G_K has a nontrivial smooth connected proper normal K-subgroup then the same holds for $G_{\overline{k}}$). In class we proved the "split" case of the following important result:

Theorem 1.1. *Let G be a nontrivial connected semisimple group over a field k. The set $\{G_i\}_{i \in I}$ of minimal nontrivial normal smooth connected k-subgroups of G is finite, they pairwise commute, and the multiplication homomorphism*

$$f : \prod G_i \to G$$

is a central isogeny. In the split case each G_i is split and absolutely simple.

Moreover, every normal smooth connected k-subgroup $N \subset G$ is the central quotient image N_J of $\prod_{i \in J} G_i$ for a unique subset $J \subset I$, with and $N_J \cap G_i$ is finite for all $i \in I - J$. In particular, each G_i is k-simple.

In §2 we give the Galois descent arguments to deduce the general case from the split case. The most crucial aspect of the proof of Theorem 1.1 in the split case was to show that if G is split with an irreducible root system then G is absolutely simple. We also saw in class that for general connected semisimple G, if it is simply connected or of adjoint type then f is an isomorphism. This has an interesting consequence:

Corollary 1.2. *Let G be a connected semisimple k-group that is either simply connected or of adjoint type. There exists a pair $(k'/k, G')$ consisting of a finite étale k-algebra k' and a smooth affine k'-group G' whose fiber over each factor field of k' is connected semisimple and absolutely simple such that there is a k-homomorphism $f : G \simeq \mathrm{R}_{k'/k}(G')$.*

The triple $(k'/k, G', f)$ is unique up to unique isomorphism; i.e., if $(k''/k, G'')$ is another such pair equipped with an isomorphism $G \simeq \mathrm{R}_{k''/k}(G'')$ then the resulting composite isomorphism

$$\mathrm{R}_{k'/k}(G') \simeq \mathrm{R}_{k''/k}(G'')$$

arises from a unique pair (α, φ) consisting of a k-algebra isomorphism $\alpha : k' \simeq k''$ and a group isomorphism $\varphi : G' \simeq G''$ over α.

The case of trivial G corresponds to the case $k' = 0$ (with G' the trivial group scheme over $\mathrm{Spec}(0) = \emptyset$), but a reader who is uneasy about that may feel free to assume $G \neq 1$.

Before we prove this corollary, we explain its importance. For *any* connected semisimple k-group G, there is a central isogenous cover by a simply connected $\widetilde{G}$ and a central isogenous quotient G/Z_G of adjoint type. The corollary says that both extremes are canonically described in terms of a finite étale Weil restriction involving *absolutely simple* connected semisimple groups (over finite separable extensions of k). This underlies the essential role of *absolutely simple* connected semisimple groups (i.e., those whose Dynkin diagram over k_s is irreducible) in the study of general connected reductive groups over fields.

Proof. In view of the uniqueness assertions all over the place, by Galois descent it suffices to treat the case $k = k_s$. But then every finite étale k-algebra is a direct product of copies of k and the Weil restrictions are direct products of the fiber groups over the factor fields. Hence, the content of the assertion is that G is a direct product of k-simple factors (k-simple is the same as absolutely simple since every connected semisimple k-group is split) and that such a decomposition unique up to rearrangement.

Put in other words, our task over $k = k_s$ is to show that if $\prod G_i \simeq \prod G'_j$ is an isomorphism with factors that are connected semisimple and k-simple then the isomorphism arises from a unique pair (α, φ) consisting of a bijection $\alpha : I \simeq J$ and isomorphisms $\varphi_i : G_i \simeq G'_{\alpha(i)}$ for all $i \in I$. (The case of empty I or J corresponds to the case $G = 1$, in which case I and J must both be empty, but a reader who finds that unsettling may assume I and J are both non-empty.) This existence and uniqueness property is immediate from the isomorphism property of f in Theorem 1.1 in the simply connected and adjoint type cases, along with the explicit description of all smooth connected normal k-subgroups of G in terms of the k-simple "factors". ■

2. Proof of Theorem 1.1

We now carry out the Galois descent arguments to prove Theorem 1.1 in general, given its validity in the split case. Let $\{H_j\}_{j \in J}$ be the finitely many simple factors of G_{k_s}. These all arise over some common finite Galois extension k'/k inside k_s, and for each $\sigma \in \mathrm{Gal}(k_s/k)$ the canonical isomorphism $\sigma^*(G_{k_s}) \simeq G_{k_s}$ carries $\sigma^*(H_j)$ onto $H_{\sigma(j)}$ for a unique $\sigma(j) \in J$. This defines a $\mathrm{Gal}(k_s/k)$-action on the finite set J for which the open subgroup $\mathrm{Gal}(k_s/k')$ acts trivially, so it is a continuous action.

Let I be the set of $\mathrm{Gal}(k_s/k)$-orbits in J, and for each $i \in I$ (i.e., this is a Galois orbit in J) let $G_i \subset G$ be the Galois descent of the smooth connected normal k-subgroup of G_{k_s} generated by the H_j's for $j \in i$. It is clear by working over k_s that the G_i's pairwise commute, and the multiplication map $\prod G_i \to G$ is a central isogeny because over k_s it is dominated by the central isogeny $\prod H_j \to G_{k_s}$.

The explicit description of all smooth connected normal k_s-subgroups of G_{k_s} in terms of the H_j's, along with the Galois-stability characterization of those such k_s-subgroups of G_{k_s} that arise from a k-subgroup of G, yields that every smooth connected normal k-subgroup N of G is generated by the G_i's that it contains. Likewise, for the smooth connected normal k-subgroup N' generated by all other G_i's (so N' and N commute with each other) it follows that the multiplication map $N \times N' \to G$ is a central isogeny. Hence, $N \cap G_{i'}$ is finite when $G_{i'} \not\subset N$.

The description of all possibilities for N implies that if $N \neq 1$ then N contains some G_i. Hence, these G_i's are indeed the minimal nontrivial smooth connected normal k-subgroups of G.

It remains to show that each G_i is k-simple. Suppose $N \subset G_i$ is a nontrivial smooth connected normal k-subgroup. Such an N commutes with $G_{i'}$ for all $i' \neq i$ (as that holds even for G_i), so N is normalized by $G_{i'}$ for *all* i' and hence is normalized by G (as the $G_{i'}$

collectively generated G). That is, N is normal in G. The minimality of G_i then forces $N = G_i$, so k-simplicity is proved.