MATH 249B. STRUCTURE OF SOLVABLE GROUPS OVER FIELDS

Introduction

Consider a smooth connected solvable group G over a field k. If k is algebraically closed then $G = T \ltimes \mathscr{R}_u(G)$ for any maximal torus T of G. Over more general k, an analogous such semi-direct product structure can fail to exist.

For example, consider an imperfect field k of characteristic $p > 0$ and $a \in k - k^p$, so $k' := k(a^{1/p})$ is a degree-p purely inseparable extension of k. Note that $k'_s := k' \otimes_k k_s = k_s(a^{1/p})$ is a separable closure of k', and $k'^p_s \subset k_s$. The affine Weil restriction $G = \mathrm{R}_{k'/k}(\mathbf{G}_\mathrm{m})$ is an open subscheme of $\mathrm{R}_{k'/k}(\mathbf{A}^1_{k'}) = \mathbf{A}^p_k$, so it is a smooth connected affine k-group of dimension $p > 1$. Loosely speaking, G is "$k'^\times$ viewed as a k-group". More precisely, for k-algebras R we have $G(R) = (k' \otimes_k R)^\times$ functorially in R. The commutative k-group G contains an evident 1-dimensional torus $T \simeq \mathbf{G}_\mathrm{m}$ corresponding to the subgroup $R^\times \subset (k' \otimes_k R)^\times$, and G/T is unipotent because $(G/T)(k_s) = (k'_s)^\times/(k_s)^\times$ is p-torsion. In particular, T is the unique maximal torus of G. Since the group $G(k_s) = k'^\times_s$ has no nontrivial p-torsion, G contains *no* nontrivial unipotent smooth connected k-subgroup. Thus, G is a commutative counterexample over k to the analogue of the semi-direct product structure for connected solvable smooth affine groups over $\overline{k}$.

The appearance of imperfect fields in the preceding counterexample is essential. To explain this, recall Grothendieck's theorem that over a general field k, if S is a maximal k-torus in a smooth affine k-group H then $S_{\overline{k}}$ is maximal in $H_{\overline{k}}$. Thus, by the conjugacy of maximal tori in $G_{\overline{k}}$, $G = T \ltimes U$ for a k-torus T and a unipotent smooth connected normal k-subgroup $U \subset G$ if and only if the subgroup $\mathscr{R}_u(G_{\overline{k}}) \subset G_{\overline{k}}$ is defined over k (i.e., descends to a k-subgroup of G). In such cases, the semi-direct product structure holds for G over k using any maximal k-torus T of G (and U is unique: it must be a k-descent of $\mathscr{R}_u(G_{\overline{k}})$). If k is perfect then by Galois descent we may always descend $\mathscr{R}_u(G_{\overline{k}})$ to a k-subgroup of G. The main challenge is the case of imperfect k.

Our exposition in §1–§4 is a refinement of Appendix B of [CGP]. The general solvable case is addressed in §5, where we include applications to general smooth connected affine k-groups. *Throughout the discussion below, k is an arbitrary field with characteristic $p > 0$.*

1. Subgroups of vector groups

The *additive group* is denoted $\mathbf{G}_\mathrm{a}$ and the *multiplicative group* is denoted $\mathbf{G}_\mathrm{m}$, always with the base ring understood from context.

Definition 1.1. A *vector group* over a field k is a smooth commutative k-group V that admits an isomorphism to $\mathbf{G}_\mathrm{a}^n$ for some $n \geqslant 0$. The $\mathbf{G}_\mathrm{m}$-scaling action arising from such an isomorphism is a *linear structure* on V.

Observe that the $\mathbf{G}_\mathrm{m}$-action on V arising from a linear structure induces the canonical $k^\times$-action on $\mathrm{Lie}(V)$ (e.g., if $\mathrm{char}(k) = p > 0$ then the composition of such a $\mathbf{G}_\mathrm{m}$-action on V with the p-power map on $\mathbf{G}_\mathrm{m}$ does not arise from a linear structure on V when $V \neq 0$).

Example 1.2. If W is a finite-dimensional k-vector space then the *associated vector group* $\underline{W}$ represents the functor $R \rightsquigarrow R \otimes_k W$ on k-algebras and its formation commutes with any extension of the ground field. Explicitly, $\underline{W} = \mathrm{Spec}(\mathrm{Sym}(W^*))$ and it has a unique linear structure relative to which the natural identification of groups $\underline{W}(k_s) \simeq W_{k_s}$ carries the linear structure over to the $k_s^\times$-action on W_{k_s} arising from the k_s-vector space structure; call this the *canonical* linear structure on $\underline{W}$. (We can use k instead of k_s in this characterization when k is infinite, as $W(k)$ is Zariski-dense in $\underline{W}$ for infinite k.) For finite-dimensional k-vector spaces W and W', the subset $\mathrm{Hom}_k(W, W') \subset \mathrm{Hom}_{k\text{-gp}}(\underline{W}, \underline{W'})$ consists of precisely the k-homomorphisms respecting the canonical linear structures.

When linear structures are specified on a pair of vector groups, a homomorphism respecting them is called *linear*. Over a field of characteristic 0 there is a unique linear structure and all homomorphisms are linear. Over a field with characteristic $p > 0$ the linear structure is not unique in dimension larger than 1 (e.g., $a.(x, y) := (ax + (a - a^p)y^p, ay)$ is a linear structure on $\mathbf{G}_\mathrm{a}^2$, obtained from the usual one via the non-linear k-group automorphism $(x, y) \mapsto (x + y^p, y)$ of $\mathbf{G}_\mathrm{a}^2$). For a finite-dimensional k-vector space W, a *linear subgroup* of $\underline{W}$ is a smooth closed k-subgroup that is stable under the $\mathbf{G}_\mathrm{m}$-action. By computing with k_s-points and using Galois descent, it is straightforward to verify that the linear subgroups of $\underline{W}$ are precisely $\underline{W'}$ for k-subspaces $W' \subset W$.

Definition 1.3. A smooth connected solvable k-group G is *k-split* if it admits a composition series

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = 1$$

consisting of smooth closed k-subgroups such that G_{i+1} is normal in G_i and the quotient G_i/G_{i+1} is k-isomorphic to $\mathbf{G}_\mathrm{a}$ or $\mathbf{G}_\mathrm{m}$ for all $0 \leqslant i < n$. (Such G_i must be connected, so each G_i is also a k-split smooth connected solvable k-group.)

In the case of tori this is a widely-used notion, and it satisfies convenient properties, such as: (i) every subtorus or quotient torus (over k) of a k-split k-torus is k-split, (ii) every k-torus is an almost direct product of its maximal k-split subtorus and its maximal k-anisotropic subtorus. However, in contrast with the case of tori, it is not true for general smooth connected solvable G that the k-split property is inherited by smooth connected normal k-subgroups:

Example 1.4 (Rosenlicht)**.** Assume k is imperfect and choose $a \in k - k^p$. The k-group

$$\mathbf{U} := \{y^p = x - ax^p\}$$

is a k-subgroup of the k-split $G = \mathbf{G}_\mathrm{a}^2$ and it becomes isomorphic to $\mathbf{G}_\mathrm{a}$ over $k(a^{1/p})$ but there is no non-constant k-morphism $f : \mathbf{A}_k^1 \to \mathbf{U}$, let alone a k-group isomorphism $\mathbf{G}_\mathrm{a} \simeq \mathbf{U}$. Indeed, the regular compactification $\overline{\mathbf{U}}$ of $\mathbf{U}$ has a unique point $\infty_\mathbf{U} \in \overline{\mathbf{U}} - \mathbf{U}$, and the regular compactification of $\mathbf{G}_\mathrm{a}$ is $\mathbf{P}_k^1$ via $x \mapsto [x, 1]$, so any non-constant map f extends to a (finite) surjective map $\mathbf{P}_k^1 \to \overline{\mathbf{U}}$ that must carry $[1, 0]$ to $\infty_\mathbf{U}$, an absurdity since $k(\infty_\mathbf{U}) = k(a^{1/p}) \neq k$.

Tits introduced an analogue for unipotent k-groups of the notion of anisotropicity for tori over a field. This rests on a preliminary understanding of the properties of subgroups of vector groups, so we take up that study now. The main case of interest to us will be imperfect ground fields, due to the fact that every unipotent smooth connected group over a perfect field is split (proved in Lecture 20 of the first course).

Definition 1.5. A polynomial $f \in k[x_1, \dots, x_n]$ is a *p-polynomial* if every monomial appearing in f has the form $c_{ij}x_i^{p^j}$ for some $c_{ij} \in k$; that is, $f = \sum f_i(x_i)$ with $f_i(x_i) = \sum_j c_{ij}x_i^{p^j} \in k[x_i]$. (In particular, $f_i(0) = 0$ for all i. Together with the identity $f = \sum f_i(x_i)$, this uniquely determines each f_i in terms of f. Note that $f(0) = 0$.)

Proposition 1.6. *A polynomial $f \in k[x_1, \dots, x_n]$ is a p-polynomial if and only if the associated map of k-schemes $\mathbf{G}_\mathrm{a}^n \to \mathbf{G}_\mathrm{a}$ is a k-homomorphism.*

Proof. This is elementary and is left to the reader. □

A nonzero polynomial over k is *separable* if its zero scheme in affine space is generically k-smooth.

Proposition 1.7. *Let $f \in k[x_1, \dots, x_n]$ be a nonzero polynomial such that $f(0) = 0$. Then the subscheme $f^{-1}(0) \subset \mathbf{G}_{\mathrm{a}}^n$ is a smooth k-subgroup if and only if f is a separable p-polynomial.*

Proof. The "if" direction is clear. For the converse, we assume that $f^{-1}(0)$ is a smooth k-subgroup and we denote it as G. The smoothness implies that f is separable. To prove that f is a p-polynomial, by Proposition 1.6 it suffices to prove that the associated map of k-schemes $\mathbf{G}_{\mathrm{a}}^n \to \mathbf{G}_{\mathrm{a}}$ is a k-homomorphism. Without loss of generality, we may assume that k is algebraically closed.

For any $\alpha \in G(k)$, $f(x + \alpha)$ and $f(x)$ have the same zero scheme (namely, G) inside $\mathbf{G}_{\mathrm{a}}^n$. Thus, $f(x + \alpha) = c(\alpha)f(x)$ for a unique $c(\alpha) \in k^{\times}$. Consideration of a highest-degree monomial term appearing in f implies that $c = 1$. Pick $\beta \in k^n$, so $f(\beta + \alpha) - f(\beta) = 0$ for all $\alpha \in G(k)$. Thus $f(\beta + x) - f(\beta)$ vanishes on G, so $f(\beta + x) - f(\beta) = g(\beta)f(x)$ for a unique $g(\beta) \in k$. Consideration of a highest-degree monomial term in f forces $g(\beta) = 1$. □

Corollary 1.8. *Let $G \subset \mathbf{G}_{\mathrm{a}}^n$ be a smooth k-subgroup of codimension 1. Then G is the zero scheme of a separable nonzero p-polynomial in $k[x_1, \dots, x_n]$.*

Proof. Since G is smooth of codimension 1 in $\mathbf{G}_{\mathrm{a}}^n$, it is the zero scheme of a separable nonzero polynomial $f \in k[x_1, \dots, x_n]$. By Proposition 1.7, f is a p-polynomial. □

Lemma 1.9. *If $f : U' \to U$ is a surjective homomorphism between smooth connected unipotent k-groups and U' is k-split then so is U.*

This result was proved in §20 of the first course; we include a proof here for convenience of the reader.

Proof. Let $\{U'_i\}$ be a descending composition series of U' over k with successive quotients U'_i/U'_{i+1} isomorphic to $\mathbf{G}_{\mathrm{a}}$. Then the k-groups $U_i = f(U'_i)$ are a composition series for U and U_i/U_{i+1} is a quotient of $U'_i/U'_{i+1} = \mathbf{G}_{\mathrm{a}}$. It therefore suffices to show that for any surjective k-homomorphism $q : \mathbf{G}_{\mathrm{a}} \to G$ with $G \neq 1$, necessarily $G \simeq \mathbf{G}_{\mathrm{a}}$. Clearly q is an isogeny. If $\ker q$ is not étale then $\ker q$ has nontrivial Frobenius kernel. But the Frobenius kernel of $\mathbf{G}_{\mathrm{a}}$ is α_p, so q factors through $\mathbf{G}_{\mathrm{a}}/\alpha_p \simeq \mathbf{G}_{\mathrm{a}}$. Hence, by induction on $\deg q$ we can assume $\ker q$ is étale. By Proposition 1.7, the smooth k-subgroup $\ker q \subset \mathbf{G}_{\mathrm{a}}$ must be the zero scheme of a 1-variable separable p-polynomial $f = \sum c_j t^{p^j}$ (so $c_0 \neq 0$). But $f : \mathbf{G}_{\mathrm{a}} \to \mathbf{G}_{\mathrm{a}}$ is then an isogeny and its kernel $\{f = 0\}$ coincides with $\ker q$, so f identifies $G = \mathbf{G}_{\mathrm{a}}/\ker q$ with $\mathbf{G}_{\mathrm{a}}$. □

Definition 1.10. If $f = \sum_{i=1}^n f_i(x_i)$ is a p-polynomial over k in n variables with $f_i(0) = 0$ for all i, then the *principal part* of f is the sum of the leading terms of the f_i.

Lemma 1.11. *Let V be a vector group of dimension $n \geqslant 1$ over k, and let $f : V \to \mathbf{G}_{\mathrm{a}}$ be a k-homomorphism. Then the following are equivalent:*

1. *there exists a non-constant k-scheme morphism $f' : \mathbf{A}_k^1 \to V$ such that $f \circ f' = 0$;*
2. *for every k-group isomorphism $h : \mathbf{G}_{\mathrm{a}}^n \simeq V$, the principal part of the p-polynomial $f \circ h \in k[x_1, \dots, x_n]$ has a nontrivial zero in k;*
3. *there exists a k-group isomorphism $h : \mathbf{G}_{\mathrm{a}}^n \simeq V$ such that $f \circ h$ "only depends on the last $n - 1$ coordinates" (i.e., $\ker(f \circ h)$ contains the first factor of $\mathbf{G}_{\mathrm{a}}^n$).*

In this lemma, it is not sufficient in (2) to consider just a single choice of h. For example, if k is imperfect and $a \in k - k^p$, then $f := y^p - (x + ax^p)$ has principal part $y^p - ax^p$ with no zeros on $k^2 - \{0\}$. Composing f with the k-automorphism $(x, y) \mapsto (x, y + x^p)$ yields the polynomial $y^p + x^{p^2} - (x + ax^p)$ whose principal part is $y^p + x^{p^2}$, which has zeros on $k^2 - \{0\}$.

