Math 249B. Standard parabolic subgroups: theory and examples

1. Introduction

Let G be a connected reductive k-group, P_0 a minimal parabolic k-subgroup, and ${}_k\Delta$ the basis of ${}_k\Phi := \Phi(G,S)$ corresponding to the positive system of roots ${}_k\Phi(P_0,S)$ for a maximal split k-torus $S \subset P_0$. A parabolic k-subgroup of G is called *standard* if it contains P_0. We have seen that each $G(k)$-conjugacy class of parabolic k-subgroups has a unique standard member, and that there is an inclusion-preserving bijection $I \mapsto {}_kP_I$ from the set of subsets $I \subset {}_k\Delta$ onto the set of standard parabolic k-subgroups (so ${}_kP_\emptyset = P_0$ and ${}_kP_{{}_k\Delta} = G$). In particular, the number of $G(k)$-conjugacy classes of parabolic k-subgroups is the number of such I, which is $2^{\#_k\Delta} = 2^{r_k(\mathscr{D}(G))}$ (where $r_k(\mathscr{D}(G)) = \dim S'$ is the k-rank of $\mathscr{D}(G)$, with $S' = (S \cap \mathscr{D}(G))^0_{\mathrm{red}}$ a maximal split k-torus of $\mathscr{D}(G)$).

Remark 1.1. The relative root system for the quotient of ${}_kP_I$ modulo its unipotent radical (equivalently, for a Levi factor) has basis given by I and the same associated coroots as for G (thus the same pairings of roots and coroots arising from I). In particular, if ${}_k\Delta$ is reduced then the Dynkin diagram for a Levi factor of ${}_kP_I$ is the same as that formed by I inside the diagram for ${}_k\Delta$ with the same edges and edge-multiplicities that join vertices from I inside the Dynkin diagram for ${}_k\Delta$. As an extreme case, for empty I this corresponds to the fact that the Levi factor of a minimal parabolic k-subgroup has k-anisotropic derived group, hence an empty relative root system.

We do not need this fact, so we refer the reader to the *proof* of Theorem 2.9 in the forthcoming handout on Tits systems and root groups for this result (where the compatibility with reflections proved there establishes the compatibility of coroots as well).

Example 1.2. Assume G is split (so we write Δ and Φ rather than ${}_k\Delta$ and ${}_k\Phi$, the root groups are commutative, and Φ is reduced). The Levi factors of the parabolic k-subgroup ${}_kP_I$ are therefore split, and their Dynkin diagram is given by the subdiagram of that for Δ having vertices in I (and the same edge multiplicities for edges of the ambient diagram joining vertices in I). Indeed, using central isogenies and derived groups allows us to reduce to the case when G is semisimple and *simply connected*. These Levi factors are special cases of torus centralizers, and rather generally for simply connected G the proof of Corollary 9.5.11 in class described a root basis inside Δ for the derived group of any torus centralizer in G; this also gave that such derived groups are also simply connected, so in such cases the isomorphism type of the derived group of the Levi factor of a parabolic k-subgroup can be read off from the Dynkin diagram (e.g., this recovers the familiar fact that parabolic subgroups for SL_n have Levi factors that are direct products of GL_{d_i}'s with product of determinants equal to 1). For maximal proper parabolic k-subgroups (usually referred to as a "maximal parabolic subgroup", in the spirit of "maximal ideal"), the derived groups of their Levi factors have diagram given by removing a single vertex (and any related edges) from the original diagram.

For which maximal (proper) parabolic k-subgroups P is $\mathscr{R}_{u,k}(P)$ abelian? If ${}_kP_{{}_k\Delta-\{a\}}$ is in the $G(k)$-conjugacy class of P then the abelian property is exactly the condition that $(U_b, U_{b'}) = 1$ for all positive non-divisible b, b' whose Δ-expansions contain a in their support. It is necessary and sufficient that $b+b' \notin \Phi$ for all such b and b' (for necessity this rests on the precise form of the Chevalley commutation relations with "universal" structure constants in

$\mathbf{Z}$ up to signs, especially when $p = \mathrm{char}(k) \in \{2, 3\}$ and Φ has an edge of multiplicity p in its Dynkin diagram).

In this handout we will determine the parabolic k-subgroups of symplectic groups, as well as of special orthogonal groups, in the latter case allowing arbitrary (finite-dimensional) non-degenerate quadratic spaces. For a specific minimal P_0, we will also describe the resulting standard parabolic k-subgroups. The case of $\mathrm{SO}(q)$'s with general (V, q) will provide further classes of examples of determining relative root systems.

The general theme for all of these calculations will be "stabilizer of an isotropic flag". As a warm-up, we now address the case of SL_n in terms of flags (with no isotropicity condition):

Proposition 1.3. *Let V be a vector space of finite dimension $n \geq 2$ over a field k, and let $G = \mathrm{SL}(V)$. For any strictly increasing flag*

$$F = \{F^1 \subsetneq \cdots \subsetneq F^r\}$$

of nonzero proper subspaces of V, the scheme-theoretic stabilizer $P_F := \mathrm{Stab}_G(F)$ is a parabolic k-subgroup and every parabolic k-subgroup of G arises in this way for a unique F.

The conjugacy class of P_F is uniquely determined by the "numerical invariants" of F: the sequence $\{\dim(F^j)\}$ of dimensions of the successive terms.

Proof. It is clear that any two flags F and F' with the same numerical invariants are $G(k)$-conjugate (use scaling along a line inside F^1 to enforce triviality of the determinant), and that the numerical invariants are preserved under $G(k)$-conjugacy In particular, if we fix an ordered basis $\{e_1, \ldots, e_n\}$ of V then P_F is $G(k)$-conjugate to $P_{F'}$ for F' a subflag of the *standard full flag* whose jth term is the span of $e_1, \ldots, e_j$ $(1 \leq j \leq n-1)$.

There are 2^{n-1} subflags of the standard full flag (among the $n-1$ nonzero proper subspaces of V occurring in the standard full flag, choose which ones to remove), and they are pairwise non-conjugate since their numerical data are pairwise distinct. Moreover, F is recovered from P_F as the unique flag with stabilizer P_F. (Explicitly, $P_F = G$ only for empty F, and otherwise F^1 is the unique irreducible subrepresentation of the natural representation of P_F on V, with $\{F^j/F^1\}_{j>1}$ a flag in V/F^1 whose stabilizer is the image of P_F in $\mathrm{SL}(V/F^1)$.)

We know that there are 2^{n-1} distinct conjugacy classes of parabolic k-subgroups since G is split semisimple with rank $n-1$, so it suffices to check that the visibly smooth k-subgroups P_F are parabolic. But each P_F is the G-stabilizer of a point on a suitable Grassmannian on which the G-action is geometrically transitive, so G/P_F is a Grassmannian. Hence, P_F is parabolic. ■

Example 1.4. Consider the standard full flag $F = \{F^j\}$ in k^n, and its stabilizer B that is the upper triangular Borel subgroup of SL_n. For each subset $I \subset \Delta := \{1, \ldots, n-1\}$, let P_I be the G-stabilizer of the flag obtained by removing from F the terms F^j for $j \in I$ (e.g., $P_\emptyset$ is the stabilizer B of the standard full flag F and $P_\Delta = G$). The terms of F_I consist of the subspaces $V_j = \mathrm{span}(e_1, \ldots, e_j)$ for $j \notin I$ $(1 \leq j \leq n-1)$, so P_I is the corresponding "staircase" subgroup of G containing B with jumps corresponding to the constituents of F_I.

Example 1.5. A maximal (proper) parabolic subgroup of SL_n is exactly the stabilizer of a single nonzero proper subspace F^1 of V (a minimal non-empty flag); this is a "staircase" with 1 step. The conjugacy class corresponds to the dimension of F^1 (between 1 and $n-1$).

By inspection of the parabolic subgroups containing the standard Borel, we see that these all have abelian unipotent radical: $\mathscr{R}_u(P_{\{F^1\}})$ is the vector group associated to $\mathrm{Hom}(V/F^1, F^1)$.