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

For (1) $\Rightarrow$ (2), assume that (1) holds and let $\varphi = h^{-1} \circ f'$. Let $\varphi_i : \mathbf{G}_a \to \mathbf{G}_a$ be the ith component of φ, and $a_i t^{s_i}$ denote the leading term of $\varphi_i(t)$, with $s_i = 0$ when $\varphi_i = 0$. For some i we have $s_i > 0$, since some φ_i is non-constant (as φ is non-constant, because of the same for f'). Let $\sum_{i=1}^n c_i x_i^{p^{m_i}}$ be the principal part of $f \circ h$, so

$$0 = f(h(\varphi(t))) = \sum_{i=1}^n c_i a_i^{p^{m_i}} t^{s_i p^{m_i}} + \dots$$

since $f \circ h \circ \varphi = f \circ h \circ h^{-1} \circ f' = f \circ f' = 0$. Let $N = \max_i \{s_i p^{m_i}\} > 0$. Define $b_i = a_i$ if $s_i p^{m_i} = N$ (so $b_i \neq 0$), and $b_i = 0$ if $s_i p^{m_i} < N$. Since the coefficient of the term of degree N in $f(h(\varphi(t)))$ must be zero, we have $\sum_{i=1}^n c_i b_i^{p^{m_i}} = 0$ with $b_i \in k$ and some b_i is nonzero, so (2) holds.

To prove (2) $\Rightarrow$ (3), assume (2) holds and let $h : \mathbf{G}_a^n \simeq V$ be any k-group isomorphism. We may assume $f \neq 0$, so the principal part of $f \circ h$ is nonzero. The proof will proceed by induction on the sum d of the degrees of nonzero terms of the principal part $\sum_{i=1}^n c_i x_i^{p^{m_i}}$ of $f \circ h$. If $c_r = 0$ for some r, we are done by interchanging x_r and x_1. So we may assume that all c_i are nonzero and, upon permuting the coordinates, that $m_1 \geqslant \dots \geqslant m_n \geqslant 0$. By (2), there exists $(a_1, \dots, a_n) \in k^n - \{0\}$ such that $\sum_{i=1}^n c_i a_i^{p^{m_i}} = 0$. Let $r \geqslant 0$ be minimal such that $a_r \neq 0$. Define the k-group isomorphism $h' : \mathbf{G}_a^n \simeq \mathbf{G}_a^n$ by $h'(y_1, \dots, y_n) = (x_1, \dots, x_n)$ with

$$x_1 = y_1, \dots, x_{r-1} = y_{r-1},$$
$$x_r = a_r y_r, \quad x_{r+1} = y_{r+1} + a_{r+1} y_r^{p^{m_r - m_{r+1}}}, \dots, x_n = y_n + a_n y_r^{p^{m_r - m_n}}$$

Thus, $f \circ h \circ h'$ is a p-polynomial with principal part

$$\sum_{i \neq r} c_i y_i^{p^{m_i}} + \sum_{i=1}^n c_i a_i^{p^{m_i}} \cdot y_r^{p^{m_r}} = \sum_{i \neq r} c_i y_i^{p^{m_i}}$$

since $\sum_{i=1}^n c_i a_i^{p^{m_i}} = 0$. The sum of the degrees of the nonzero terms of the principal part of $f \circ h \circ h'$ is strictly smaller than d since $c_r \neq 0$, so the induction hypothesis applies.

Finally, we assume (3) and prove (1). Let $h : \mathbf{G}_a^n \to V$ be a k-isomorphism such that $\ker(f \circ h)$ contains the first factor of $\mathbf{G}_a^n$. Define $\varphi : \mathbf{G}_a \to \mathbf{G}_a^n$ by $\varphi(t) = (t, 0, 0, \dots, 0)$. Finally, let $f' = h \circ \varphi$. Then $f \circ f' = f \circ h \circ \varphi = 0$. $\square$

Lemma 1.12. *If a p-polynomial $\sum_{i=1}^n c_i x_i^{p^{m_i}}$ over k has a zero in $K^n - \{0\}$ for a Galois extension K/k then it has a zero in $k^n - \{0\}$.*

Proof. The proof is by induction on n. The terms may be ordered so that $m_1 \geqslant m_2 \geqslant \cdots$. If $n = 1$, then since $c_1 a_1^{p^{m_1}} = 0$ with $a_1 \in K^\times$ we see that $c_1 = 0$, so $c_1 x_1^{p^{m_1}}$ has a zero in $k^\times$.

Now suppose $n > 1$ and that $\sum_{i=1}^n c_i a_i^{p^{m_i}} = 0$ with $a_i \in K$ not all zero. Let $a = (a_1, \dots, a_n)$. If $a_n = 0$ then the theorem is true by the induction hypothesis. If $a_n \neq 0$, we may assume $a_n = 1$ by replacing a_i with $a_i / a_n^{p^{m_n - m_i}}$ for all i. For all $\sigma \in \mathrm{Gal}(K/k)$, the point $a - \sigma(a)$ is a zero of $\sum c_i x_i^{p^{m_i}}$. If not all a_i belong to k then $a - \sigma(a) \neq 0$, so since $a_n - \sigma(a_n) = 0$ we may again apply the inductive hypothesis. $\square$

Lemma 1.13. *Let V be a vector group over k, K/k a Galois extension, and $f : V \to \mathbf{G}_a$ a k-homomorphism. The equivalent conditions* (1), (2), *and* (3) *of Lemma* 1.11 *hold over K if and only if they hold over k.*

Proof. It is clear that if (1) holds over k then it also holds over K. On the other hand, by Lemma 1.12, (2) is true over k if it is true over K. $\square$

Lemma 1.14. *Every smooth p-torsion commutative affine k-group G embeds as a k-subgroup of a vector group over k. Moreover, G admits an étale isogeny onto a vector group over k, and if G is connected and $k = \overline{k}$ then G is a vector group over k.*

Proof. We first construct the embedding into a vector group over k, and then at the end use this to make the étale isogeny. Consider the canonical k-subgroup inclusion $G \hookrightarrow \mathrm{R}_{k'/k}(G_{k'})$ for any finite extension field k'/k. Since $\mathrm{R}_{k'/k}(\mathbf{G}_{\mathrm{a}}) \simeq \mathbf{G}_{\mathrm{a}}^{[k':k]}$, it is harmless (for the purpose of finding an embedding into a vector group over k) to replace k with a finite extension. If $G_{\overline{k}}$ embeds as a subgroup of $\mathbf{G}_{\mathrm{a}}^N$ over $\overline{k}$, the embedding descends to a finite extension k'/k inside $\overline{k}$. Hence, for the construction of the embedding into a vector group we can now assume that k is algebraically closed.

The component group G/G^0 is a power of $\mathbf{Z}/p\mathbf{Z}$. Thus, since G is commutative and p-torsion, the connected-étale sequence of G splits. That is, $G = G^0 \times (\mathbf{Z}/p\mathbf{Z})^n$ for some $n \geqslant 0$. The finite constant k-group $\mathbf{Z}/p\mathbf{Z}$ is a k-subgroup of $\mathbf{G}_{\mathrm{a}}$, so we can assume that G is connected. We shall prove that G is a vector group. Since $k = \overline{k}$ and the unipotent G is nilpotent, it has a composition series whose successive quotients are $\mathbf{G}_{\mathrm{a}}$. By induction on $\dim G$, it suffices to prove that a commutative extension U of $\mathbf{G}_{\mathrm{a}}$ by $\mathbf{G}_{\mathrm{a}}$ over k is a split extension if $p \cdot U = 0$.

Let W_2 be the additive k-group of Witt vectors of length 2, so there is a canonical exact sequence of k-groups

$$0 \to \mathbf{G}_{\mathrm{a}} \to W_2 \to \mathbf{G}_{\mathrm{a}} \to 0.$$

It is a classical fact (see [Ser, Ch. VII.9, Lemma 3]) that every commutative extension U of $\mathbf{G}_{\mathrm{a}}$ by $\mathbf{G}_{\mathrm{a}}$ over k is obtained by pullback of this Witt vector extension along a (unique) k-homomorphism $f : \mathbf{G}_{\mathrm{a}} \to \mathbf{G}_{\mathrm{a}}$. In other words, there is a unique pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{G}_{\mathrm{a}} & \longrightarrow & U & \longrightarrow & \mathbf{G}_{\mathrm{a}} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \mathbf{G}_{\mathrm{a}} & \longrightarrow & W_2 & \longrightarrow & \mathbf{G}_{\mathrm{a}} & \longrightarrow & 0 \end{array}$$

and we claim that if U is p-torsion then $f = 0$ (so the top row is a split sequence). Clearly $f'(U) \subset W_2[p]$, but the maximal smooth k-subgroup of $W_2[p]$ is the kernel term $\mathbf{G}_{\mathrm{a}}$ along the bottom row. Hence, $f'(U)$ is killed by the quotient map along the bottom row, so $f = 0$.

Now return to the setting of a general ground field k, and fix a k-subgroup inclusion of G into a vector group V, say with codimension c. Choose a linear structure on V (in the sense of Definition 1.1). Then $W \mapsto \mathrm{Lie}(W)$ is a bijection between the set of linear subgroups of V and the set of linear subspaces of $\mathrm{Lie}(V)$. Hence, if we choose W so that $\mathrm{Lie}(W)$ is complementary to $\mathrm{Lie}(G)$ then the natural map $G \to V/W$ is an isomorphism on Lie algebras, so it is an étale isogeny. Since W is a linear subgroup of V, the quotient V/W is a vector group over k. ☐

Proposition 1.15. *Let $V_1, \dots, V_n$ be k-groups isomorphic to $\mathbf{G}_{\mathrm{a}}$, and let $V = \prod_{i=1}^n V_i$. Let U be a smooth k-subgroup of V such that U_{k_s} as a k_s-subgroup of V_{k_s} is generated by images of k_s-scheme morphisms $\mathbf{A}^1_{k_s} \to V_{k_s}$ that pass through 0.*

There exists a k-group automorphism $h : V \simeq V$ such that $h(U)$ is the direct product of some of the V_i inside V. In particular, U is a vector group over k and is a k-group direct factor of V.

Proof. The proof is by induction on n and is trivial for $n = 1$. Now consider $n > 1$. The case $U = V$ is trivial, so we can assume $\dim U \leqslant n - 1$. First assume that $\dim U = n - 1 > 0$. By Corollary 1.8, U is the kernel of a k-homomorphism $f : V \to \mathbf{G}_{\mathrm{a}}$. By hypothesis, there exists a non-constant k_s-scheme morphism $\mathbf{A}^1_{k_s} \to U_{k_s}$, so by Lemma 1.11 (applied over k_s) and Lemma

1.13 there exists a k-group automorphism h' of V such that $h'(U) \supset V_1$. But then $h'(U) = V_1 \times U'$, where U' denotes the projection of $h'(U)$ into $V' = \prod_{i=2}^n V_i$. Applying the induction hypothesis to V' and U', we are done.

Suppose now that $\dim U < n - 1$, and let U' denote the projection of U into the product V' as defined above. By the inductive hypothesis, after relabeling $V_2, \dots, V_n$ there exists a k-group automorphism $h_1 : V' \to V'$ such that $h_1(U') = \prod_{i=2}^r V_i$ for some $r < n$. Setting

$$h' = \mathrm{id}_{V_1} \times h_1 : V \simeq V,$$

we then have $h'(U) \subset \prod_{i=1}^r V_i$, and we can again apply induction. The proof is now complete. □

Corollary 1.16. *In a smooth p-torsion commutative affine k-group G, every smooth k-subgroup that is a vector group is a k-group direct factor.*

Proof. This is a consequence of Proposition 1.15, provided that G is a k-subgroup of a vector group. Such an embedding is provided by Lemma 1.14. □

The following proposition is a useful refinement of Lemma 1.14.

Proposition 1.17. *Let k be an infinite field of characteristic $p > 0$ and let U be a smooth p-torsion commutative affine k-group. Then U is k-isomorphic to a k-subgroup of codimension* 1 *in a k-vector group. In particular, U is isomorphic* (*as a k-group*) *to the zero scheme of a separable nonzero p-polynomial over k.*

This proposition is also true for finite k if U is connected since then U is a vector group; see Corollary 2.8.

Proof. By Lemma 1.14, U can be identified with a k-subgroup of a k-vector group V. Let $m = \dim V - \dim U$. If $m \leqslant 1$ then we are done by Corollary 1.8, so we assume $m > 1$. We will show that U can be embedded in a k-vector group W with $\dim W = \dim V - 1$, which will complete the argument via induction on m. The vector group W will arise as a quotient of V.

The k-linear subspace $\mathrm{Lie}(U)$ in $\mathrm{Lie}(V)$ has codimension m. Fix a choice of linear structure on V (in the sense of Definition 1.1). Since $m \geqslant 2$, the Zariski closure $\mathbf{G}_{\mathrm{a}}.U\,(\subset V)$ of the image of the multiplication map $\mathbf{G}_{\mathrm{a}} \times U \to V$ is a closed subscheme of V with nonzero codimension. By irreducibility of V, the union $\mathrm{Lie}(U) \cup (\mathbf{G}_{\mathrm{a}}.U)$ inside V is a proper closed subscheme of V.

Since $V(k)$ is Zariski-dense in V (as k is infinite), there exists $v \in V(k)$ with

$$v \notin \mathrm{Lie}(U) \cup (\mathbf{G}_{\mathrm{a}}.U).$$

Let $L \subset V$ be the k-subgroup corresponding to the line $kv \subset V(k)$. Consider the canonical k-homomorphism $\phi : V \to W := V/L$, and let $\psi = \phi|_U$. We shall prove $\ker \psi = 1$, from which it follows that ψ identifies U with a k-subgroup of W.