2. Symplectic groups

Let $G = \mathrm{Sp}(\psi)$ for a symplectic space (V, ψ) of (necessarily even) dimension $2n > 0$ over a field k. An increasing flag $F = \{F^j\}$ of nonzero proper subspaces of V is called *isotropic* if each F^j is isotropic in the sense that $\psi|_{F^j \times F^j} = 0$. The action of G on V carries an isotropic flag to an isotropic flag preserving its numerical invariants (i.e., the sequence of dimensions of its successive terms).

By non-degeneracy of ψ, if $W \subset V$ is a subspace then $(V/W)^* \simeq W^\perp$, so

$$\dim W + \dim W^\perp = \dim V = 2n.$$

In particular, if W is isotropic, so $W \subset W^\perp$, then $\dim W \leq n$ and ψ induces a symplectic form on $W^\perp/W$. It then follows from the structure of symplectic spaces (applied to $W^\perp/W$) that any isotropic subspace W is contained in an n-dimensional isotropic subspace, so the latter are called *maximal isotropic*. By similar reasoning, inductive considerations show that any flag of isotropic subspaces may be extended to one in which the successive dimensions are $1, 2, \ldots, n$; the latter is therefore called a *maximal isotropic* flag. The maximal ones are also exactly the isotropic flags consisting of n terms F^j.

Given a maximal isotropic flag $F = \{F^1 \subsetneq \cdots \subsetneq F^n\}$ (so $\dim F^j = j$), by choosing compatible bases for the F^j's and compatible lifts to $(F^j)^\perp$ of the basis of each $(F^j)^\perp/F^n = (F^j)^\perp/(F^n)^\perp$ dual to the chosen basis of F^n/F^j, we can identify (V, ψ, F) with the standard example in which $V = k^{2n}$, ψ corresponds to the block matrix

$$\psi_{\mathrm{std}} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

and F^j is the span of $e_1, \ldots, e_j$. Consequently, for any two isotropic flags F and F' with the same numerical invariants, if we extend each to a maximal isotropic flag we see that there is an automorphism of (V, ψ) carrying F to F'. Hence, the corresponding G-stabilizers P_F and $P_{F'}$ are $G(k)$-conjugate and moreover G acts geometrically transitively on each Grassmannian of isotropic flags with specified numerical invariants. It follows that G/P_F is proper for each F, so P_F is parabolic *provided* that it is smooth.

There are 2^n distinct subflags F of a maximal isotropic flag $F_{\max}$ (and the stabilizers of all such subflags clearly contain $P_{F_{\max}}$): the count is based on keeping track of which among the n terms are dropped from $F_{\max}$ to obtain F. But there are also 2^n distinct $G(k)$-conjugacy classes of parabolic k-subgroups of G since G is split semisimple with maximal tori of dimension n (the root system is C_n). Thus, if the stabilizers of the subflags of one maximal isotropic flag are smooth and pairwise distinct then we will have found all parabolic k-subgroups and moreover the numerical invariants of a flag will be *determined* by the associated stabilizer. This motivates:

Proposition 2.1. *For each isotropic flag $F \subset V$, the stabilizer scheme P_F is smooth* (*so it is a parabolic k-subgroup*) *and the only isotropic flag with stabilizer P_F is F. Every parabolic k-subgroup of G arises in this manner.*

In particular, the $G(k)$-conjugacy class of P_F determines and is determined by the numerical invariants of F.

Proof. We may assume $V = k^{2n}$ equipped with the standard symplectic form $\psi_{\rm std}$, and we consider the maximal isotropic flag $F_{\max}$ whose terms are the spans

$$V_j = \mathrm{span}(e_n, \dots, e_j)$$

for $1 \le j \le n$ (so $V_{n-j+1} = \mathrm{span}(e_n, \dots, e_j)$ and $V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n$).

Pick $I \subset \{1, \dots, n\}$ and let $F = F_I = \{F^1 \subsetneq F^2 \subsetneq \dots\}$ be the subflag of $F_{\max}$ obtained by omitting V_j's for precisely $j \in I$ (e.g., $F_\emptyset = F_{\max}$). Define the maximal torus

$$\mathrm{GL}_1^n = T = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \subset \mathrm{Sp}_{2n} = G$$

(with diagonal $t \in \mathrm{GL}_n$). Let

$$\mu_I : \mathrm{GL}_1 \to \mathrm{GL}_1^n$$

be a cocharacter given by $z \mapsto (z^{m_1}, \dots, z^{m_n})$ with $m_n \le \dots \le m_1 \le m_0 := 0$ and $m_{n-j+1} = m_{n-j}$ $(1 \le j \le n)$ precisely when $j \in I$ (i.e., when V_j does not occur in F_I).

Using the identification of T with GL_1^n as defined above, consider $\lambda_I : \mathrm{GL}_1 \to T$ defined via μ_I; i.e.,

$$\lambda_I(z) = \mathrm{diag}(z^{-m_1}, \dots, z^{-m_n}, z^{m_1}, \dots, z^{m_n})$$

with

$$m_n \le \dots \le m_1 \le 0 \le -m_1 \le \dots \le -m_n$$

(so $m_1 < -m_1$ if and only if $m_1 < 0 =: m_0$). Then $P_{\mathrm{GL}_{2n}}(\lambda_I)$ is the subgroup scheme of GL_{2n} that preserves F_I and the $\psi_{\rm std}$-orthogonal complements $(F^j)^\perp$ for the terms F^j occurring in F_I. But preservation of each $(F^j)^\perp$ is *automatic* for points of $G = \mathrm{Sp}_{2n}$ that preserve F_I, so $P_{F_I} = P_G(\lambda_I)$ as k-group schemes. Thus, P_{F_I} is parabolic (in particular, smooth).

Let $\Delta = \{a_1, \dots, a_n\}$ be the basis of $\Phi(G,T)$ as in the type-C case of the handout on classical groups, with the indexing by $\{1, \dots, n\}$ as defined there. We claim that P_{F_I} is the parabolic corresponding to the subset $\{a_i\}_{i \in I} \subset \Delta$ under the general dictionary for describing standard parabolic k-subgroups via subsets of a root basis. First note that for each I the pairing $\langle a_j, \lambda_I \rangle$ is non-negative for all $1 \le j \le n$ (equal to $m_{n-j} - m_{n-j+1} \ge 0$ if $1 \le j < n$ and equal to $-2m_1 \ge 0$ if $j = n$) and it is positive for all j when $I = \emptyset$. Hence, each P_{F_I} contains U_a for all roots $a \in \mathbf{Z}_{\ge 0} \cdot \Delta$, and $U_G(\lambda_\emptyset)$ contains U_a for all such a. In particular, $\lambda_\emptyset$ is *regular* and positive with respect to Δ. Thus, $P_G(\lambda_\emptyset)$ is the Borel subgroup containing T that corresponds to the basis Δ (so $U_G(\lambda_\emptyset)$ is directly spanned by the U_a's for Δ-positive a) and every P_{F_I} is then "standard" (as $P_G(\lambda_\emptyset) = T \ltimes U_G(\lambda_\emptyset) \subset P_G(\lambda_I) = P_{F_I}$ for all I). Inspection of the unipotent radicals of standard parabolic subgroups in general shows that the subset of Δ corresponding to P_{F_I} is the *complement* of the set of $a \in \Delta$ for which $U_a \subset \mathscr{R}_{u,k}(P_{F_I}) = U_G(\lambda_I)$; i.e., this is the set of $a \in \Delta$ such that $\langle a, \lambda_I \rangle = 0$. Thus, by definition of F_I, we need to show that $\langle a_j, \lambda_I \rangle = 0$ (i.e., $m_{n-j} = m_{n-j+1}$, where $m_0 := 0$) precisely when $j \in I$. This in turn is immediate from the definition of λ_I.

The preceding argument shows that the parabolics P_{F_I} exhaust without repetition the standard parabolic subgroups, so for any isotropic flag F the $G(k)$-conjugacy class of k-subgroup $P_F \subset G$ determines the numerical invariants of F (as the F_I's for varying I exhaust without repetition all possible numerical invariants).

It remains to show that an isotropic flag F is uniquely determined by its G-stabilizer P_F. Suppose F' is an isotropic flag such that $P_{F'} = P_F$. In particular, F and F' have the *same* numerical invariants, so there exists $g \in G(k)$ satisfying $g(F) = F'$. Hence, $gP_Fg^{-1} = P_{F'}$. But $P_{F'} = P_F$, so $g \in N_{G(k)}(P_F) = P_F(k)$. This says that $g(F) = F$, so $F' = g(F) = F$. ■

Example 2.2. It follows from Proposition 2.1 (exercise!) that a maximal (proper) parabolic subgroup of $G = \mathrm{Sp}(\psi) = \mathrm{Sp}_{2n}$ is exactly the stabilizer of a *minimal* non-empty isotropic flag; i.e., the G-stabilizer of a nonzero isotropic subspace $F^1 \subset V$. There are n such conjugacy classes, corresponding to a choice of vertex to remove from the diagram, and to the dimension of F^1 (between 1 and n). Which F^1, if any, correspond to parabolics with abelian unipotent radical? For the case $n = 1$ (i.e., SL_2) the abelian property always holds (unipotent radical of a Borel subgroup of SL_2), so let's consider $n \geq 2$.

Inspection of the table for type C_n in Bourbaki for $n \geq 2$ shows that for the unique long root $a_n \in \Delta$, the a_n-coefficient in every positive root b is either 0 or 1. Thus, if b and b' are positive roots whose Δ-expansions contain a_n in their supports then $b + b'$ is not a root. Hence, by Example 1.2, the maximal (proper) parabolic subgroup corresponding to $\Delta - \{a_n\}$ has abelian unipotent radical. In contrast, inspection of the Bourbaki table for C_n with $n \geq 2$ shows that for each short root $a_i \in \Delta$ there exist positive roots b and b' whose Δ-expansions contain a_i in their support and for which $b + b'$ is a root.

Thus, there is exactly one conjugacy class of parabolic subgroups with abelian unipotent radical, corresponding to the subset $\Delta - \{a_n\} \subset \Delta$ (with a_n long); this also works for $n = 1$ with the unique root in Δ understood to be long (reasonable, as it is divisible by 2 in the weight lattice). These are called the *Siegel parabolics*. In terms of the notation in the proof of Proposition 2.1, these correspond to $m_n = m_{n-1} = \cdots = m_1 < 0$, which is to say isotropic subspaces $F^1 \subset V$ with dimension n (as also works for $n = 1$). [By the same method, if $F^1 \subset V$ is an isotropic subspace of dimension $1 \leq i < n$ then the conjugacy class of $P_{\{F^1\}}$ corresponds to the condition $m_n = \cdots = m_{n-i+1} < m_{n-i} = \cdots = m_1 = 0$.] We conclude that for any $n \geq 1$, the Siegel parabolic subgroups of Sp_{2n} are the stabilizers of the maximal isotropic subspaces. For the standard maximal isotropic subspace $F^1 = V_1 = \mathrm{span}(e_1, \ldots, e_n)$ of $(k^{2\mathrm{n}}, \psi_{\mathrm{std}})$, by inspection $\mathscr{R}_u(P_{\{F^1\}})$ consists of the block matrices $\left(\begin{smallmatrix} 1_n & M \\ 0 & 1_n \end{smallmatrix}\right)$ with M a symmetric $n \times n$ matrix.

In the special case $n = 2$ (i.e., Sp_4) there is one other conjugacy class of maximal (proper) parabolic subgroups, corresponding to $\Delta - \{a_1\}$ for the unique short node in the Dynkin diagram. These are called the *Klingen* (or *Jacobi*) parabolic subgroups of Sp_4, and they are the stabilizers of lines (all of which are isotropic in a symplectic space). Using the standard choice in the proof of Proposition 2.1, the standard Klingen parabolic is the stabilizer of $V_2 = ke_2$, and its unipotent radical is

$$\begin{pmatrix} (u^{-1})^{\mathrm{t}} & mu \\ 0 & u \end{pmatrix}$$

for unipotent $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2$ and symmetric $m = \left(\begin{smallmatrix} 0 & y \\ y & z \end{smallmatrix}\right) \in \mathrm{Mat}_2$. (This description encodes parameterizations of the 3 root groups directly spanning that unipotent radical, corresponding to x, y, z.) The same terminology is used for GSp_4 (whose parabolic subgroups are in natural bijective correspondence with those of the derived group Sp_4).

3. Special orthogonal groups

We next turn to the study of $G = \mathrm{SO}(q)$ for a non-degenerate quadratic space (V, q) of dimension $n \geq 3$; we assume $n \geq 3$ so that G is connected semisimple (it is a torus when $n = 2$). Note that typically G is not k-split; this will introduce some new issues not encountered in the symplectic case. Let $B_q : V \times V \to k$ be the associated symmetric bilinear form $(v, v') \mapsto q(v + v') - q(v) - q(v')$, so B_q is non-degenerate except when n is odd and $\mathrm{char}(k) = 2$ (in which case B_q has a 1-dimensional defect space $V^\perp$, with B_q inducing a symplectic form on $V/V^\perp$). When we speak of *orthogonality* of subspaces of V, we mean with respect to B_q.

By Exercise 5 in HW6 of the previous course, (V, q) is isotropic (i.e., $q(v) = 0$ for some nonzero $v \in V$) if and only if G is k-isotropic, in which case a nonzero solution to $q = 0$ lies in a hyperbolic plane $H \subset V$, with $V = H \perp H^\perp$ as quadratic spaces. (This much only requires $n \geq 2$, rather than $n \geq 3$.) For an analysis of proper parabolic k-subgroups of the connected semisimple group G, we therefore may and do now assume that (V, q) contains a hyperbolic plane, so G has positive k-rank. Note that in such a hyperbolic plane there are exactly two lines on which the restriction of q vanishes.

Definition 3.1. A subspace $W \subset V$ is *isotropic* if $q|_W = 0$. (This is equivalent to $B_q|_{W \times W} = 0$ except when n is odd and $\mathrm{char}(k) = 2$.)

Eventually we will prove that parabolic k-subgroups of G arise from isotropic flags in V, as in the symplectic case. However, the bijective correspondence as in the symplectic case will break down when $n \geq 4$ is even and G is k-split.

Remark 3.2. Since $\mathrm{SO}(q) = \mathrm{SO}(cq)$ inside $\mathrm{GL}(V)$ for $c \in k^\times$, if $n = 2m + 1 \geq 3$ is odd then it can happen then $\mathrm{SO}(q)$ is split even though q is not split. For example, consider $q = ax_0^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m}$. Then discriminant considerations show that q is not split if a is not a square. However, $(1/a)q$ is split and has the same special orthogonal group.

In general, since $n \geq 3$, the k-group $\mathrm{SO}(q)$ determines (V, q) up to conformal isometry. To prove this fact (which we do not need), note that the conformal automorphism group of (V, q) is $\mathrm{GO}(q)$, and that there is a natural action of $\mathrm{PGO}(q) = \mathrm{GO}(q)/\mathrm{GL}_1$ on $\mathrm{SO}(q)$ via conjugation. The remarkable fact is that this action identifies $\mathrm{PGO}(q)$ with the automorphism scheme of $\mathrm{SO}(q)$ (we only need this on k_s-valued points) for $n \geq 3$, first proved by Dieudonné away from characteristic 2 (for a modern proof, see Lemma C.3.13 in the article *Reductive group schemes* in the 2011 Luminy summer school Proceedings). Using Hilbert 90 then gives the assertion.

Proposition 3.3. *Let $r > 0$ be the k-rank of G. Every maximal mutually orthogonal collection of hyperbolic planes in V has size r.*

Note that a split subspace of dimension $2r$ is an orthogonal sum of hyperbolic planes by another name.

Proof. An orthogonal sum of hyperbolic planes has vanishing defect space, so by Witt's Extension Theorem (see Theorem 3.1 in the handout on root systems for classical groups) the action of $\mathrm{O}(q)(k)$ is transitive every split non-degenerate subspace of a given even dimension $2m > 0$. Thus, we just have to build such a subspace of dimension $2r$ and then show that its orthogonal complement is anisotropic (and so contains no hyperbolic plane).

Let $S \subset G$ be a maximal split k-torus, and $T \subset G$ a maximal k-torus containing S. By inspection of a split quadratic space over $\overline{k}$ we see that there is a collection of nontrivial T_{k_s}-weights on V_{k_s} that is a basis for $\mathrm{X}(T_{k_s})_{\mathbf{Q}}$. Since the restriction map $\mathrm{X}(T_{k_s}) \to \mathrm{X}(S)$ is surjective and the restriction to S_{k_s} of each T_{k_s}-weight on V_{k_s} is an S-weight on V, there exists a subset $\{\chi_1, \dots, \chi_r\}$ of $\mathrm{X}(S)$ that is a basis of $\mathrm{X}(S)_{\mathbf{Q}}$.