It suffices to show that $\mathrm{Lie}(\psi)$ is injective (so $\ker \psi$ is étale) and that $\psi|_{U(\overline{k})}$ is injective. The map $\mathrm{Lie}(\psi)$ has kernel $L \cap \mathrm{Lie}(U) = \{0\}$, so it is indeed injective. If $\psi|_{U(\overline{k})}$ is not injective then the line L would lie in $\mathbf{G}_{\mathrm{a}}.U$ since $\mathbf{G}_{\mathrm{a}}.U$ is stable under the $\mathbf{G}_{\mathrm{a}}$-multiplication on V. But the point $v \in L(k)$ does not lie in $(\mathbf{G}_{\mathrm{a}}.U)(\overline{k})$, due to how we chose v, so indeed $\psi|_{U(\overline{k})}$ is injective. □

2. Wound unipotent groups

A smooth connected unipotent k-group U is analogous to an anisotropic torus if U does not contain $\mathbf{G}_{\mathrm{a}}$ as a k-subgroup. This concrete viewpoint is inconvenient for developing a general theory, but eventually we will prove that it gives the right concept. A more convenient definition to get the theory of such U off the ground requires going beyond the category of k-groups, as follows.

Definition 2.1. A smooth connected unipotent k-group U is *k-wound* if every map of k-schemes $\mathbf{A}^1_k \to U$ is a constant map to a point in $U(k)$. Equivalently, $U(k) = U(k[x])$.

By considering translation by k-points, it is equivalent to say that every map of pointed k-schemes $(\mathbf{A}^1_k, 0) \to (U, 1)$ is constant.

Remark 2.2. An analogous definition for tori using $\mathbf{A}^1 - \{0\}$ recovers the usual notion of anisotropicity: if F is any field (possibly of characteristic 0) and T is an F-torus, then the condition $T(F[x, 1/x]) = T(F)$ (i.e., the constancy of any F-scheme map $\mathbf{G}_{\mathrm{m}} \to T$, or equivalently the triviality of any map of pointed F-schemes $(\mathbf{G}_{\mathrm{m}}, 1) \to (T, 1)$) characterizes F-anisotropicity of T.

Indeed, F-anisotropicity is equivalent to the vanishing of $\mathrm{Hom}_{F\text{-gp}}(\mathbf{G}_{\mathrm{m}}, T)$, so we just need to check that in general a map of pointed F-schemes $(\mathbf{G}_{\mathrm{m}}, 1) \to (T, 1)$ is a homomorphism. By extending scalars we may assume $F = \overline{F}$, so T is a power of $\mathbf{G}_{\mathrm{m}}$, and this reduces us to the case $T = \mathbf{G}_{\mathrm{m}}$. An endomorphism of the pointed F-scheme $(\mathbf{G}_{\mathrm{m}}, 1)$ is the "same" as an element $u \in F[x, 1/x]^{\times}$ satisfying $u(1) = 1$, and such units are precisely $u = x^n$ for $n \in \mathbf{Z}$.

The main reason that we go beyond the category of k-groups in Definition 2.1 is due to the intervention of a non-homomorphic conjugation morphism φ' that arises in the proof of Proposition 3.2 below. The interested reader can easily check that all appearances of maps from $\mathbf{A}^1$ in §1–§2 can be replaced with homomorphisms from $\mathbf{G}_{\mathrm{a}}$ without affecting the proofs there.

Remark 2.3. The definition of "wound" makes sense in characteristic 0, where it is only satisfied by $U = 1$ (since a nontrivial smooth connected unipotent group in characteristic 0 contains $\mathbf{G}_{\mathrm{a}}$ as a subgroup over the ground field). Thus, although we only work with ground fields of positive characteristic, it is convenient in practice (for handling some trivialities) to make the convention that "wound" means "trivial" for smooth connected unipotent groups in characteristic 0.

Whereas anisotropicity for a torus over a field is insensitive to purely inseparable extension of the ground field but is often lost under a separable algebraic extension of the ground field, the k-wound property behaves in the opposite manner: we will prove that it is insensitive to a separable extension on k (such as scalar extension from a global field to a completion), but it is often lost under a purely inseparable extension on k.

Example 2.4. Assume k is imperfect and choose $a \in k - k^p$. The k-group $U = \{y^p = x - ax^p\}$ becomes isomorphic to $\mathbf{G}_{\mathrm{a}}$ over the purely inseparable extension $k(a^{1/p})$ but by Example 1.4 it is k-wound. Observe that the isogeny $y : U \to \mathbf{G}_{\mathrm{a}}$ is étale, so applying an étale isogeny can destroy the wound property. (Although y is étale, its extension to a degree-p finite flat covering $\overline{U} \to \mathbf{P}^1_k$ between regular compactifications is not étale: explicitly, at the point at infinity the ramification index is 1 but the residue field extension is $k(a^{1/p})/k$.) Hence, for problems involving wound unipotent groups one must be more attentive to the use of isogenies than is usually necessary when working with tori.

Note that the wound k-group U is a *k-subgroup* of the k-split group $\mathbf{G}_{\mathrm{a}}^2$. In the opposite direction, there also exist nontrivial k-split *quotients* of k-wound groups modulo *smooth connected* k-subgroups. For instance, in [Oes, Ch. V, 3.5] there is an example over any imperfect field k of a 2-dimensional k-wound smooth connected p-torsion commutative affine group G admitting a 1-dimensional (necessarily k-wound) smooth connected k-subgroup G' such that $G/G' \simeq \mathbf{G}_{\mathrm{a}}$ as k-groups.

Example 2.5. Assume k is infinite. By Corollary 1.8, smooth p-torsion commutative affine k-groups G are precisely the zero schemes of separable nonzero p-polynomials f over k. Since G is connected if and only if it is geometrically irreducible (as for any k-group scheme of finite type), we see that G is connected if and only if f is irreducible over k, as well as if and only if f is absolutely irreducible over k. Assume G is connected.

If the principal part f_{prin} of f has no zero on $k^n - \{0\}$ then by Lemma 1.11 it follows that G is k-wound. The converse is false, as we saw following the statement of Lemma 1.11. However, if f_{prin} has a zero on $k^n - \{0\}$ then the calculation in the proof of (2) $\Rightarrow$ (3) in Lemma 1.11 (taking h to be the identity map of $\mathbf{G}_{\text{a}}^n$) shows that we can find a p-polynomial $F \in k[x_1, \dots, x_n]$ having zero scheme k-isomorphic to G as a k-group (so F is absolutely irreducible over k) with the sum of the degrees of the monomials appearing in F_{prin} strictly less than the corresponding sum for f_{prin}. Continuing in this way, we eventually arrive at a choice of f having zero scheme G (as a k-group) such that f_{prin} has no zeros on $k^n - \{0\}$. In this sense, the zero schemes of absolutely irreducible p-polynomials f over k for which f_{prin} has no nontrivial k-rational zero are precisely the p-torsion commutative k-wound smooth connected unipotent k-groups (up to k-isomorphism).

Theorem 2.6. *Every smooth connected p-torsion commutative affine k-group U is a direct product $U = V \times W$ of a vector group V and a smooth connected unipotent k-group W such that W_{k_s} is k_s-wound. In this decomposition, the subgroup V is uniquely determined: V_{k_s} is generated by the images of k_s-scheme morphisms $\varphi : \mathbf{A}^1_{k_s} \to U_{k_s}$ passing through the identity.*

Proof. By Galois descent, there is a unique smooth connected k-subgroup V of U such that V_{k_s} is generated by the images of k_s-scheme morphisms $\varphi : \mathbf{A}^1_{k_s} \to U_{k_s}$ that pass through the identity. By Lemma 1.14, we can identify U with a k-subgroup of a vector group over k. Thus, by Proposition 1.15, V is a vector group over k and (by Corollary 1.16) we have $U = V \times W$ as k-groups for some k-subgroup W of U. Since U is a smooth connected unipotent k-group, so is its direct factor W. Clearly, W_{k_s} is k_s-wound (due to the definition of V).

Now we prove that V in this decomposition is unique. Consider any decomposition of k-groups $U = V' \times W'$, where V' is a vector group over k and W' is a smooth connected unipotent k-subgroup of U such that W'_{k_s} is k_s-wound. The image of any k_s-scheme morphism $\varphi : \mathbf{A}^1_{k_s} \to U_{k_s}$ passing through the identity is contained in V'_{k_s} because otherwise the composite of φ and the canonical projection $U_{k_s} \to W'_{k_s}$ would be a non-constant k_s-scheme morphism from $\mathbf{A}^1_{k_s}$ to W'_{k_s} (contradicting that W'_{k_s} is assumed to be k_s-wound). Hence, $V \subset V'$, so $V' = V \times V'_1$ with V'_1 the image of the vector group V' under the projection $U \twoheadrightarrow W$. Since W_{k_s} is k_s-wound and V' is a vector group, $V'_1 = 0$. That is, $V' = V$. $\square$

In Theorem 2.6, the group W as an abstract k-group is unique up to isomorphism, since it is identified with the quotient U/V modulo the uniquely determined k-subgroup V. However, the decomposition of U as $V \times W$ is not unique when $V, W \neq 0$. That is, there may be more than one k-homomorphic section to $U \to U/V = W$, or in other words $\text{Hom}_k(W, V)$ may be nontrivial. For example, over an imperfect field consider $U = \mathbf{G}_{\text{a}}^2 \times \mathbf{U}$ where $\mathbf{U}$ is as in Example 1.4. Clearly $\text{Hom}_k(\mathbf{U}, \mathbf{G}_{\text{a}}^2)$ is nontrivial.

Corollary 2.7. *A smooth connected p-torsion commutative affine k-group U is k-wound if and only if U_{k_s} is k_s-wound, and also if and only if there are no nontrivial k-homomorphisms $\mathbf{G}_{\text{a}} \to U$. The k-group U is a vector group over k if and only if U_{k_s} is a vector group over k_s.*

Proof. This is immediate from Theorem 2.6. $\square$

Corollary 2.8. *If k is perfect then a smooth connected p-torsion commutative affine k-group is a vector group.*

Proof. By Corollary 2.7, we may assume that k is algebraically closed. This case is part of Lemma 1.14. $\square$

To get results on k-wound groups beyond the commutative p-torsion case, we need to study smooth connected p-torsion central k-subgroups in a general smooth connected unipotent k-group U. This is taken up in the next section. We end this section with some examples.

Example 2.9. Let k be a field and let G be a commutative smooth connected affine k-group containing no nontrivial unipotent smooth connected k-subgroup. The commutativity ensures that there exists a unipotent smooth connected k-subgroup $\mathscr{R}_{u,k}(G)$ in G containing all other such k-subgroups, and by Galois descent $\mathscr{R}_{u,k}(G)_{k_s} = \mathscr{R}_{u,k_s}(G_{k_s})$. Assume $\mathscr{R}_{u,k}(G) = 1$. (By the argument near the start of the Introduction, such a G with $\mathscr{R}_u(G_{\overline{k}}) \neq 1$ is $\mathrm{R}_{k'/k}(T')$ for any nontrivial purely inseparable finite extension k'/k and a nontrivial k'-torus T'.)

For the maximal k-torus T in G, consider the smooth connected commutative unipotent quotient $U = G/T$. We claim that U is k-wound. Since $\mathscr{R}_{u,k_s}(G_{k_s}) = \mathscr{R}_{u,k}(G)_{k_s} = 1$, we may assume $k = k_s$, so T is k-split. By definition, we need to prove that any map of k-schemes $f : \mathbf{A}^1_k \to U$ is constant.

Consider the pullback $G \times_U \mathbf{A}^1_k$. This is a T-torsor over $\mathbf{A}^1_k$, so it is trivial since T is split and $\mathrm{Pic}(\mathbf{A}^1_k) = 1$. A choice of splitting defines a k-scheme morphism $\widetilde{f} : \mathbf{A}^1_k \to G$ over f, so it suffices to prove that $\widetilde{f}$ is constant. Using a translation, we may assume $\widetilde{f}(0) = 1$. We claim that for any smooth connected commutative k-group C and any k-scheme morphism $h : \mathbf{A}^1_k \to C$ satisfying $h(0) = 1$, the smooth connected k-subgroup of C generated by the image of h is unipotent. Applying this to G would then force $\widetilde{f} = 1$ since $\mathscr{R}_{u,k}(G) = 1$, so we would be done.

To prove our claim concerning C we may assume $k = \overline{k}$, so C is a direct product of a torus and a unipotent group. Using projections to factors, it suffices to treat the case $C = \mathbf{G}_{\mathrm{m}}$. In this case h is a nowhere-vanishing polynomial in one variable with value 1 at the origin, so $h = 1$.

Example 2.10. Here is an example (due to Gabber) of a 2-dimensional *non-commutative* wound smooth connected unipotent group U over an arbitrary imperfect field k of characteristic $p > 0$. Choose $a \in k - k^p$, and consider the smooth connected k-subgroups of $\mathbf{G}_{\mathrm{a}}^2$ defined by

$$G = \{x = x^{p^2} + ay^{p^2}\}, \;\; C^{\pm} = \{x = \pm(x^p + ay^p)\}.$$

Their closures in $\mathbf{P}^2_k$ are regular with a unique point at infinity, and this point is not k-rational, so these groups are wound. We will construct a non-commutative central extension U of G by C^-, so U must be a k-wound smooth connected unipotent k-group. (The construction will work "universally" over the polynomial ring $\mathbf{F}_p[a]$, yielding the desired k-group via base change.)

Define the k-morphism $f : G \to C^+$ by $(x, y) \mapsto (x^{p+1}, xy^p)$ and consider the symmetric bi-additive 2-coboundary $b = -\mathrm{d}f : G \times G \to C^+$ defined by

$$b(g, g') = f(g + g') - f(g) - f(g') = (xx'^p + x^p x', xy'^p + x'y^p)$$

for points $g = (x, y)$ and $g' = (x', y')$ of G. The related map $b^- : G \times G \to C^-$ defined by

$$b^-((x, y), (x', y')) = (xx'^p - x^p x', xy'^p - x'y^p)$$

is easily checked to be an alternating bi-additive 2-cocycle, so if $p \neq 2$ then b^- is *not* symmetric. Thus, if $p \neq 2$ then the associated k-group U with underlying scheme $C^- \times G$ and composition law

$$(c, g)(c', g') = (c + c' + b^-(g, g'), g + g')$$

is a non-commutative central extension of G by C^- (with identity $(0, 0)$ and inversion $-(c, g) = (-c, -g)$).