As is well-known, the S-weight spaces in V for distinct S-weights are mutually linearly independent (since S is split). The S-invariance of q implies that for each $\chi \in \mathrm{X}(S)$ the quadratic form q vanishes on each $V(\chi)$ (since for $v \in V(\chi)$ and $s \in S$ we have $q(v) = q(s.v) = q(\chi(s)v) = \chi^2(s)q(v)$, forcing $q(v) = 0$). Note that if n is odd and $\mathrm{char}(k) = 2$, so there is a defect line $V^\perp$, the action of G on V preserves $V^\perp$ and hence is trivial on the line $V^\perp$ (as G has no nontrivial characters), so S acts trivially on $V^\perp$. Thus, B_q sets up a perfect pairing between each $V(\chi)$ and its orthogonal complement.

For any $\chi, \chi' \in \mathrm{X}(S)$ and $v \in V(\chi)$, $v' \in V(\chi')$, and $s \in S$, we have

$$B_q(v, v') = B_q(s.v, s.v') = B_q(\chi(s)v, \chi'(s)v') = \chi(s)\chi'(s)B_q(v, v').$$

Thus, if $\chi'\chi \neq 1$ then $B_q(v, v') = 0$. It follows that B_q defines a perfect duality between $V(\chi)$ and $V(\chi^{-1})$ for any $\chi \in \mathrm{X}(S)$. In particular, each $V(\chi_j^{-1})$ is nonzero and the weight spaces $V(\chi_j^{\pm 1})$ are collectively linearly independent.

Choose nonzero $v_j^{\pm} \in V(\chi_j^{\pm 1})$, so the span H_j of v_j^+ and v_j^- is a hyperbolic plane with $H_1, \dots, H_r$ mutually orthogonal. For $W = H_1 \perp \dots \perp H_r$ we have $V = W \perp W^\perp$, so G contains $\mathrm{SO}(W) \times \mathrm{SO}(W^\perp)$. But $\mathrm{SO}(W)$ obviously contains a split torus of dimension r, yet r is the k-rank of G, so $\mathrm{SO}(W^\perp)$ must be anisotropic. Hence, $W^\perp$ is anisotropic; i.e., q has no nontrivial zeros on $W^\perp$. In particular, there is no hyperbolic plane contained in $W^\perp$. It then follows from Witt's Theorem that W is maximal as an orthogonal sum of hyperbolic planes in V. ■

Corollary 3.4. *For $n \geq 3$, the maximal isotropic subspaces $W \subset V$ have dimension r, and $G(k)$ acts transitively on the set of these except when $n = 2r$, in which case there are exactly two $G(k)$-orbits of such subspaces.*

Every maximal split k-torus $S \subset G$ acts on V with nontrivial weights occurring in r pairs of opposite weights $\{\chi_j^{\pm 1}\}$, each with a 1-dimensional weight space and with $\{\chi_j\}$ a basis for $\mathrm{X}(S)$. Every maximal isotropic subspace of V arises from such an S as the span of S-weight spaces for one from each of the r pairs of opposite nontrivial S-weights on V.

In Example 3.5 we give a direct geometric characterization of the two orbits when $n = 2r$.

Proof. An isotropic subspace has vanishing intersection with the defect line $V^\perp$ when n is odd with $\mathrm{char}(k) = 2$, so by Witt's Theorem any two isotropic subspaces of the same dimension are related through the action of $\mathrm{O}(q)(k)$. Thus, if one r-dimensional isotropic subspace is maximal then all maximal isotropic subspaces have dimension r. We have found a decomposition

$$V = H_1 \perp \cdots \perp H_r \perp V'$$

where the H_i are hyperbolic planes and V' is (non-degenerate and) anisotropic. Each H_j contains exactly two isotropic lines $\{L_j, L'_j\}$. Let $W = L_1 \perp \cdots \perp L_r$, an isotropic subspace of dimension r, and let $W' = L'_1 \perp \cdots \perp L'_r$.

To prove that W is maximal isotropic, we have to show that any isotropic vector $v \in W^\perp - W$ must vanish. It is harmless to change v modulo W since W is isotropic, so we may assume $v = w' + v'$ for some $w' \in W'$ and $v' \in V'$. But W' is orthogonal to V', so

$$0 = q(v) = q(w') + B_q(w', v') + q(v') = q(w') + q(v').$$

Since W' is isotropic we have $q(w') = 0$, so v' is isotropic in V'. But V' is anisotropic, so $v = w' \in W'$.

But W' is in perfect duality with W under B_q via the construction using the isotropic lines in the hyperbolic planes $H_1, \dots, H_r$, so the condition $w' = v \in W^\perp$ with $w' \in W'$ forces $w' = 0$. This completes the proof that maximal isotropic subspaces have dimension r.

For W as just built, there is an evident split torus $S_0 = \mathrm{GL}_1^r \subset G$ for which the jth factor acts on the line L_j through usual scaling, on L'_j through scaling via inversion, and trivially on V'. Identify S_0 as a split maximal k-torus in the split subgroup $\mathrm{SO}(kv' \perp (\perp H_j)) = \mathrm{SO}_{2r+1} \subset G$ when n is odd or when n is even with $r < n/2$, and in the split group $\mathrm{SO}_n = G$ when n is even and $r = n/2$. Inspection of the Weyl group in the Bourbaki tables for types B_r and D_r (separate care for $\mathrm{B}_1 = \mathrm{A}_1$ and $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$) and the explicit description of split maximal tori and the roots for types B and D in the handout on root systems for classical groups (using type-D only when $n = 2r$) show that $N_G(S_0)(k)$ acts by whatever collection of inversions we wish along the r evident GL_1-factors *except* that when $n = 2r$ only an even number of such inversions can be realized.

We conclude from the transitivity of the $G(k)$-action on the set of split maximal k-tori that transitivity holds on the set of such tori equipped with a choice of one from each pair of opposite non-trivial weights on V *except* that for even $n \geq 4$ and $r = n/2$ there are at most two orbits. Such data yields a specific maximal isotropic subspace, and *all arise in this way* since $\mathrm{O}(q)(k)$ normalizes G and (by Witt's Extension Theorem) acts transitively on the set of such subspaces, so $G(k)$ acts transitively on the set of maximal isotropic subspaces except possibly when $n = 2r \geq 4$, in which case there are at most two $G(k)$-orbits.

Finally, we check for $n = 2r$ (so $G = \mathrm{SO}_{2r}$ is split and $\mathrm{O}(q) = \mathrm{O}_{2r}$ is an extension of $\mathbf{Z}/2\mathbf{Z}$ by G in all characteristics) that there are indeed two distinct $G(k)$-orbits of such subspaces. Since $\mathrm{O}_{2r}(k)$ acts transitively, it suffices to show that the O_{2r}-stabilizer of such a subspace is contained inside $\mathrm{SO}_{2r} = \mathrm{O}_{2r}^0$. Indeed, once this is proved it follows that the O_{2r}-homogenous space of maximal isotropic subspaces inside the Grassmannian of r-planes inside k^{2r} has two connected components, so each $\mathrm{SO}_{2r}(k)$-orbit is constrained to lie in one component and the effect of any $\rho \in \mathrm{O}_{2r}(k) - \mathrm{SO}_{2r}(k)$ (such as reflection in a non-isotropic vector) must move an $\mathrm{SO}_{2r}(k)$-orbit to another such orbit.

Letting $W \subset k^{2r}$ be a maximal isotropic subspace, our uniform description of these in terms of hyperbolic planes provides an isotropic complement W' that is in perfect duality with W via B_q. Consequently, if $g \in \mathrm{O}_{2r}(k)$ preserves W then its effect on $V/W = W^*$ must be inverse-dual to $g|_W$ via g-equivariance of B_q. Hence, $\det(g) = \det(g|W)\det((g|_W)^*)^{-1} = 1$, so if $\mathrm{char}(k) \neq 2$ then $g \in \mathrm{SO}_{2r}(k)$. To handle characteristic 2, we can argue in another way that is characteristic-free: for W of dimension r, we may identify our quadratic space with $W \oplus W^*$ on which $q(w, \ell)$ is equal to $\ell(w)$, and then the stabilizer of W inside $\mathrm{O}(q)$ is easily computed to consist of block matrices

$$\begin{pmatrix} M & MT \\ 0 & (M^*)^{-1} \end{pmatrix}$$

with $M \in \mathrm{GL}(W)$ and $T \in \mathrm{Hom}(W^*, W)$ satisfying $\ell(T(\ell))) = 0$ for all $\ell \in W^*$. This is clearly linear condition on T; explicitly, relative to a basis of W and its dual basis for W^*, it says that the matrix for T has vanishing diagonal and vanishing sum for ij and ji entries for all $i < j$. In this way we see that the W-stabilizer in $\mathrm{O}(q)$ is connected (a semi-direct product of $\mathrm{GL}(W)$ against a linear subrepresentation of $\mathrm{Hom}(W^*, W)$), so it lies inside $\mathrm{O}(q)^0 = \mathrm{SO}(q)$. ■

Example 3.5. When $n = 2r \geq 4$ (so $G = \mathrm{SO}_{2r}$ is k-split of type D_r), among the maximal isotropic subspaces $W \subset V$ we have seen that there are two $G(k)$-orbits. How do we detect when two such subspaces $W, W' \subset V$ lie in the same $G(k)$-orbit? We claim W and W' are in the same orbit if and only if the common codimension of $W \cap W'$ inside W and W' is even.

Let c be the common codimension of $L := W \cap W'$ in W and W'. Then $W, W' \subset L^\perp$, and W/L and W'/L are c-dimensional maximal isotropic subspaces in the non-degenerate quadratic space $L^\perp/L$ with dimension $\dim(V) - 2\dim(L)$ that is even. (There is a natural quadratic form on $L^\perp/L$ since L is isotropic, and it is non-degenerate because B_q induces a perfect pairing on this space.) If $c \geq 2$, then we claim that by applying to W' a composition of an even number of reflections in non-isotropic vectors of $L^\perp$ (giving an element of $G(k)$) brings us to the case in which the codimension becomes 0 for even c and become 1 for odd c. Note that such reflections have trivial effect on L, so for this purpose we may work instead with $L^\perp/L$ in place of V, so $W \cap W' = 0$ and $\dim W = \dim W' = c \geq 2$.

Now B_q sets up a perfect duality between W and W'. By choosing a basis of W and equipping W' with the B_q-dual basis, we may express V as an orthogonal sum of c hyperbolic planes $H_1, \dots, H_c$ such that for the two isotropic lines $\{L_j, L'_j\}$ of each H_j we may label them to ensure that $W = \bigoplus L_j$ and $W' = \bigoplus L'_j$. A reflection in a non-isotropic vector of H_j swaps L_j and L'_j, and the composition of such reflections for an even number of j's lies in $G(k)$. We can take this even number to be c when c is even, and to be $c - 1$ when c is odd. This provides $g \in G(k)$ such that $g(W) = W'$ when c is even and $g(W) \cap W'$ has codimension 1 in each of $g(W)$ and W' when c is odd.

It remains to show that if $W \cap W'$ has codimension 1 in W and W' then W and W' are not in the same $G(k)$-orbit. The $\mathrm{O}(q)(k)$-stabilizer of W is contained in G (see the proof of Corollary 3.4), all elements of $\mathrm{O}(q)(k)$ carrying W to W' lie in the same $G(k)$-coset. In particular, to prove that W and W' are not in the same $G(k)$-orbit it suffices to find $\gamma \in \mathrm{O}(q)(k) - G(k)$ such that $\gamma(W) = W'$. We will find a reflection in a non-isotropic vector that carries W to W', thereby completing the proof.

Since $L := W \cap W'$ has dimension $r - 1$, the quotients W/L and W'/L are distinct isotropic lines in the non-degenerate quadratic space $L^\perp/L$ of dimension 2, so $L^\perp/L$ is a hyperbolic plane. A reflection in a suitable non-isotropic vector in a hyperbolic plane swaps the two isotropic lines, so we get the desired γ as a reflection in a non-isotropic vector.

We define a flag $F = (0 \subsetneq F^1 \subsetneq \cdots \subsetneq F^m \subsetneq V)$ to be *isotropic* if each F^j is isotropic (so $\dim F^m \leq r \leq n/2$). Witt's Extension Theorem ensures that the maximal isotropic flags are precisely the full flags in maximal isotropic subspaces (which we know have dimension exactly $r > 0$). But we have proved that $G(k)$ acts transitively on the set of maximal isotropic subspaces except when $n = 2r \geq 4$, in which case there are two $G(k)$-orbits.

Lemma 3.6. *The group $G(k)$ acts transitively on the set of maximal isotropic flags except that if $n = 2r \geq 4$ then there are two orbits (determined by the orbit of the maximal member of the flag).*

Proof. Let W be the maximal isotropic subspace occurring in a maximal isotropic flag. We can choose another maximal isotropic subspace W' in perfect duality with W under B_q (by considerations with Witt's Extension Theorem and mutually orthogonal hyperbolic planes). Then $\mathrm{GL}(W)$ naturally embeds into $\mathrm{SO}(q)$ by making $g \in \mathrm{GL}(W)$ act as follows: the usual action on W, the inverse-dual action on $W' \simeq W^*$, and trivially on the anisotropic $(W \oplus W')^{\perp}$. Since $\mathrm{GL}(W)$ acts transitively on the set of flags of nonzero proper subspaces of W with a given set of numerical invariants, it follows that $G(k)$ acts transitively on the set of maximal isotropic flags except for the obstruction at the top layer: if $n = 2r \geq 4$ then there are two orbits (determined by the orbit of the maximal member of the flag). ■

Any isotropic subspace of dimension *less than* r can be embedded into one of dimension r, and this leads to a refinement of the preceding lemma.

Proposition 3.7. *The action on $G(k)$ on the set of isotropic flags in V with a given set of numerical invariants is transitive except when $n = 2r \geq 4$ and the maximal subspace in the flag has dimension r. In the latter case there are exactly two $G(k)$-orbits with the given numerical invariants.*

Proof. Assume that we are not in the case $n = 2r \geq 4$. Since any isotropic flag F can be put into a maximal one, and F is recovered as a subflag using the given numerical invariants, transitivity in the maximal case in Lemma 3.6 does the job.

Now suppose $n = 2r \geq 4$ (so $r \geq 2$). By the same reasoning, there are at most two $G(k)$-orbits of isotropic flags with a given set of numerical invariants, and certainly when the flags under consideration contain an r-dimensional member there are exactly 2 such orbits.

It remains to show that when the members of the flags all have dimension $< r$ then there is only one $G(k)$-orbit. It is harmless to restrict attention to flags

$$F = (F^1 \subsetneq \cdots \subsetneq F^{r-1})$$

with $\dim F^j = j$ for all $1 \leq j \leq r-1$. For such an F, let W be a maximal isotropic subspace containing F^{r-1} as a hyperplane. We can choose another r-dimensional isotropic subspace W' complementary to W and in perfect duality with W via B_q.

Choose a basis $\{e_1, \ldots, e_r\}$ of W with $e_j \in F^j$, and let $\{e'_1, \ldots, e'_r\}$ be the dual basis of W', so $H_j = ke_j + ke'_j$ is a hyperbolic plane. Reflection in a non-isotropic vector of H_r has the effect of swapping the two isotropic lines ke_r and ke'_r and has no effect on $H_1, \ldots, H_{r-1}$. Hence, the original flag F is unaffected by that reflection, but the enlargements F' and F'' of F by appending $W = F^{r-1} + ke_r$ or $F^{r-1} + ke'_r$ are swapped.