To handle the case $p = 2$, we consider a variant on this construction. For any p and $\zeta \in \mathbf{F}_{p^2} - \mathbf{F}_p$ consider the bi-additive map $b_\zeta : G \times G \to C^+$ over $\mathbf{F}_{p^2}[a]$ defined by $b_\zeta(g, g') = b(g, \zeta g') = b(\zeta^p g, g')$. This is easily seen to be a 2-cocycle that is not symmetric, so it defines a non-commutative central extension U_ζ of G by C^+ over $\mathbf{F}_{p^2}[a]$ ($U_\zeta = C^+ \times G$ as $\mathbf{F}_{p^2}[a]$-schemes, equipped with the composition law $(c,g)(c',g') = (c + c' + b_\zeta(g,g'), g + g')$, identity $(0,0)$, and inversion $-(c,g) = (-c - b_\zeta(g,-g), -g)$). Taking $p = 2$, so $C^+ = C^-$ and each ζ is a primitive cube root of unity, we have $\zeta^{-1} = \zeta + 1$ and $b_{\zeta+1} = b_\zeta + b = b_\zeta - \mathrm{d}f$, so for each ζ we obtain an isomorphism of central extensions $U_\zeta \simeq U_{\zeta+1}$ via $(c,g) \mapsto (c + f(g), g)$. Letting σ be the nontrivial automorphism of $\mathbf{F}_4$, upon fixing ζ we have built an $\mathbf{F}_4[a]$-isomorphism $[\sigma] : U_\zeta \simeq U_{\zeta+1} = \sigma^*(U_\zeta)$ corresponding to the automorphism $(c,g) \mapsto (c+f(g), g)$ of $C^- \times G$. By inspection, the automorphism $\sigma^*([\sigma]) \circ [\sigma]$ of U_ζ is the identity map. Thus, $[\sigma]$ defines a descent datum on the central extension U_ζ relative to the quadratic Galois covering $\mathrm{Spec}(\mathbf{F}_4[a]) \to \mathrm{Spec}(\mathbf{F}_2[a])$. The descent is a non-commutative central extension of G by C^- over $\mathbf{F}_2[a]$, so it yields the desired k-group by base change.

3. The cckp-kernel

In a smooth connected unipotent k-group U, any two smooth connected p-torsion central k-subgroups generate a third such subgroup. Hence, the following definition makes sense.

Definition 3.1. The maximal smooth connected p-torsion central k-subgroup of U is the *cckp-kernel*.

Note that if $U \neq 1$ then its cckp-kernel is nontrivial, since the latter contains the cckp-kernel of the last nontrivial term of the descending central series of U. By Galois descent and specialization (as in the proof of [CGP, 1.1.9(1)]), the formation of the cckp-kernel commutes with any separable extension on k. However, its formation generally does *not* commute with purely inseparable extension on k; see Exercise U.2(ii).

Proposition 3.2. *Let U be a smooth connected unipotent k-group, and let k'/k be a separable extension. Let F denote the cckp-kernel of U. Then U is k-wound if and only if $U_{k'}$ is k'-wound, and the quotient U/F is k-wound whenever U is k-wound. Also, the following conditions are equivalent:*

1. *U is k-wound,*
2. *U does not have a central k-subgroup k-isomorphic to $\mathbf{G}_{\mathrm{a}}$,*
3. *the cckp-kernel F of U is k-wound.*

This proposition implies that U is k-wound if and only if U admits no nontrivial k-homomorphism from $\mathbf{G}_{\mathrm{a}}$. Such a characterization of the k-wound property is analogous to the characterization of anisotropic tori over a field in terms of homomorphisms from $\mathbf{G}_{\mathrm{m}}$ over the ground field.

Proof. Obviously (1) $\Rightarrow$ (2). By Theorem 2.6, (2) and (3) are equivalent. Also, by specialization (as in the proof of [CGP, 1.1.9(1)]), if U_K is not K-wound for some separable extension K/k then the same holds with K/k taken to be some finite separable extension. Thus, to prove the equivalence of (1), (2), and (3) and the fact that $U_{k'}$ is k'-wound whenever U is k-wound, it suffices to show that if U_{k_s} is not k_s-wound then the cckp-kernel F of U is not k-wound.

Let $\varphi : \mathbf{A}^1_{k_s} \to U_{k_s}$ be a non-constant k_s-scheme morphism. Composing with a $U(k_s)$-translation if necessary, we may assume $\varphi(0) = 1$. We may choose such a φ so that $\varphi(\mathbf{A}^1_{k_s})$ is central. Indeed, suppose $\varphi(\mathbf{A}^1_{k_s})$ is non-central, so U is not commutative and there exists $g \in U(k_s)$ not centralizing $\varphi(\mathbf{A}^1_{k_s})$. The k_s-scheme morphism $\varphi' : \mathbf{A}^1_{k_s} \to U_{k_s}$ defined by $\varphi'(x) = g^{-1}\varphi(x)^{-1}g\varphi(x)$ (which is generally not a homomorphism even when φ is a homomorphism) carries 0 to 1, so it is then non-constant, and its image lies in derived group $\mathscr{D}(U_{k_s}) = \mathscr{D}(U)_{k_s}$. The k-subgroup $\mathscr{D}(U)$ has smaller dimension than U and is nontrivial since the smooth connected k-group U is not commutative. Hence, by iteration with the descending central series of U, the required non-constant φ with

$\varphi(\mathbf{A}^1_{k_s})$ central is eventually obtained. We may also assume that $\varphi(\mathbf{A}^1_{k_s})$ is p-torsion by replacing the original φ with $p^e \cdot \varphi$ for some $e \geqslant 0$.

The nontrivial k_s-subgroup generated by $\varphi(\mathbf{A}^1_{k_s})$ lies in the cc$k_s p$-kernel of U_{k_s}; i.e., it lies in F_{k_s}. Thus F_{k_s} is not k_s-wound, so by Corollary 2.7 the k-group F is not k-wound.

It remains to show that if U is k-wound then U/F is k-wound. For this we may, in view of the preceding conclusions, assume that $k = k_s$. Suppose that U is k-wound and U/F is not k-wound. Thus, there exists a central k-subgroup A of U/F that is k-isomorphic to $\mathbf{G}_{\rm a}$. Let π denote the canonical homomorphism $U \to U/F$. The k-subgroup scheme $\pi^{-1}(A)$ in U is an extension of A by F, so it is smooth, connected, and unipotent.

We claim that $\pi^{-1}(A)$ is central in U. If not, we get a non-constant k-scheme morphism $\varphi : \mathbf{A}^1_k \to F$ (contradicting that U is k-wound) as follows. Choose $g \in U(k)$ not centralizing $\pi^{-1}(A)$ (recall $k = k_s$), identify $\mathbf{G}_{\rm a}$ with $A = \pi^{-1}(A)/F$, and define $\varphi : \pi^{-1}(A)/F \to F$ by $xF \mapsto gxg^{-1}x^{-1}$. Thus, $\pi^{-1}(A)$ is central in U. Similarly, $\pi^{-1}(A)$ is p-torsion because otherwise we would get a non-constant k-scheme morphism $\psi : \mathbf{A}^1_k \to F$ via $\psi(xF) = x^p$. We have shown that $\pi^{-1}(A)$ lies in the cckp-kernel F of U, so the given inclusion $F \subset \pi^{-1}(A)$ is an equality. Hence, $A = 1$, which is absurd since $A \simeq \mathbf{G}_{\rm a}$. □

Corollary 3.3. *Let U be a k-wound smooth connected unipotent k-group. Define the ascending chain of smooth connected normal k-subgroups $\{U_i\}_{i \geqslant 0}$ as follows: $U_0 = 1$ and U_{i+1}/U_i is the cckp-kernel of the k-wound group U/U_i for all $i \geqslant 0$. These subgroups are stable under k-group automorphisms of U, their formation commutes with any separable extension of k, and $U_i = U$ for sufficiently large i.*

Moreover, if H is a smooth k-group acting on U then H carries each U_i into itself.

Proof. Well-posedness of the definition (e.g., that U/U_1 is k-wound) and compatibility with separable extension on k follow from Proposition 3.2. By dimension considerations, $U_i = U$ for sufficiently large i since the cckp-kernel of a nontrivial smooth connected unipotent k-group is nontrivial.

Finally, if H is a smooth k-group acting on U then we need to prove that H carries each U_i into itself. For this we may extend scalars to k_s, so k is separably closed. Then the H-stability of U_i is equivalent to the $H(k)$-stability of U_i, and this latter property is a special case of each U_i being stable under all k-automorphisms of U. □

As an application of the structure of k-wound groups we can unify the definitions of "wound" for unipotent groups and "anisotropic" for tori (see Remark 2.2):

Corollary 3.4. *A unipotent smooth connected k-group U is k-wound if and only if $U(k[x, 1/x]) = U(k)$. More generally, if $h \in k[x]$ is nonzero and separable then U is k-wound if and only if $U(k[x][1/h]) = U(k)$.*

Proof. The equality $U(k[x, 1/h]) = U(k)$ clearly forces U to be k-wound. For the converse, suppose U is k-wound, so U_{k_s} is k_s-wound (Proposition 3.2). Thus, to prove that $U(k[x][1/h]) = U(k)$ we may replace k with k_s (by Galois descent). Hence, now $h = c\prod(x - a_i)$ for $c \in k^\times$ and pairwise distinct $a_i \in k$. For each i, the k-wound property implies $U(k(\!(x - a_i)\!)) = U(k[\![x - a_i]\!])$ by [Oes, V, §8] (whose proof rests on the existence of a composition series for the k-wound U with successive quotients that are commutative p-torsion wound hypersurface groups; see Corollary 3.3, Proposition 1.17, and Example 2.5). Writing $h = (x - a_i)q_i$, inside $k(\!(x - a_i)\!)$, we have $k[x][1/h] \cap k[\![x - a_i]\!] = k[x][1/q_i]$. Thus, $U(k[x][1/h]) = \bigcap_i U(k[x][1/q_i]) = U(k[x])$ since $\gcd_i(q_i) = 1$, and $U(k[x]) = U(k)$ since U is k-wound. □

Remark 3.5. It is well-known that if F is a non-archimedean local field and T is an F-torus then $T(F)$ is compact if and only if T is F-anisotropic. (To prove compactness of $T(F)$ for F-anisotropic T, identify $\mathrm{X}(T_{F_s})$ with a quotient of a direct sum of copies of the regular representation of $\mathrm{Gal}(F'/F)$ over $\mathbf{Z}$ for a finite Galois splitting field F'/F of T. This identifies T with an F-subgroup of $\mathrm{R}_{F'/F}(\mathbf{G}_\mathrm{m})^N$ for some $N \geqslant 1$. By F-anisotropicity, T lies in $(T^1_{F'/F})^N$, where $T^1_{F'/F}$ is the F-torus $\ker(\mathrm{R}_{F'/F}(\mathbf{G}_\mathrm{m}) \to \mathbf{G}_\mathrm{m})$ of "norm-1 units". Since $T^1_{F'/F}(F) = \mathscr{O}^\times_{F'}$, we are done.)

There is a similar equivalence in the unipotent case, as follows. We restrict attention to unipotent smooth connected U over a local function field k, since in characteristic 0 the split condition always holds for unipotent groups and hence compactness cannot hold when the unipotent group is nontrivial. Over such k, the equivalence of k-woundness for U and compactness for $U(k)$ is [Oes, VI, §1] (whose proof ultimately reduces to an explicit calculation with wound hypersurface groups over $k = \mathbf{F}_q(\!(t)\!)$, using the "principal part" criterion at the end of Example 2.5).

Remark 3.6. The separability condition on h in Corollary 3.4 cannot be relaxed For example, if $p = 2$ and $a \in k - k^2$ then the k-wound group $U = \{y^2 = x - ax^2\}$ is a smooth plane conic with $U(k) \neq \emptyset$, so U is k-rational. Explicitly, $U \simeq \mathrm{Spec}\, k[t, 1/(t^2 - a)]$ via $t \mapsto (1/(t^2 - a), t/(t^2 - a))$.

We will now prove a structure theorem that is analogous to the unique presentation of a torus over a field as an extension of an anisotropic torus by a split torus.

Theorem 3.7. *Let U be a unipotent smooth connected k-group. There exists a unique k-split smooth connected normal k-subgroup $U_{\mathrm{split}} \subset U$ such that U/U_{split} is k-wound.*

The subgroup U_{split} contains the image of every k-homomorphism from a k-split smooth connected unipotent k-group into U. Also, the kernel of every k-homomorphism from U into a k-wound smooth connected unipotent k-group contains U_{split}, and the formation of the k-subgroup U_{split} is compatible with any separable extension of k.

Proof. The proof is by induction on $\dim U$. If U is k-wound then $U_{\mathrm{split}} := \{1\}$ satisfies the requirements and is unique as such. Assume that U is not k-wound, and let A be a smooth central k-subgroup isomorphic to $\mathbf{G}_\mathrm{a}$ (Proposition 3.2). Let $H = U/A$. By induction, there exists a smooth connected normal k-subgroup H_{split} in H with the desired properties in relation to H (in the role of U). Let U_{split} be the corresponding subgroup of U containing A. It is k-split, and $U/U_{\mathrm{split}} \simeq H/H_{\mathrm{split}}$ is k-wound.

Let U' be a smooth connected unipotent k-group having a composition series

$$U' = U'_0 \supset U'_1 \supset \cdots$$

with successive quotients k-isomorphic to $\mathbf{G}_\mathrm{a}$, and let $\varphi : U' \to U$ be a k-homomorphism. There exists a minimal i such that $\varphi(U'_i) \subset U_{\mathrm{split}}$. If $i > 0$ then there is induced a k-homomorphism $\mathbf{G}_\mathrm{a} \simeq U'_{i-1}/U'_i \to U/U_{\mathrm{split}}$ with nontrivial image. This contradicts that U/U_{split} is k-wound. Thus, $i = 0$; i.e., $\varphi(U') \subset U_{\mathrm{split}}$. It follows in particular that U_{split} is unique. Also, for any k-homomorphism $\varphi : U \to U''$ into a k-wound smooth connected unipotent k-group U'' we have $\varphi(U_{\mathrm{split}}) \subset U''_{\mathrm{split}} = \{1\}$. This says that $\ker \varphi$ contains U_{split}.

The last assertion of the theorem follows from Proposition 3.2. Indeed, if k'/k is a separable extension and $U' := U_{k'}$ then $(U_{\mathrm{split}})_{k'} \subset U'_{\mathrm{split}}$ and the k'-split quotient $U'_{\mathrm{split}}/(U_{\mathrm{split}})_{k'}$ is a k'-subgroup of the k'-group $(U/U_{\mathrm{split}})_{k'}$ that is k'-wound (by Proposition 3.2). This forces $U'_{\mathrm{split}} = (U_{\mathrm{split}})_{k'}$. ∎

Example 3.8. An elementary non-commutative example of Theorem 3.7 over any imperfect field k of characteristic $p > 0$ is obtained via a central pushout construction, as follows. Let $U_3 \subset \mathrm{GL}_3$

be the standard upper triangular unipotent subgroup. Its scheme-theoretic center is the group $Z \simeq \mathbf{G}_{\mathrm{a}}$ consisting of points

$$u(x) = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Viewing U_3 as a central extension of $\mathbf{G}_{\mathrm{a}}^2$ by Z, let U be the pullback along the inclusion $y : U' \hookrightarrow \mathbf{G}_{\mathrm{a}}^2$ where U' is a 1-dimensional k-wound group as in Example 2.4. A straightforward calculation shows that U is a non-commutative 2-dimensional smooth connected k-subgroup of U_3 that is neither split nor wound (since it contains a central $Z = \mathbf{G}_{\mathrm{a}}$ and admits the wound quotient U'). Thus, $Z = U_{\mathrm{split}}$ and the sequence $1 \to Z \to U \to U/Z \to 1$ cannot split since Z is the center of U and U/Z is commutative.

Corollary 3.9. *A unipotent smooth connected k-group U is k-split if and only if $U \simeq \mathbf{A}_k^n$ as k-schemes for some $n \geqslant 0$. It is also equivalent for there to be a dominant k-morphism $V = \mathbf{A}_k^d - Z \to U$ for a generically smooth closed subscheme $Z \subset \mathbf{A}_k^d$.*

Before we prove this corollary, we make some observations. The dominance condition on $\mathbf{A}_k^d - Z \to U$ forces $Z \neq \mathbf{A}_k^d$, and by Remark 3.6 we cannot remove the generic smoothness condition on Z. Also, Corollary 3.9 has no analogue for tori, since any torus T over any field F is unirational (by using an isogeny-splitting of the inclusion of F-tori $T \hookrightarrow \mathrm{R}_{F'/F}(T_{F'})$ for a finite separable splitting field F'/F of T). Finally, the proof of sufficiency below for the second criterion in Corollary 3.9 uses Bertini's Theorem in the affine setting over k_s but the only Z that we actually use in later applications is a (possibly empty) union of hyperplane slices in distinct coordinate directions, for which linear algebra works equally well in place of Bertini's Theorem.

Proof. First assume that U is k-split, and let $n = \dim U$. We seek to prove that $U \simeq \mathbf{A}_k^n$ as k-schemes. The cases $n \leqslant 1$ are obvious, so we may assume $n > 1$. Thus, there is a k-split smooth connected normal k-subgroup $U' \subset U$ such that $U/U' \simeq \mathbf{G}_{\mathrm{a}}$. By induction, $U' \simeq \mathbf{A}_k^{n-1}$ as k-schemes. We claim that the U'-torsor $U \to \mathbf{G}_{\mathrm{a}} = \mathbf{A}_k^1$ for the étale topology is trivial. More generally, for any *affine* k-scheme X the cohomology set $\mathrm{H}^1(X_{\text{ét}}, U')$ classifying U'-torsors for the étale topology on X is trivial. Indeed, using a composition series for U' over k reduces this to the case of $\mathrm{H}^1(X_{\text{ét}}, \mathbf{G}_{\mathrm{a}})$, and by étale descent theory for quasi-coherent sheaves this coincides with $\mathrm{H}^1(X_{\mathrm{Zar}}, \mathscr{O}) = 0$. We conclude that as k-schemes, $U \simeq U' \times (U/U') \simeq \mathbf{A}_k^n$, as desired.

For the converse, suppose there is a dominant k-morphism $f : V = \mathbf{A}_k^d - Z \to U$ for a generically smooth closed subscheme $Z \subset \mathbf{A}_k^d$. To prove that U is k-split, we may replace U with the k-wound quotient U/U_{split} from Theorem 3.7 to reduce to the case that U is k-wound. In such cases we seek to prove that $U = 1$, so it suffices to prove that the dominant f is a constant map into $U(k)$. It is harmless to extend scalars to k_s, so $V(k)$ is Zariski-dense in V. Since Z is generically smooth and $Z \neq \mathbf{A}_k^d$, by Bertini's Theorem over k there exists a dense open locus Ω in the $2(d-1)$-dimensional quasi-projective variety $\mathbf{Gr}_d$ of affine lines in $\mathbf{A}_k^d$ such that the closed subscheme $Z_K \bigcap \ell$ in ℓ is 0-dimensional and K-smooth for all K/k and affine lines ℓ in K^d corresponding to a point in $\Omega(K)$. (If Z is a union of several affine hyperplanes then linear algebra gives the same conclusion, without using Bertini's Theorem.) Such a closed subscheme is K-étale, so for each affine line $\ell \simeq \mathbf{A}_k^1$ corresponding to a point in $\Omega(k)$ the open locus $V \bigcap \ell$ in ℓ is the complement of the zero locus on ℓ of a separable polynomial. Hence, by Corollary 3.4 and the k-wound hypothesis on U, f has constant restriction to $V \bigcap \ell$ for all $\ell \in \Omega(k)$.

To prove the constancy of f, it suffices to prove the constancy of f on $V'(k)$ for a dense open $V' \subset V$ (since $k = k_s$). The idea is that for a generic pair of distinct points v and v' in V, the line ℓ joining them should correspond to a point in Ω and hence the constancy of f on $V \bigcap \ell$ forces $f(v) = f(v')$. To make this idea rigorous, consider the $2d$-dimensional variety $X = V \times V - \Delta$ of ordered pairs of distinct points in V. There is an evident morphism $X \to \mathbf{Gr}_d$ assigning to any $(v, v') \in X$ the unique line joining them, and all fibers are 2-dimensional, so for dimension reasons this map is dominant. Hence, there is a dense open locus $X' \subset X$ that is carried into Ω. For all $(v, v') \in X'(k)$, the unique line $\ell \subset k^d$ passing through v and v' corresponds to a point in $\Omega(k)$, so f is constant on $V \bigcap \ell$. In particular, $f(v) = f(v')$. The projection $\mathrm{pr}_1 : X' \to V$ is dominant, so its image contains a dense open subset of V. We may choose $v_0 \in V(k)$ in this image, so the open subset $V' := X' \bigcap (\{v_0\} \times V)$ in V (via pr_2) is non-empty and therefore dense. Clearly $f(v') = f(v_0)$ for all $v' \in V'(k)$. □

Remark 3.10. The above cohomological proof that $U \simeq \mathbf{A}_k^n$ as k-schemes for k-split unipotent smooth connected k-groups U generalizes to show that any k-split solvable smooth connected affine k-group is k-isomorphic to $\mathbf{A}_k^{n,m} := \mathbf{A}_k^n \times (\mathbf{A}_k^1 - \{0\})^m$ for some $n, m \geqslant 0$. (This result is due to Rosenlicht, who gave a non-cohomological proof; see Lemma 2 to Theorem 2 in [Ros]. A generalization to homogeneous spaces under such groups is [Ros, Thm. 5].) To carry out this generalization, first note that a composition series expressing the k-split property reduces the problem to proving that for a k-split solvable smooth connected k-group G, every G-torsor over $\mathbf{G}_{\mathrm{a}}$ or $\mathbf{G}_{\mathrm{m}}$ for the étale topology is a trivial torsor; i.e., $\mathrm{H}^1((\mathbf{G}_{\mathrm{a}})_{\text{ét}}, G)$ and $\mathrm{H}^1((\mathbf{G}_{\mathrm{m}})_{\text{ét}}, G)$ vanish.

As in the proof of Corollary 3.9, by using a composition series expressing the k-split property of G, the low-degree 6-term exact sequence in non-abelian cohomology associated to a short exact sequence of smooth affine group schemes reduces the vanishing assertion to the special cases $G = \mathbf{G}_{\mathrm{a}}$ and $\mathbf{G}_{\mathrm{m}}$. The case $G = \mathbf{G}_{\mathrm{a}}$ was addressed more generally in the proof of Corollary 3.9. The case $G = \mathbf{G}_{\mathrm{m}}$ follows from the general equality $\mathrm{H}^1(X_{\text{ét}}, \mathbf{G}_{\mathrm{m}}) = \mathrm{Pic}(X)$ (via descent theory for line bundles) and the PID property for $k[x]$ and $k[x, 1/x]$.

Corollary 3.11. *If G is a k-split solvable smooth connected affine k-group then $\mathscr{D}(G)$ is k-split.*

Proof. By the structure theory over $\overline{k}$, $\mathscr{D}(G)$ is unipotent. Hence, by Corollary 3.9 it suffices to construct a dominant k-morphism $\mathbf{A}_k^n - Z \to \mathscr{D}(G)$ for some $n \geqslant 1$ and some geometrically reduced closed subscheme $Z \subset \mathbf{A}_k^n$. Since the product of several varieties $\mathbf{A}_k^{n_i} - Z_i$ with generically smooth Z_i has the form $\mathbf{A}_k^{\sum n_i} - Z$ for a generically smooth closed subscheme Z, and the geometric points of $\mathscr{D}(G)$ can be expressed as a product of a universally bounded number of commutators (depending on G), by considering such a product morphism for a sufficiently large set of commutators we are reduced to constructing a dominant k-morphism $\mathbf{A}_k^N - Z \to G$ for some $N \geqslant 1$ and generically smooth Z. By Remark 3.10 there is a k-scheme isomorphism $\mathbf{A}_k^{n,m} \simeq G$, so we are done. □

Let G be a smooth connected affine k-group. The *k-unipotent radical* $\mathscr{R}_{u,k}(G)$ is the maximal normal unipotent smooth connected k-subgroup of G, and the *k-split unipotent radical* $\mathscr{R}_{us,k}(G)$ is the maximal normal k-split unipotent smooth connected k-subgroup of G. For any extension field K/k clearly $\mathscr{R}_{u,k}(G)_K \subset \mathscr{R}_{u,K}(G_K)$ inside G_K. This inclusion is an equality when K/k is separable [CGP, 1.1.9(1)], but generally not otherwise (e.g., for a nontrivial purely inseparable extension k'/k of degree $p = \mathrm{char}(k)$ and G equal to the Weil restriction $\mathrm{R}_{k'/k}(\mathbf{G}_{\mathrm{m}})$ we have $\mathscr{R}_{u,k}(G) = 1$ but $\mathscr{R}_{u,k'}(G_{k'}) = \mathbf{G}_{\mathrm{a}}^{p-1}$; see [CGP, 1.1.3, 1.6.3]).

Corollary 3.12. *For any smooth connected affine k-group G, $\mathscr{R}_{us,k}(G) = \mathscr{R}_{u,k}(G)_{\mathrm{split}}$. In particular, $\mathscr{R}_{u,k}(G)/\mathscr{R}_{us,k}(G)$ is k-wound, and the formation of $\mathscr{R}_{us,k}(G)$ commutes with separable extension on k.*

Proof. By Galois descent, $\mathscr{R}_{us,k_s}(G_{k_s})$ descends to a smooth connected unipotent normal k-subgroup of G. This descent is k-split, since the k-split property of smooth connected unipotent k-groups is insensitive to separable extension on k (due to Theorem 3.7). Thus, the descent is contained in $\mathscr{R}_{us,k}(G)$, so the inclusion $\mathscr{R}_{us,k}(G)_{k_s} \subset \mathscr{R}_{us,k_s}(G_{k_s})$ is an equality. In other words, the formation of $\mathscr{R}_{us,k}(G)$ is compatible with separable algebraic extension on k. Hence, to prove the compatibility with general separable extension on k and the agreement with the maximal k-split smooth connected k-subgroup of $\mathscr{R}_{u,k}(G)$, we may assume $k = k_s$. But $\mathscr{R}_{u,k}(G)_{\rm split}$ is a characteristic k-subgroup of G, so it is normal due to the Zariski-density of $G(k)$ in G when $k = k_s$. This proves that $\mathscr{R}_{u,k}(G)_{\rm split} \subset \mathscr{R}_{us,k}(G)$, so equality holds.

The compatibility of the formation of $\mathscr{R}_{us,k}(G)$ with respect to separable extension on k now follows from such a compatibility for the formation of $U_{\rm split}$ in Theorem 3.7 and the formation of $\mathscr{R}_{u,k}(G)$ [CGP, 1.1.9(1)]. ☐

4. Torus actions on unipotent groups

Consider the action of a k-torus T on a smooth connected unipotent k-group U. This induces a linear representation of T on $\mathrm{Lie}(U)$, so if T is k-split then we get a weight space decomposition of $\mathrm{Lie}(U)$. If U is a vector group then it is natural to wonder if this decomposition of $\mathrm{Lie}(U)$ can be lifted to the group U. When $\dim U > 1$, the T-action may not respect an initial choice of linear structure on U (in the sense of Definition 1.1) since $\mathrm{char}(k) = p > 0$, so we first seek a T-equivariant linear structure.