In this way we have made representatives of the two $G(k)$-orbits of maximal isotropic flags that each contain F as their subflag in dimensions $< r$. But *every* flag with the same numerical invariants as F can be enlarged to a maximal isotropic flag and hence is in the same $G(k)$-orbit as one of F' or F''. Yet the subflags of F' and F'' in dimension $< r$ coincide with F! This establishes the desired transtivity. ■

We are nearly ready to prove the orthogonal analogue of Proposition 2.1, but first we shall address $n = 2r = 4$ because this illustrates very directly a special feature of the general case with $n = 2r \geq 4$ (so $r \geq 2$) that we shall see later: isotropic flags F containing members with dimensions $r-1$ and r have the *same* stabilizer as the subflag in dimensions $< r$. The reason that $n = 2r = 4$ is very accessible is that $\mathrm{SO}_4 = \mathrm{SL}_2 \times^{\mu_2} \mathrm{SL}_2$ via the action of $\mathrm{SL}_2 \times^{\mu_2} \mathrm{SL}_2$ on $(\mathfrak{gl}_2, \det)$ by left and right multiplication of SL_2 on $\mathfrak{gl}_2$, using $(g,h).M = gMh^{-1}$ (put another way, the diagram D_2 consists of two isolated points).

Example 3.8. Assume $n = 2r = 4$. For the isotropic line $L = \{\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)\}$ and isotropic plane $W = \{\left(\begin{smallmatrix} 0 & * \\ 0 & * \end{smallmatrix}\right)\}$, direct calculation shows that their respective scheme-theoretic stabilizers are the parabolic subgroups $B' := B \times^{\mu_2} B$ and $P := \mathrm{SL}_2 \times^{\mu_2} B$ for the upper triangular Borel subgroup $B \subset \mathrm{SL}_2$. Since $B' \subset P$, the stabilizer of the maximal isotropic flag $F_0 = (L \subset W)$ is also equal to B'. Similarly, the isotropic plane $W' = \{\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)\}$ has stabilizer $B \times^{\mu_2} \mathrm{SL}_2$.

We have obtained all 3 proper parabolic k-subgroups containing B', so every parabolic k-subgroup of SO_4 arises as the stabilizer of an isotropic flag. Since the $G(k)$-conjugacy classes of $B \times^{\mu_2} \mathrm{SL}_2$ and $\mathrm{SL}_2 \times^{\mu_2} B$ are distinct, the isotropic planes W and W' are *not* in the same $G(k)$-orbit, so these represent the two distinct $G(k)$-orbits of maximal isotropic subspaces. (Alternatively, $W \cap W'$ has odd codimension in each of W and W', so by Example 3.5 the $G(k)$-orbits of W and W' are distinct.)

Likewise, $F_0 := (L \subset W)$ and $F_0' := (L \subset W')$ must represent the two $G(k)$-orbits of maximal isotropic flags, so we conclude that the scheme-theoretic G-stabilizer of *every* isotropic flag is a parabolic k-subgroup. Note in particular that the two $G(k)$-conjugacy classes of maximal parabolic k-subgroups correspond to the two $G(k)$-orbits of maximal isotropic subspaces.

Observe that the flag F_0 and the flag consisting just of the isotropic subspace L have different numerical invariants but the *same* stabilizer, and likewise F_0' also has this stabilizer (by inspection or because $\mathrm{O}(q)(k)$ acts transitively on the set of maximal isotropic subspaces). More generally, an isotropic line and both flags extending it with an isotropic plane are the *only cases* of distinct isotropic flags in $(\mathfrak{gl}_2, \det)$ with the same stabilizer. Indeed, by our knowledge of the $G(k)$-orbits of isotropic flags and the $G(k)$-conjugacy classes of parabolic k-subgroups it suffices to check that (i) L is the only isotropic line with stabilizer B' and (ii) W is the only isotropic plane with stabilizer P. For the diagonal k-torus $T \subset \mathrm{SL}_2$ the split k-torus $S := T \times^{\mu_2} T$ is maximal in G and is contained in B and P. The weight spaces for the S-action on $\mathfrak{gl}_2$ are the 4 isotropic lines given by the standard matrix entries, so (i) and (ii) are easy case-checking with these lines.

The relationship between isotropic flags and parabolic k-subgroups in special orthogonal groups is somewhat lengthy to sort out when $n = 2r \geq 4$, due to the need to grapple with the absence of a bijection between isotropic flags and parabolic k-subgroups for such n and r. To be more precise, it is true for all $n \geq 3$ that the (scheme-theoretic) stabilizer in $\mathrm{SO}(q)$

of any isotropic flag is parabolic and that every parabolic k-subgroup arises in this way. However, when $n = 2r \geq 4$ some parabolics arise as the stabilizer of *three* distinct flags, and informally this is "cancelled" by the fact that in related cases the numerical invariants of a flag can fail to characterize its $\mathrm{SO}(q)(k)$-orbit (as opposed to its $\mathrm{O}(q)(k)$-orbit). Keeping track of these breakdowns is a source of case-checking headaches in the following proof.

Theorem 3.9. *Let $F \subset V$ be an isotropic flag, and $\nu(F) \subset \{1, \dots, r\}$ its set of numerical invariants.*

(i) *The G-stabilizer P_F is a parabolic k-subgroup of G, and every parabolic k-subgroup of G arises in this manner.*
(ii) *The stabilizer $P_F \subset G$ determines F except that if $n = 2r \geq 4$ and $r-1$ is the largest element of $\nu(F)$ then $P_F = P_{F'}$ for both isotropic flags F' obtained from F by inserting an r-dimensional isotropic subspace.*
(iii) *The $G(k)$-conjugacy class of P_F determines the $G(k)$-orbit of F except that if $n = 2r \geq 4$ and $\nu \subset \{1, \dots, r-1\}$ is a subset containing $r-1$ then the $G(k)$-orbit of F's with $\nu(F) = \nu$ and both $G(k)$-orbits of F's with $\nu(F) = \nu \cup \{r\}$ all have the same $G(k)$-orbit of stabilizers.*

In (ii) there are exactly two possibilities for F' because if W is the $(r-1)$-dimensional member of F then the r-dimensional member of F' corresponds to an isotropic line in the quadratic space $W^{\perp}/W$ that is a hyperbolic plane.

Proof. We have already shown that a $G(k)$-orbit of isotropic flags consists of exactly those flags with fixed numerical invariants with the exception that if $n = 2r \geq 4$ then there are two $G(k)$-orbits of isotropic flags containing an r-dimensional member and having specified numerical invariants. In particular, (iii) is a formal consequence of (ii), so we may and do now focus on (i) and (ii).

Let's show that to prove (i) and (ii) we may assume q is split. To prove the parabolicity of P_F in (i) it suffices to check after scalar extension to k_s (S_{k_s} is usually not maximal split but F_{k_s} remains an isotropic flag). Likewise, to prove (ii) it suffices to check over k_s because if there is an r-dimensional isotropic subspace of V then $n = 2r \geq 4$ and q is split (so every $(r-1)$-dimensional isotropic subspace lies in an r-dimensional one). Once (ii) is proved, the second assertion in (i) for a given parabolic k-subgroup P follows from the case over k_s because we can apply Galois descent to the *unique* isotropic flag in V_{k_s} giving rise to P_{k_s} which doesn't have both $r-1$ and r among its numerical invariants (uniqueness by (ii)). Over a separably closed field k we can replace q with a $k^{\times}$-multiple so that it is split. Hence, now we may and so assume q is split (and allow k to be arbitrary).

To be explicit, it is convenient to use the coordinatizations

$$q_{2m} = x_0 x_{2m-1} + \cdots + x_{m-1} x_m, \;\; q_{2m+1} = x_0 x_{2m} + \cdots + x_{m-1} x_{m+1} + x_m^2$$

so that an m-dimensional split maximal torus $T \subset G = \mathrm{SO}(q_n)$ is given by the diagonal

$$\mathrm{diag}(t_1, \dots, t_m, t_m^{-1}, \dots, t_1^{-1}), \;\; \mathrm{diag}(t_1, \dots, t_m, 1, t_m^{-1}, \dots, t_1^{-1})$$

for $n = 2m, 2m+1$ respectively. Note that $r = m$.

Due to the special behavior of B_1 (for $n = 3$) relative to higher-rank type-B and of D_2 (for $n = 4$) relative to higher-rank type-D, we first treat $n = 3, 4$ directly, using the identification

$\mathrm{PGL}_2 \simeq \mathrm{SO}_3$ via the action of PGL_2 on $(\mathfrak{sl}_2, \det)$ by conjugation and the identification $\mathrm{SL}_2 \times^{\mu_2} \mathrm{SL}_2 \simeq \mathrm{SO}_4$. In dimension 3 the only non-empty isotropic flags are isotropic lines, and in dimension 4 the non-empty isotropic flags are: isotropic lines, isotropic planes, and pairs $(L \subset W)$ consisting of an isotropic line in an isotropic plane.