For example, if $U = \mathbf{G}_{\rm a}^2$ with its usual linear structure and $T = \mathbf{G}_{\rm m}$ with the action $t.(x,y) = (tx, (t^p - t)x^p + ty)$ then the T-action is not linear and the action on $\mathrm{Lie}(U) = k^2$ has the single weight given by the identity character of T. But note that if we transport the T-action by the additive automorphism $(x,y) \mapsto (x, y + x^p)$ of U then the action becomes $t.(x,y) = (tx, ty)$, which is linear.

Tits proved rather generally that if a k-split T acts on U with only nontrivial weights on $\mathrm{Lie}(U)$, then there are nontrivial constraints on the possibilities for U as a k-group and that (after passing to a suitable characteristic composition series for U) the action can always be described in terms of linear representations of T. To explain his results in this direction, we begin with the following result that generalizes Lemma 1.14 by incorporating a torus action.

Proposition 4.1. *Let U be a smooth p-torsion commutative affine k-group equipped with an action by an affine k-group scheme T of finite type. There exists a linear representation of T on a finite dimensional k-vector space V and a T-equivariant isomorphism of U onto a k-subgroup of V.*

Proof. Let $\mathbf{Hom}(U, \mathbf{G}_{\rm a})$ be the covariant functor assigning to any k-algebra R the R-module $\mathrm{Hom}_R(U_R, \mathbf{G}_{\rm a})$ of R-group morphisms $\phi : U_R \to \mathbf{G}_{\rm a}$ (with R-module structure defined via the R-linear structure on the R-group $\mathbf{G}_{\rm a}$). There is a natural R-linear injection $\mathbf{Hom}(U, \mathbf{G}_{\rm a})(R) \hookrightarrow R[U_R] = R \otimes_k k[U]$ defined by $\phi \mapsto \phi^*(x)$ (where x is the standard coordinate on $\mathbf{G}_{\rm a}$), and its image is the R-submodule of "group-like" elements: those f satisfying $m_R^*(f) = f \otimes 1 + 1 \otimes f$ (where $m : U \times U \to U$ is the group law). This is an R-linear condition on f and is functorial in R, so by k-flatness the R-module of group-like elements over R is J_R where $J \subset k[U]$ is the k-subspace of group-like elements over k. In particular, the natural map $R \otimes_k \mathrm{Hom}(U, \mathbf{G}_{\rm a}) \to \mathrm{Hom}_R(U_R, \mathbf{G}_{\rm a})$ is an isomorphism.

The (left) T-action on U defines a left T-action on $\mathbf{Hom}(U, \mathbf{G}_{\rm a})$ (via $(t.\phi)(u) = \phi(t^{-1}.u)$) making the k-linear inclusion $\mathrm{Hom}(U, \mathbf{G}_{\rm a}) \hookrightarrow k[U]$ a T-equivariant map. Thus, $\mathrm{Hom}(U, \mathbf{G}_{\rm a})$ is the directed union of T-stable finite-dimensional k-subspaces, due to the same property for $k[U]$ [Bo, 1.9–1.10]. By Lemma 1.14 there is a k-subgroup inclusion $j : U \hookrightarrow \mathbf{G}_{\rm a}^n$ for some $n \geqslant 1$. Let $W \subset \mathrm{Hom}(U, \mathbf{G}_{\rm a})$

be a T-stable finite-dimensional k-subspace containing $j^*(x_1), \dots, j^*(x_n)$. The canonical map $U \to \underline{W^*} = \mathrm{Spec}(\mathrm{Sym}(W))$ is a T-equivariant closed immersion that is a k-homomorphism (since W consists of group-like elements in $k[U]$ that generate $k[U]$ as a k-algebra). □

We now apply our work with wound groups to analyze the structure of smooth connected unipotent k-groups equipped with a sufficiently nontrivial action by a k-torus.

Proposition 4.2. *Let T, U, and V be as in Proposition 4.1, with T a k-torus, and let $V_0 \times V'$ be the unique T-equivariant k-linear decomposition of V with $V_0 = V^T$ (so V' is the span of the isotypic k-subspaces for the nontrivial irreducible representations of T over k that occur in V). The product map*

$$\iota : (U \cap \underline{V}_0) \times (U \cap \underline{V}') \to U$$

is an isomorphism and there is a T-equivariant k-linear decomposition $V' = V_1' \times V_2'$ of V' and a T-equivariant k-automorphism α of the additive k-group $\underline{V}$ such that

$$\alpha(U) = (\alpha(U) \cap \underline{V}_0) \times \underline{V}_1'.$$

In particular, if $V^T = 0$ then the k-group U is a vector group admitting a T-equivariant linear structure.

Proof. Clearly $\underline{V_0} = Z_{\underline{V}}(T)$ as k-subgroups of $\underline{V}$, so $U_0 := U \cap \underline{V}_0$ is $Z_U(T)$. This is smooth since U is smooth. We will first prove that ι is an isomorphism, so $U \cap \underline{V}'$ is smooth.

Since the formation of V' clearly commutes with scalar extension on k, to establish that ι is an isomorphism we may assume k is algebraically closed. Choose $s \in T(k)$ such that for every weight χ of T in V', $\chi(s) \neq 1$. Consider the k-linear map $f : V \to V$ defined by $f(v) = s \cdot v - v$. It is obvious that f maps V onto V' with $\ker f = Z_V(s) = V_0$ and that the restriction of f to V' is a linear automorphism. The image $f(U)$ is a smooth k-subgroup of $\underline{V}'$, and it lies in U due to the T-stability of U inside $\underline{V}$. By definition, $\underline{V}'$ has a T-equivariant composition series whose successive quotients are 1-dimensional vector groups with a nontrivial T-action. Hence, all T-stable k-subgroup schemes of $\underline{V}'$ are connected. In particular, $f(U)$ is connected.

Since $U_0 \cap f(U) = 0$ (as $\underline{V_0} \cap \underline{V}' = 0$), under addition $U_0 \times f(U)$ is a k-subgroup of U. Thus, $f : U \to f(U)$ is a map onto a k-subgroup of U and the restriction of this map to $f(U)$ is therefore an endomorphism $f(U) \to f(U)$ with trivial kernel. But $f(U)$ is smooth and connected, so this endomorphism is an automorphism. In other words, $f : U \twoheadrightarrow f(U)$ is a projector up to an automorphism of $f(U)$. Since $U \cap \ker f = U \cap \underline{V}_0 = U_0$, this shows that the k-subgroup inclusion $U_0 \times f(U) \hookrightarrow U$ is an isomorphism, so $f(U) = U \cap \underline{V}'$. This completes the proof that ι is an isomorphism.

Let $U' = U \cap \underline{V}'$ and define $V_1' = \mathrm{Lie}(U')$. Then V_1' is a T-stable k-linear subspace of V'. Complete reducibility of k-linear representations of T provides a T-stable k-linear complement V_2' of V_1' in V'. Using the decomposition $\underline{V}' = \underline{V_1'} \times \underline{V_2'}$, the projection $U' \to \underline{V_1'}$ is an isomorphism on Lie algebras, so it is étale. By T-equivariance, the finite étale kernel is T-stable and therefore centralized by the connected T. But $Z_{\underline{V}'}(T) = 0$, so this kernel vanishes. Hence, $U' \to \underline{V_1'}$ is an isomorphism. It follows that the k-subgroup $U' \subset \underline{V}' = \underline{V_1'} \times \underline{V_2'}$ is the graph of a T-equivariant k-homomorphism $g : \underline{V_1'} \to \underline{V_2'}$. The T-equivariant k-automorphism α of $\underline{V}$ may be taken to be the automorphism that is the identity on $\underline{V}_0$ and is the inverse of the map $(v_1, v_2) \mapsto (v_1, g(v_1) + v_2)$ on $\underline{V_1'} \times \underline{V_2'}$. □

Theorem 4.3. *Let T be a k-torus and U a smooth p-torsion commutative affine k-group. Suppose that there is given an action of T on U over k. Then $U = U_0 \times U'$ with $U_0 = Z_U(T)$ and U'*

a T-stable k-subgroup that is a vector group admitting a linear structure relative to which T acts linearly. Moreover, U' is uniquely determined and is functorial in U.

Proof. By Propositions 4.1 and 4.2 we get the existence of U'. To prove the uniqueness and functoriality of U', we may assume $k = k_s$. Under the decomposition of U' into weight spaces relative to a T-equivariant linear structure on U', all T-weights must be nontrivial due to the definition of U_0. Hence, the canonical map $T \times U \to U$ defined by $(t, u) \mapsto t.u - u$ has image U'. This proves the uniqueness and functoriality of U'. □

If U in Theorem 4.3 is k-wound, then it must coincide with U_0 and so have trivial T-action. This is a special case of the following general consequence of invariance of the wound property with respect to separable extension of the ground field (Proposition 3.2):

Corollary 4.4. *Let T be a k-torus and U a k-wound smooth connected unipotent k-group. The only T-action on U is the trivial one.*

Proof. Our aim is to prove that the k-subgroup scheme $Z_U(T)$ is equal to U. For the k-group $G = U \rtimes T$, we have that the torus centralizer $Z_G(T)$ is equal to $Z_U(T) \rtimes T$. But $Z_G(T)$ is smooth and connected, so the same holds for $Z_U(T)$. Since $Z_U(T)$ is a scheme-theoretic centralizer, $\mathrm{Lie}(Z_U(T))$ is the T-centralizer in $\mathrm{Lie}(U)$. Hence, to prove that $Z_U(T) = U$ it suffices (by smoothness and connectedness of U) to prove that T acts trivially on $\mathrm{Lie}(U)$.

By Proposition 3.2, we may extend scalars to k_s, so T is k-split. Consider the composition series $\{U_i\}$ from Corollary 3.3. This is T-equivariant, and each U_{i+1}/U_i is k-wound, commutative, and p-torsion. The Lie algebras $\mathrm{Lie}(U_i)$ provide a T-equivariant filtration on $\mathrm{Lie}(U)$ whose successive quotients are the $\mathrm{Lie}(U_{i+1}/U_i)$. By complete reducibility for the T-action on $\mathrm{Lie}(U)$, to prove triviality of the action it suffices to treat the successive quotients of a T-stable composition series of k-subspaces of $\mathrm{Lie}(U)$. Hence, it suffices to treat each U_{i+1}/U_i separately in place of U, so we may assume that the k-wound U is commutative and p-torsion. Applying the decomposition in Theorem 4.3, we have $U = Z_U(T) \times U'$ where U' is a vector group. Since U is wound, we conclude that $U' = 1$, so the T-action on U is trivial. □

5. Solvable groups

By Theorem 3.7, if U is a unipotent smooth connected k-group then there is a unique k-split smooth connected k-subgroup U_{split} such that U/U_{split} is k-wound. For tori the analogous assertion using an anisotropic quotient is elementary. We shall establish a common generalization for solvable smooth connected affine k-groups G. This rests on the following common generalization of the wound condition in the unipotent case and the anisotropicity condition for tori:

Definition 5.1. A solvable smooth connected affine k-group G is *k-wound* if $G(k[x, 1/x]) = G(k)$.

By Remark 2.2, if G is a torus then this coincides with k-anisotropicity. By Corollary 3.4, if G is unipotent then this coincides with Definition 2.1.

An obvious but useful reformulation of Definition 5.1 is that the specialization homomorphism $G(k[x, 1/x]) \to G(k)$ at $x = 1$ has trivial kernel. For example, this immediately implies:

Lemma 5.2. *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of solvable smooth connected k-groups. If G' and G'' are k-wound then so is G.*

The converse of Lemma 5.2 fails in the commutative unipotent case, as we noted in Example 2.4.

Remark 5.3. A delicate aspect of Definition 5.1 is that it is generally poorly behaved with respect to any nontrivial extension of the ground field. More specifically, in the unipotent case the separable extensions preserve the woundness property and the purely inseparable ones can destroy it, whereas in the torus case the purely inseparable extensions preserve the woundness (i.e., anisotropicity) property and the separable ones can destroy it.

For a k-wound solvable smooth connected affine k-group G, it is obvious that any smooth connected k-subgroup is k-wound and that if G' is a k-split solvable smooth connected affine k-group then $\mathrm{Hom}_{k\text{-gp}}(G', G) = 1$ (e.g., argue by induction on $\dim G'$, using a composition series over k whose successive quotients are $\mathbf{G}_{\mathrm{a}}$ or $\mathbf{G}_{\mathrm{m}}$). In particular, if we drop the k-wound hypothesis on G then there is at most one k-split smooth connected normal k-subgroup $G_s \subset G$ such that G/G_s is k-wound.

Since any quotient of a k-split solvable smooth connected affine k-group is k-split, it is elementary that there exists a unique maximal k-split *normal* smooth connected k-subgroup $G_{\mathrm{split}} \subset G$. In Theorem 5.4 we will show that G_{split} is the semi-direct product of a k-split torus against a k-split unipotent smooth connected normal k-subgroup of G_{split}. (This is proved by more classical methods in [Bo, 15.4(i)].)

The only possibility for G_s is G_{split}, so G_s exists if and only if G/G_{split} is k-wound (in which case G_{split} remains maximal in G even without the normality requirement as a k-subgroup of G). The main result of this section is:

Theorem 5.4. *For any solvable smooth connected affine k-group G, the k-group G/G_{split} is a central extension of a k-wound unipotent group by a k-wound torus (so G/G_{split} is k-wound). In particular, G is k-wound if and only if $G_{\mathrm{split}} = 1$. The k-group G_{split} is the semi-direct product of a maximal k-split torus against a normal k-split unipotent smooth connected k-subgroup.*

The natural map $G \to G/G_{\mathrm{split}}$ is initial among k-homomorphisms from G to k-wound solvable smooth connected affine k-groups and the natural map $G_{\mathrm{split}} \to G$ is final among k-homomorphisms to G from k-split smooth connected affine k-groups.