We treated $n = 4$ in Example 3.8, so suppose $n = 3$. A direct calculation with the isotropic line $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ inside $\mathfrak{sl}_2$ shows that its scheme-theoretic stabilizer in PGL_2 is the upper triangular Borel subgroup B of PGL_2. To show that this isotropic line is the only one preserved by B, consideration of the action on $\mathfrak{sl}_2$ by the diagonal torus T of B shows that the only other possible lines are $\mathrm{Lie}(T)$ and the Lie algebra of the unipotent radical of the opposite Borel subgroup with respect to T, neither of which is normalized by B.

Now we may and do assume $n \geq 5$, so either $n = 2m+1$ with $m \geq 2$ or $n = 2m$ with $m \geq 3$. Let $W_j = \mathrm{span}(e_0, \ldots, e_{j-1})$ for $1 \leq j \leq m$, so $W := W_m$ is the standard maximal isotropic subspace of k^n and $F_{\mathrm{std}} := (W_j)_{1 \leq j \leq m}$ is a maximal isotropic flag. Note that $W^\perp = W$ for even n and $W^\perp = W \oplus ke_m$ for $n = 2m + 1$. Consider cocharacters $\lambda \in \mathrm{X}_*(T)$ defined by

$$\lambda : z \to \begin{cases} (z^{h_1}, \ldots, z^{h_m}, z^{-h_m}, \ldots, z^{-h_1}), & m \text{ even;} \\ \\ (z^{h_1}, \ldots, z^{h_m}, 1, z^{-h_m}, \ldots, z^{-h_1}), & m \text{ odd,} \end{cases}$$

where $h_1 \geq \cdots \geq h_m \geq 0 =: h_{m+1}$. Say that λ is *of type* I for a subset $I \subset \{1, \ldots, m\}$ if for $1 \leq j \leq m$ we have $h_j = h_{j+1}$ exactly for $j \in I$.

For each I, let F_I be the flag consisting of W_j's for $j \notin I$ and fix λ_I of type I. By a variant of the argument in the symplectic case, $P_{\mathrm{GL}_n}(\lambda_I)$ is the stabilizer in GL_n of the flag consisting of the members of F_I and their B_q-orthogonal complements. Hence, the parabolic k-subgroup $P_G(\lambda_I) = G \cap P_{\mathrm{GL}_n}(\lambda_I)$ coincides with the G-stabilizer scheme P_{F_I} of F_I. As we vary I, the F_I's realize all possible numerical invariants. Since the numerical invariants determine the $\mathrm{O}(q)(k)$-orbit of a flag (not always the $G(k)$-orbit!), and $\mathrm{O}(q)(k)$ normalizes G inside $\mathrm{O}(q)$, it follows that P_F is parabolic for all isotropic flags F.

Each of the 2^m parabolic subgroups P_{F_I} contains $P_{F_\emptyset}$, and $P_{F_\emptyset}$ is a Borel subgroup because $\lambda_\emptyset$ is regular (as we see by composing λ_I with the list of positive roots for types B and D in the handout on root systems for classical groups). Moreover, the T-root groups in $\mathscr{R}_u(P_{F_\emptyset}) = U_G(\lambda_\emptyset)$ include the ones deemed positive for types B and D in the handout on root systems for classical groups, so by comparing counts with the size of a positive system of roots these are exactly the root groups in that unipotent radical. That is, the Borel subgroup $P_{F_\emptyset} \supset T$ corresponds to the root basis Δ as defined for types B and D in that handout. Now we treat the remaining cases $n = 2m + 1 \geq 5$ and $n = 2m \geq 6$ separately.

Odd $\mathbf{n = 2m + 1 \geq 5}$. We claim that the P_{F_I}'s exhaust the 2^m standard parabolics relative to the Borel $P_{F_\emptyset}$ (so *every* parabolic k-subgroup arises as P_F for some isotropic flag F, with the $G(k)$-conjugacy class of P_F characterized by the numerical invariants of F). For this it suffices to show that $P_{F_I} \neq P_{F_{I'}}$ for $I \neq I'$. Once that is done, the fact that the subgroup $P_F \subset G$ determines the isotropic flag F for odd n goes via Chevalley's self-normalizing theorem for parabolics as in the symplectic case, so we would be done.

So now it suffices to prove the more precise result (with $n = 2m + 1 \geq 5$) that P_{F_I} corresponds to the subset I of the basis Δ as specified with roots indexed by $\{1, \ldots, m\}$

for type B_m ($m \geq 2$) in the handout on root systems for classical groups. Since we have shown that $P_{F_\emptyset}$ corresponds to the root basis Δ, as in the symplectic case our task reduces to checking that the roots $a_j \in \Delta$ ($1 \leq j \leq m$) satisfying $\langle a_j, \lambda_I \rangle > 0$ are exactly those j not in I, or equivalently those j for which $h_j > h_{j+1}$. This goes similarly to the symplectic case.

Even $\mathbf{n = 2m \geq 6}$. This goes similarly to type B except that extra care is required when analyzing the contribution from the roots denoted a_{m-1} and a_m in the D_m-diagram ($m \geq 3$) from the handout on root systems for split groups.

First, let's check that $P_F = P_{F'}$ when $m-1$ is maximal in $\nu(F)$ and F' is either of the two m-dimensional isotropic subspaces containing F; this will account for the existence of the exceptional case in (ii). Letting W be the $(m-1)$-dimensional member of F, the flag F' corresponds to one of the isotropic lines in the hyperbolic plane $H := W^\perp/W$. Clearly P_F preserves W and hence acts on the hyperbolic plane H. Consequently, to prove that the evident containment $P_{F'} \subset P_F$ of parabolic subgroups is an equality (equivalently, P_F preserves F') it suffices to check that the image of P_F in $O(H)$ preserves both isotropic lines.

We know that the stabilizer of every isotropic flag is a parabolic subgroup, and in particular is *connected*, so the image of P_F in $O(H)$ lands inside $SO(H)$. But as a subgroup of $GL(H) = GL_2$ the subgroup $SO(H)$ is the unique split maximal torus in $SL(H)$ *preserving* each of the two isotropic lines (as we see by computing in a basis arising from the isotropic lines).

Now we may and do focus on isotropic flags F such that if $m-1 \in \nu(F)$ then $m \notin \nu(F)$ (equivalently, $\nu(F)$ doesn't contain both $m-1$ and m), to bypass the duplication problem revealed by the preceding calculation of stabilizers of isotropic flags (corresponding to the exceptional situation in (ii)).

Definition 3.10. An isotropic flag F is *good* if $\nu(F)$ doesn't contain $\{m-1, m\}$.

The good F satisfying $m \in \nu(F)$ (hence $m-1 \notin \nu(F)$) require extra care when counting possibilities. The good news is:

Lemma 3.11. *There are 2^m distinct $G(k)$-orbits of good flags.*

Proof. First consider good F with $m-1 \in \nu(F)$, so $m \notin \nu(F)$ by design. We claim that there are 2^{m-2} $G(k)$-orbits of such F. Using notation as in our treatment of odd $n \geq 5$ (so F_I is the "standard" isotropic flag whose set of numerical invariants is the *complement* of I in $\{1, \dots, m\}$!), since these flags F have no m-dimensional member each such $G(k)$-orbit contains a unique isotropic flag F_I with $m-1 \notin I$ but $m \in I$. In other words, $I = J \cup \{m\}$ for an arbitrary subset J of $\{1, \dots, m-2\}$. For the $G(k)$-orbits of F with $m-1 \notin \nu(F)$, among those with also $m \notin \nu(F)$ we again get 2^{m-2} orbits, namely the orbit of F_I for $I = J \cup \{m-1, m\}$ with J an arbitrary subset of $\{1, \dots, m-2\}$. We have obtained 2^{m-1} orbits so far.