Example 5.5. If F is a perfect field (perhaps of characteristic 0) and G is a solvable smooth connected affine F-group then $G = T \ltimes U$ for an F-torus T and an F-split unipotent smooth connected F-group U. Thus, $G_{\mathrm{split}} := T_{\mathrm{split}} \ltimes U$ is an F-split normal smooth connected F-subgroup such that $G/G_{\mathrm{split}} = T/T_{\mathrm{split}}$ is an F-anisotropic F-torus. It follows that Theorem 5.4 is only interesting when k is imperfect. Likewise, Theorem 5.4 is only nontrivial when $\mathscr{R}_u(G_{\overline{k}})$ is not defined over k as a $\overline{k}$-subgroup of $G_{\overline{k}}$ (e.g., $G = \mathrm{R}_{k'/k}(\mathbf{G}_{\mathrm{m}})$ for a nontrivial purely inseparable finite extension k'/k).

Remark 5.6. Although Definition 5.1 goes beyond the category of k-groups (using k-scheme morphisms from $\mathbf{A}^1_k - \{0\}$), it is natural to wonder if it can be expressed within the category of k-groups, as in the case of tori and unipotent groups. That is, if G is a solvable smooth connected affine k-group and $\mathrm{Hom}_{k\text{-gp}}(\mathbf{G}_{\mathrm{a}}, G) = 1$ and $\mathrm{Hom}_{k\text{-gp}}(\mathbf{G}_{\mathrm{m}}, G) = 1$ (equivalently, $\mathrm{Hom}_{k\text{-gp}}(G', G) = 1$ for all k-split solvable smooth connected affine k-groups G') then is G a k-wound group? This will be immediate once we prove that G/G_{split} is always k-wound.

Lemma 5.7. *Let U be a k-split unipotent smooth connected k-group, and M a (finite type) k-group scheme of multiplicative type. Any exact sequence of affine finite type k-groups*

$$1 \to M \to G \to U \to 1$$

is uniquely split: $G = M \times U$ as k-groups.

Proof. By the uniqueness claim and Galois descent, we may and do assume $k = k_s$. Hence, M is Cartier dual to a finitely generated $\mathbf{Z}$-module (so M is a k-subgroup of a split k-torus). The uniqueness of the splitting amounts to the assertion that $\mathrm{Hom}_{k\text{-gp}}(U, M) = 1$, which is obvious (e.g., use an inclusion of M into a k-torus). For the existence, we first note that G must be a central extension of U by M, since the conjugation action of $G/M = U$ on the commutative normal subgroup M defines a homomorphism from U to the automorphism functor of M, and any such homomorphism is trivial since U is connected and $\underline{\mathrm{Aut}}_{M/k}$ is represented by a constant k-group. Thus, we aim to prove the triviality of the pointed set $\mathrm{Ex}_k(U, M)$ of central extensions of U by M (in the category of affine k-group schemes of finite type).

By using a composition series of U over k with successive quotients isomorphic to $\mathbf{G}_{\mathrm{a}}$, the low-degree δ-functoriality involving $\mathrm{Hom}_{k\text{-gp}}(\cdot, M)$ and $\mathrm{Ex}_k(\cdot, M)$ (or direct bare-hands arguments with exact sequences and splittings thereof) reduces the problem to the case $U = \mathbf{G}_{\mathrm{a}}$. That is, we seek to prove the vanishing of $\mathrm{Ex}_k(\mathbf{G}_{\mathrm{a}}, M)$. Any central extension G of $\mathbf{G}_{\mathrm{a}}$ by M is commutative since the commutator of G factors through a bi-additive pairing $b : \mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{a}} \to M$ that is necessarily trivial since for all $u \in \mathbf{G}_{\mathrm{a}}(k) = k$ the map $b(u, \cdot) : \mathbf{G}_{\mathrm{a}} \to M$ is a k-homomorphism and hence trivial.

Since M is a product of $\mathbf{G}_{\mathrm{m}}$'s and μ_n's, by low-degree δ-functoriality considerations in the second variable (rather than the first) it suffices to separately treat the cases $M = \mu_n$ and $M = \mathbf{G}_{\mathrm{m}}$. The Kummer sequence $1 \to \mu_n \to \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}} \to 1$ and the vanishing of $\mathrm{Hom}_{k\text{-gp}}(\mathbf{G}_{\mathrm{a}}, \mathbf{G}_{\mathrm{m}})$ reduce us to the special case $M = \mathbf{G}_{\mathrm{m}}$. That is, we want $\mathrm{Ex}(\mathbf{G}_{\mathrm{a}}, \mathbf{G}_{\mathrm{m}}) = 1$.

Consider a central extension G of $\mathbf{G}_{\mathrm{a}}$ by $\mathbf{G}_{\mathrm{m}}$, so G is commutative. By viewing G as a $\mathbf{G}_{\mathrm{m}}$-torsor over the affine line (for the étale topology, and hence the Zariski topology due to descent theory for line bundles), we see that the quotient map $\pi : G \to \mathbf{G}_{\mathrm{a}}$ admits a k-scheme section σ. Using translation by a point in $\mathbf{G}_{\mathrm{m}}(k) = (\ker \pi)(k)$ we can arrange that $\sigma(0) = e \in G(k)$. Hence, the resulting identification of G with the pointed k-scheme $(\mathbf{G}_{\mathrm{m}} \times \mathbf{A}^1_k, (1, 0))$ carries the group law on G over to a composition law $(c, x) \cdot (c', x') = (cc' f(x, x'), x + x')$ for a symmetric polynomial $f : \mathbf{A}^2_k \to \mathbf{G}_{\mathrm{m}}$ satisfying $f(0, 0) = 1$. The only such f is $f = 1$. $\square$

Proof of Theorem 5.4. There are no nontrivial k-homomorphisms from a k-split solvable smooth connected affine k-group to a k-wound solvable smooth connected affine k-group, so the only task is to establish the central extension structure of G/G_{split} and the semi-direct product structure of G_{split}.

First consider $H = G_{\mathrm{split}}$. The derived group $\mathscr{D}(H)$ is unipotent (as we may check over $\overline{k}$) and k-split (Corollary 3.11), and any maximal k-torus of H maps isomorphically onto a maximal k-torus of $H/\mathscr{D}(H)$. Thus, to prove that H is a semi-direct product of a maximal k-split torus against a normal k-split unipotent subgroup U (in which case the k-torus H/U is k-split, so all maximal k-tori in H are k-split), we may pass to the k-split commutative $C = H/\mathscr{D}(H)$. This has a unique maximal k-torus T and the quotient $U = C/T$ is k-split unipotent, so by Lemma 5.7 there exists a unique decomposition $C = T \times U$. Thus, T is a quotient of the k-split C, so it is k-split.

It remains to understand the structure of G/G_{split}, which is to say that we can assume $G_{\mathrm{split}} = 1$. By Lemma 5.2 it remains to show that G is a central extension of a k-wound unipotent group by a k-wound torus. Since G is solvable, $\mathscr{D}(G)$ is unipotent (as we may check over $\overline{k}$). Thus, the formation of $\mathscr{D}(G)_{\mathrm{split}}$ commutes with separable extension on k (even though such extension may ruin the hypothesis that $G_{\mathrm{split}} = 1$). By computing with $G(k_s)$-conjugation on $\mathscr{D}(G)_{k_s}$, it follows that $\mathscr{D}(G)_{\mathrm{split}}$ is normal in G. But we have arranged that $G_{\mathrm{split}} = 1$, so $\mathscr{D}(G)_{\mathrm{split}} = 1$. Hence, by the structure theory in the unipotent case, $\mathscr{D}(G)$ is k-wound.

Let T be a maximal k-torus in G. Since $\mathscr{D}(G)$ is k-wound unipotent, the conjugation action by T on $\mathscr{D}(G)$ is trivial (Corollary 4.4). Since T maps isomorphically onto its image $\overline{T}$ in the commutative $G/\mathscr{D}(G)$ (due to the unipotence of $\mathscr{D}(G)$), the k-subgroup $T \times \mathscr{D}(G)$ in G is *normal*. Thus, the $G(k_s)$-action via conjugation on the normal k_s-subgroup $T_{k_s} \times \mathscr{D}(G)_{k_s}$ of G_{k_s} preserves the unique maximal k_s-torus T_{k_s}, so T is normal in G. The connectedness of G then forces T to be central in G. Since $G_{\rm split} = 1$, so $T_{\rm split} = 1$, we see that T is k-anisotropic. The formation of T as the maximal central torus commutes with scalar extension on k, even though such scalar extension may ruin the anisotropicity property of T.

The quotient $U = G/T$ now makes sense and is unipotent. It remains to prove that U is k-wound. By the structure theory in the unipotent case, it suffices to show that $U_{\rm split} = 1$. The preimage G' of $U_{\rm split}$ in G is an extension of $U_{\rm split}$ by T, so by Lemma 5.7 there is a unique k-group decomposition $G' = U_{\rm split} \times T$. The formation of G' commutes with scalar extension to k_s, as does the formation of $U_{\rm split} \subset U$, so the same holds for the unique subgroup of G' isomorphically lifting $U_{\rm split}$. That is, the unique product decomposition of G' commutes with scalar extension to k_s, so consideration of $G(k_s)$-conjugation on G'_{k_s} shows that $U'_{\rm split}$ is normal in G. But $G_{\rm split} = 1$, so $U'_{\rm split} = 1$. □

Corollary 5.8. *Let G be a solvable smooth connected affine k-group, and k'/k a regular field extension (i.e., separable with k algebraically closed in k'). The natural inclusion $(G_{\rm split})_{k'} \subset (G_{k'})_{\rm split}$ is an equality.*

Proof. The structure of $G/G_{\rm split}$ in Theorem 5.4 reduces the problem to verifying that if k'/k is a regular extension and G is k-wound unipotent (resp. a k-anisotropic k-torus) then $G_{k'}$ is k'-wound unipotent (resp. a k'-anisotropic k'-torus). The unipotent case follows from Proposition 3.2 since k'/k is separable. To handle the torus case, by consideration of Galois lattice character groups it suffices to prove the surjectivity of the restriction map $\mathrm{Gal}(k'_s/k') \to \mathrm{Gal}(k_s/k)$ relative to an embedding $k_s \to k'_s$ over $k \to k'$. The k'-algebra $k' \otimes_k k_s$ is a field contained in k'_s that is moreover Galois over k with Galois group $\mathrm{Gal}(k_s/k)$ in the evident manner, so we are done. □

In Remark 3.10, we saw that every k-split solvable smooth connected k-group is isomorphic as a k-scheme to $\mathbf{A}_k^{n,m} = \mathbf{A}_k^n \times (\mathbf{A}_k^1 - \{0\})^m$ for some $n, m \geqslant 0$. Here is a converse result for solvable groups in the spirit of the splitting criterion for unipotent groups in Corollary 3.9.

Corollary 5.9. *A solvable smooth connected k-group G is k-split if and only if there is a dominant k-morphism $f : \mathbf{A}_k^{n,m} \to G$.*

Proof. The implication "⇒" was shown in Remark 3.10, and for the converse we will use Theorem 5.4. Assuming such an f exists, to prove that G is split we may compose f with the quotient map $G \to G/G_{\rm split}$ to reduce to the case that G is k-wound, so G is an extension of a k-wound unipotent smooth connected k-group U by a k-anisotropic torus T. Our aim is to prove that $G = 1$. The composite map $\mathbf{A}_k^{n,m} \to U$ is dominant, so U is k-split by Corollary 3.9. But U is k-wound, so $U = 1$. That is, $G = T$ is a k-anisotropic torus.

Since the units in $k[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ are precisely the monomials $c \prod x_i^{e_i}$ with $c \in k^\times$ and $e_i \in \mathbf{Z}$, the same argument as in Remark 2.2 shows that any k-morphism $\mathbf{A}_k^{0,m} = (\mathbf{A}_k^1 - \{0\})^m \to T$ is a constant map to some $t \in T(k)$. Thus, the case $n = 0$ is settled. The anisotropicity has done its work, as it now suffices to show that for any k-torus T whatsoever and any k-morphism $f : \mathbf{A}_k^{n,m} \to T$, there is a (unique) factorization of f through the projection $\mathbf{A}_k^{n,m} \to \mathbf{A}_k^{0,m}$. This says that $f^* : k[T] \to k[\mathbf{A}_k^{n,m}]$ lands in the Laurent polynomial subalgebra $k[\mathbf{A}_k^{0,m}]$, for which it is harmless to check after extending scalars to k_s or even $\overline{k}$. Now $T = (\mathbf{G}_{\rm m})^N$ for some $N \geqslant 0$, so we are reduced to the case $T = \mathbf{G}_{\rm m}$. Any unit on $\mathbf{A}_k^{n,m}$ is the pullback of a unit on $\mathbf{A}_k^{0,m}$, so we are done. □

We end our discussion with some applications to general smooth connected affine k-groups G. Our interest is in variants of the k-subgroups $\mathscr{R}_{u,k}(G)$ and $\mathscr{R}_{us,k}(G)$ considered in Corollary 3.12. Define the *k-radical* $\mathscr{R}_k(G)$ to be the maximal normal solvable smooth connected k-subgroup of

G, and the *k-split radical* $\mathscr{R}_{s,k}(G)$ to be the maximal normal k-split solvable smooth connected k-subgroup of G. Obviously $G/\mathscr{R}_k(G)$ has trivial k-radical and $G/\mathscr{R}_{s,k}(G)$ has trivial k-split radical. Beware that $G/\mathscr{R}_k(G)$ may *not* be equal to its own derived group (in contrast with $G_{\overline{k}}/\mathscr{R}(G_{\overline{k}})$). Equivalently, there exist G such that $\mathscr{R}_k(G) = 1$ but $G \neq \mathscr{D}(G)$; see [CGP, 11.2.1] for many such G over any imperfect field k.

A *pseudo-reductive* k-group is a smooth connected affine k-group G such that $\mathscr{R}_{u,k}(G) = 1$. By Galois descent and Theorem 3.7, $\mathscr{R}_k(G)_{k_s} = \mathscr{R}_{k_s}(G_{k_s})$ and $\mathscr{R}_{us,k}(G)_{k_s} = \mathscr{R}_{us,k_s}(G_{k_s})$. There is no analogue of these equalities for $\mathscr{R}_{s,k}$.

Proposition 5.10. *Let G be a smooth connected affine k-group. Then $\mathscr{R}_k(G) = 1$ if and only if G is pseudo-reductive and has no nontrivial central k-torus.*

Proof. In either direction, G is pseudo-reductive, so we may and do assume that G is pseudo-reductive. Since pseudo-reductivity is inherited by smooth connected normal k-subgroups (as explained near the beginning of [CGP, 1.1]), $\mathscr{R}_k(G)$ is solvable and pseudo-reductive. But a solvable pseudo-reductive group is commutative [CGP, 1.2.3], so $\mathscr{R}_k(G)$ is commutative. The unique maximal k-torus S in $\mathscr{R}_k(G)$ must be normal in G and hence central (due to the connectedness of G), and $S \neq 1$ if $\mathscr{R}_k(G) \neq 1$ since $\mathscr{R}_k(G)$ cannot be unipotent when it is nontrivial (due to the pseudo-reductivity of G). Since any central k-torus in G lies in $\mathscr{R}_k(G)$, S is the maximal central k-torus in G. Thus, $\mathscr{R}_k(G) = 1$ if and only if $S = 1$. $\square$

As an application of Corollary 5.8, we can settle the following natural question: clearly $\mathscr{R}_{s,k}(G) \subset \mathscr{R}_k(G)_{\text{split}}$, but is this containment is an equality? It is equivalent to ask if $\mathscr{R}_k(G)_{\text{split}}$ is normal in G, or if the k-radical of $G/\mathscr{R}_{s,k}(G)$ is k-wound. In the proof of the unipotent analogue in Corollary 3.12 it was harmless to extend scalars to k_s, but that technique is not available in the present circumstances (and $G(k)$ might fail to be Zariski-dense in G). Nonetheless, we can prove an affirmative answer:

Proposition 5.11. *For a smooth connected affine k-group G, $\mathscr{R}_{s,k}(G) = \mathscr{R}_k(G)_{\text{split}}$.*

Proof. This amounts to the assertion that the action map

$$G \times \mathscr{R}_k(G)_{\text{split}} \to G$$

defined by $(g, h) \mapsto ghg^{-1}$ factors through $\mathscr{R}_k(G)_{\text{split}}$. By Zariski-density considerations it suffices to check this at the generic point η of G, which is to say that for $K = k(G)$ the K-group $(\mathscr{R}_k(G)_{\text{split}})_K$ is carried into itself under conjugation by the K-point $\eta \in G(K)$. More generally, we claim that the K-subgroup $(\mathscr{R}_k(G)_{\text{split}})_K$ inside G_K is stable under conjugation by the entire group $G(K)$. Since $(\mathscr{R}_k(G)_{\text{split}})_K = \mathscr{R}_K(G_K)_{\text{split}}$ (Corollary 5.8), it remains to note that for any smooth connected solvable group H over a field F, the closed F-subgroup H_{split} is obviously normalized by $H(F)$. $\square$

Corollary 5.12. *For any smooth connected affine k-group G, if $\mathscr{R}_{us,k}(G) = 1$ then $\mathscr{R}_{s,k}(G)$ is the maximal central k-split torus in G. In particular, $\mathscr{R}_{s,k}(G) = 1$ if and only if $\mathscr{R}_{us,k}(G) = 1$ with G containing no nontrivial k-split central k-torus.*

Proof. Consider the k-split solvable smooth connected affine k-group $R := \mathscr{R}_k(G)_{\text{split}} = \mathscr{R}_{s,k}(G)$. By the semi-direct product structure of split solvable smooth connected affine groups as in Theorem 5.4, R is the semi-direct product of a split torus against a normal split unipotent smooth connected k-subgroup U that must be $\mathscr{R}_{us,k}(R) = \mathscr{R}_{u,k}(R)$. Since the k_s-subgroup $U_{k_s} = \mathscr{R}_{u,k_s}(R_{k_s})$ is stable under all k_s-automorphisms of R_{k_s}, the normality of R_{k_s} in G_{k_s} implies that U_{k_s} is normal in G_{k_s}, so U is normal in G. Thus, $U \subset \mathscr{R}_{us,k}(G) = 1$, proving that R is a split torus. But the torus R is normal in the connected k-group G, so R is central in G. This proves that R is the maximal central k-split torus in G. $\square$

Corollary 5.13. *Let G be a smooth connected affine k-group. The following three conditions are equivalent:*

(1) *$G/\mathscr{R}_{s,k}(G)$ contains a nontrivial k-split solvable smooth connected k-subgroup,*
(2) *$G/\mathscr{R}_k(G)$ contains a nontrivial k-split solvable smooth connected k-subgroup,*
(3) *G contains a proper pseudo-parabolic k-subgroup.*

In (1) *and* (2) *it is equivalent to contain $\mathbf{G}_{\rm m}$ as a non-central k-subgroup.*

The notion of pseudo-parabolicity is defined in [CGP, 2.2.1]; it coincides with parabolicity in the connected reductive case [CGP, 2.2.9]. A typical example of a pseudo-parabolic k-subgroup that is not parabolic is $P := \mathrm{R}_{k'/k}(P') \subset \mathrm{R}_{k'/k}(G') =: G$ for a nontrivial purely inseparable finite extension k'/k and a proper parabolic k'-subgroup P' in a connected reductive k'-group G'. (Such P are precisely the pseudo-parabolic k-subgroups of G, by [CGP, 11.4.4]. The non-parabolicity of P, which is to say the non-properness of $G/P \simeq \mathrm{R}_{k'/k}(G'/P')$, follows from [CGP, A.5.6] since $\dim G'/P' > 0$.) By [CGP, 2.2.10], condition (3) is equivalent to the same for the maximal pseudo-reductive quotient $G/\mathscr{R}_{u,k}(G)$, and if G is pseudo-reductive then (3) is equivalent to saying that G has no non-central k-split torus [CGP, 2.2.3(1)].

Proof. The kernel $\mathscr{R}_k(G)/\mathscr{R}_{s,k}(G) = \ker(G/\mathscr{R}_{s,k}(G) \to G/\mathscr{R}_k(G))$ is k-wound since $\mathscr{R}_{s,k}(G) = \mathscr{R}_k(G)_{\rm split}$ (Proposition 5.11), so a nontrivial k-homomorphism from $\mathbf{G}_{\rm a}$ or $\mathbf{G}_{\rm m}$ to $G/\mathscr{R}_{s,k}(G)$ yields a nontrivial composite homomorphism to $G/\mathscr{R}_k(G)$. Hence, (1) implies (2).

To prove that (2) implies (3), we may replace G with $G/\mathscr{R}_{u,k}(G)$, so G is pseudo-reductive. The hypothesis in (2) says that the pseudo-reductive $G/\mathscr{R}_k(G)$ contains $\mathbf{G}_{\rm a}$ or $\mathbf{G}_{\rm m}$ as a k-subgroup. By [CGP, C.3.8], if a pseudo-reductive k-group contains $\mathbf{G}_{\rm a}$ as a k-subgroup then it contains a non-central $\mathbf{G}_{\rm m}$ as a k-subgroup. Since $\mathbf{G}_{\rm m}$ as a k-subgroup of $G/\mathscr{R}_k(G)$ cannot be central (as $\mathscr{R}_k(G/\mathscr{R}_k(G)) = 1$), it suffices to prove that if $G/\mathscr{R}_k(G)$ contains a non-central $\mathbf{G}_{\rm m}$ then so does G. The preimage H in G of such a $\mathbf{G}_{\rm m}$ is a smooth k-subgroup, so a maximal k-torus T in H must map onto this $\mathbf{G}_{\rm m}$. Hence, T contains a k-subgroup $\mathbf{G}_{\rm m}$ that is not in $\mathscr{R}_k(G)$ and thus is not central in G. The existence of a non-central $\mathbf{G}_{\rm m}$ in the pseudo-reductive k-group G is equivalent to (3), by [CGP, 2.2.3(2)].

Finally, we show that (3) implies (1). It is harmless to replace G with $G/\mathscr{R}_{us,k}(G)$, so $\mathscr{R}_{us,k}(G) = 1$. Thus, $\mathscr{R}_{s,k}(G)$ is the maximal k-split central k-torus in G (Corollary 5.12), so $G/\mathscr{R}_{s,k}(G)$ contains no non-trivial normal k-split k-tori (as a normal k-split k-torus in $G/\mathscr{R}_{s,k}(G)$ has preimage in G that is a k-split normal k-torus, and such a normal torus must be central due to connectedness of G, contradicting the maximality of $\mathscr{R}_k(G)$). From the definition of pseudo-parabolicity, (3) implies that G contains a non-central $\mathbf{G}_{\rm m}$. Its image in $G/\mathscr{R}_{s,k}(G)$ is a non-central k-subgroup isomorphic to $\mathbf{G}_{\rm m}$. ∎

Proposition 5.14. *Let G be a smooth connected affine k-group. The solvable smooth connected normal k-subgroup $R := \mathscr{R}_k(G)/\mathscr{R}_{u,k}(G)$ in the maximal pseudo-reductive quotient $G' := G/\mathscr{R}_{u,k}(G)$ is a central k-subgroup, and if N is a normal closed k-subgroup scheme of $\mathscr{R}_k(G)$ then the formation of images and preimages under $G \to G/N$ defines a bijection between the sets of pseudo-parabolic k-subgroups of G and G/N.*

Proof. Clearly $R = \mathscr{R}_k(G')$, so to prove the centrality of R in G' we can replace G with G' to reduce to the case when G is pseudo-reductive. Thus, by [CGP, Lemma 1.2.1], to prove the triviality of the smooth connected commutator (R, G) it suffices to prove that $(R, G)_{\overline{k}} \subset \mathscr{R}_u(G_{\overline{k}})$. In other words, we claim that $R_{\overline{k}}$ has central image in the connected reductive group $H := G_{\overline{k}}/\mathscr{R}_u(G_{\overline{k}})$. But this image is a solvable smooth connected normal subgroup of H, so it is a central torus in H due to the reductivity of H.

The centrality of R in G' implies that it lies in every pseudo-parabolic k-subgroup of G' (as pseudo-parabolic subgroups always contain the scheme-theoretic center). For any 1-parameter k-subgroup $\lambda : \mathbf{G}_{\mathrm{m}} \to G'/R$ there exists $n \geqslant 1$ such that λ^n lifts to a 1-parameter k-subgroup $\mathbf{G}_{\mathrm{m}} \to G'$ (since split tori lift lift up to isogeny through any *smooth* surjective k-homomorphism between smooth connected affine k-groups), so it follows formally from [CGP, 2.1.7, 2.1.9] and the role of 1-parameter subgroups in the definition of pseudo-parabolicity that the formation of images and preimages under the quotient map $G' \to G'/R$ defines a bijective correspondence between the sets of parabolic k-subgroups of G' and G'/R. But the formation of images and preimages under $G \to G'$ likewise defines a bijection between the sets of pseudo-parabolic k-subgroups of G and G' (see [CGP, Prop. 2.2.10]), so we conclude that the same holds for the formation of images and preimages under $G \to G/\mathscr{R}_k(G) = G'/R$. The analogous such bijectivity for the map $G \to G/N$ is therefore reduced to verifying that the evident containment $\mathscr{R}_k(G)/N \subset \mathscr{R}_k(G/N)$ inside G/N is an equality. This equality holds because the normal smooth connected k-subgroup

$$\mathscr{R}_k(G/N)/(\mathscr{R}_k(G)/N) \subset (G/N)/(\mathscr{R}_k(G)/N) = G/\mathscr{R}_k(G)$$

is solvable and hence trivial (due to the definition of $\mathscr{R}_k(G)$). □

Exercises on unipotent groups

U.1. Let U be a unipotent smooth connected commutative group scheme over a field k, and assume U is p-torsion if $\mathrm{char}(k) = p > 0$.

(i) If $\mathrm{char}(k) > 0$ and U is k-split, use Corollary 1.16 to prove that U is a vector group.

(ii) Assume $\mathrm{char}(k) = 0$ (so all k-group schemes of finite type are smooth, by Cartier's theorem). Prove that any short exact sequence $0 \to \mathbf{G}_{\mathrm{a}} \to G \to \mathbf{G}_{\mathrm{a}} \to 0$ with commutative G is split. Deduce that $U \simeq \mathbf{G}_{\mathrm{a}}^N$, and prove that any action on U by a k-split torus T respects this linear structure. Also prove that every unipotent k-group is *connected* and k-split.

(iii) Prove that any commutative extension of $\mathbf{G}_{\mathrm{a}}$ by $\mathbf{G}_{\mathrm{m}}$ is uniquely split over k. (Hint: first make a scheme splitting using that $\mathrm{Pic}(\mathbf{G}_{\mathrm{a}}) = 1$.)

U.2. Let k be an imperfect field of characteristic $p > 0$. Let k''/k be a purely inseparable finite extension such that $k''^{p^2} \subset k$ and $k' := k'' \cap k^{1/p} \neq k''$. Let $U = \mathrm{R}_{k''/k}(\mathbf{G}_{\mathrm{m}})/\mathbf{G}_{\mathrm{m}}$.

(i) For any smooth connected affine k'-group G', prove that the natural map $\mathrm{R}_{k'/k}(G')_{k'} \to G'$ defined functorially on k'-algebras by $G'(k' \otimes_k A') \to G'(A')$ is a smooth surjection with k'-split unipotent smooth connected kernel. Describe $(U_{k'})_{\mathrm{split}}$ and $U_{k'}/(U_{k'})_{\mathrm{split}}$, and show each is p-torsion and nontrivial. Deduce that $U_{k'} \to U_{k'}/(U_{k'})_{\mathrm{split}}$ has no k'-homomorphic section, so $U_{k'}$ is *not* a direct product of split and wound k'-groups.

(ii) Show that $(U_{k'})_{\mathrm{split}}$ is the cc$k'p$-kernel of $U_{k'}$ whereas $\mathrm{R}_{k'/k}(\mathbf{G}_{\mathrm{m}})/\mathbf{G}_{\mathrm{m}}$ is the cckp-kernel of U. (Hint: compute on k'_s-points and k_s-points respectively.) Why does this illustrate the failure of the formation of the cckp-kernel to commute with non-separable extension on k, and why is the non-smoothness of the p-torsion a necessary condition for any such example?

(iii) Does there exist a unipotent smooth connected k-group that is not an extension of a k-split group by a k-wound group, perhaps even a commutative example?

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