Finally, consider orbits of F with $m-1 \notin \nu(F)$ but $m \in \nu(F)$. We just need to show that there are 2^{m-1} orbits of such F. Using $O_{2m}(k)$-orbits rather than $G(k)$-orbits, there are unique orbit representatives of the form F_I with $I = J \cup \{m-1\}$ for $J \subset \{1, \dots, m-2\}$; this is 2^{m-2} possibilities for I, but each such $O_{2m}(k)$-orbit consists of two $G(k)$-orbits by keeping track of the $G(k)$-orbit of the m-dimensional member of the flag (of which there are exactly 2 possibilities, and $G(k)$ has index 2 in $O_{2m}(k)$). Hence, there are $2 \cdot 2^{m-2} = 2^{m-1}$ different $G(k)$-orbits obtained from this class of good isotropic flags. ∎

The existence aspect of the exceptional case in (ii) has been confirmed, and we have also verified the first assertion in (i) already. Hence, the count in the preceding lemma and the fact that there are exactly 2^m distinct $G(k)$-conjugacy classes of parabolic k-subgroups of G (corresponding to subsets of a fixed root basis) together imply that the proof of (i) and (ii) for $n = 2m \geq 6$ is reduced to proving that the map $F \mapsto P_F$ from the set of *good* isotropic flags into the set of parabolic k-subgroups of G is injective. We will do this by arguing similarly to the case of odd $n \geq 5$ (with λ_I and F_I as defined there); the restriction to good flags will counteract the effect of the existence of two $G(k)$-orbits of m-dimensional isotropic subspaces.

In our proof (for all $n \geq 3$) that P_F is parabolic for *every* isotropic flag F we saw that every F is in the $\mathrm{O}_{2m}(k)$-orbit of F_I for a necessarily unique I, namely the complement of $\nu(F)$ in $\{1, \dots, m\}$. Clearly F is good precisely when I meets $\{m-1, m\}$, which is to say I contains $m-1$ or m (or both), and those with $m \notin I$ (i.e., $m \in \nu(F)$) correspond to two distinct $G(k)$-orbits of good isotropic flags. We label Δ as $\{a_1, \dots, a_m\}$ via the type-D_m case in the handout on root systems for classical groups ($m \geq 3$).

Lemma 3.12. *Identify $\Delta = \{a_1, \dots, a_m\}$ with $\{1, \dots, m\}$ in the evident manner.*

(1) *In the 2^{m-1} cases for which $m \in I$ (all good), the subset of Δ corresponding to P_{F_I} is I when $m-1 \in I$ and is $I - \{m\}$ when $m-1 \notin I$.*
(2) *For the 2^{m-2} subsets $J \subset \{1, \dots, m-2\}$, inside the $\mathrm{O}_{2m}(k)$-conjugacy class of the stabilizer $P_{F_{J\cup\{m-1\}}}$ of the good isotropic flag $F_{J\cup\{m-1\}}$ the $G(k)$-conjugacy class of $P_{F_{J\cup\{m-1\}}}$ corresponds to the subset $J \cup \{m-1\} \subset \Delta$ and the other $G(k)$-conjugacy class corresponds to the subset $J \cup \{m\} \subset \Delta$.*

This lemma addresses at most 2^m conjugacy classes of parabolic k-subgroups, and yields each of the 2^m subsets of Δ, so in fact the 2^m conjugacy classes considered really are pairwise distinct.

Proof. For any $I \subset \{1, \dots, m\}$ we have seen that $P_{F_I} = P_G(\lambda_I)$ and that this contains the Borel subgroup $P_G(\lambda_\emptyset)$ containing the diagonal torus $T \subset G = \mathrm{SO}_n$ and corresponding as such to Δ. The subset I' of Δ corresponding to P_{F_I} is therefore the *complement* of the set of i such that $U_{a_i} \subset \mathscr{R}_{u,k}(P_{F_I}) = U_G(\lambda_I)$, which is to say the complement of the set of i such that $\langle a_i, \lambda_I \rangle > 0$. By definition of λ_I, the complement of I consists of those $1 \leq j \leq m$ such that $h_j > h_{j+1}$.

The description of the simple roots $a_i \in \Delta$ in the handout on root systems for classical groups for type-D_m ($m \geq 3$) gives that if $i < m$ then $\langle a_i, \lambda_I \rangle > 0$ precisely when $h_i > h_{i+1}$, which is to say $i \notin I$. Hence, for $i < m$ we conclude that $i \notin I'$ if and only if $i \in I$. In other words, apart from possibly m, the subset of Δ corresponding to P_{F_I} agrees with I.

Likewise, $\langle a_m, \lambda_I \rangle > 0$ (i.e., $m \notin I'$) precisely when $h_{m-1} + h_m > 0$. Since $h_{m-1} \geq h_m \geq 0 =: h_{m+1}$, clearly $h_{m-1} + h_m > 0$ if and only if either $h_m > 0$ (i.e., $m \notin I$) or $h_{m-1} > h_m = 0$ (i.e., $m-1 \notin I$ and $m \in I$). In particular, if $m \in I$ then $m \notin I'$ precisely when $m-1 \notin I$, or equivalently $m \in I'$ if and only if $m-1 \in I$. This gives (1).

Next, consider (2). Letting $I = J \cup \{m-1\}$, so $m-1 \in I$ and $m \notin I$, we have $h_{m-1} = h_m > 0$. Thus, $\langle a_m, \lambda_I \rangle > 0$, so $m \notin I'$. This forces $I' = I$, as desired. The other $G(k)$-conjugacy class of parabolics in the $\mathrm{O}_{2r}(k)$-conjugacy class of $P_{F_{J\cup\{m-1\}}}$ is obtained by conjugating against any $\rho \in \mathrm{O}_{2r}(k) - G(k)$. We can take ρ to be any reflection in a non-isotropic vector. Relative to the standard basis of $V = k^{2m}$ on which $q = q_{2m}$, choose ρ to be reflection in the unit vector $e_m + e_{2m}$ since this leaves e_i unchanged for $i \neq m, 2m$ and negates e_m and e_{2m}. Hence, this preserves T and leaves $a_1, \dots, a_{m-2}$ invariants but swaps a_{m-1} and a_m. Consequently, the subset of Δ associated to the ρ-conjugate of $P_{F_{J\cup\{m-1\}}}$ is $J \cup \{m\}$. ■

By Lemma 3.12, we get all 2^m distinct subsets of Δ by combining the output from the parabolics P_{F_I} for the $2^m - 2^{m-2}$ distinct good F_I (goodness omits those $I \subset \{1, \dots, m-2\}$) and the parabolics $P_{\rho(F_I)}$ for a fixed $\rho \in \mathrm{O}_{2n}(k) - G(k)$ and every $I \subset \{1, \dots, m\}$ containing $m-1$ but not m. This completes the proof of the exhaustive second assertion in (i), and to establish (ii) we can focus on good isotropic flags (the ones avoiding the exceptional situation in (ii)).

Consider isotropic flags F for which $m \notin \nu(F)$ (hence good). We shall prove that the $G(k)$-conjugacy class of P_F determines the $G(k)$-orbit of F (stronger than determining the numerical invariants when $m \in \nu(F)$!). Once this is proved, we can conclude (ii) similarly to the symplectic case (as we did for odd n): if F' and F are good with $P_{F'} = P_F$ then $F' = g(F)$ for some $g \in G(k)$, so $P_F = P_{g(F)} = gP_Fg^{-1}$, and hence $g \in P_F(k)$ by Chevalley's self-normalizing theorem. That forces $F' = g(F) = F$ as desired.

If $m \in \nu(F)$ then the $\mathrm{O}_{2m}(k)$-conjugacy class of P_F consists of two $G(k)$-conjugacy classes, and if $m \notin \nu(F)$ then the $\mathrm{O}_{2m}(k)$-conjugacy class is also a $G(k)$-conjugacy class (by Proposition 3.7). Hence, we can say conversely that the properties $m \in \nu(F)$ and $m \notin \nu(F)$ are characterized by the $G(k)$-conjugacy class of P_F by analyzing if this $G(k)$-conjugacy class is also an $\mathrm{O}_{2m}(k)$-conjugacy class. It is therefore legitimate to determine the $G(k)$-orbit of a good isotropic F from the $G(k)$-conjugacy class of P_F by treating the collection of all such flags F containing an m-dimensional member *separately* from the collection of those which do not contain an m-dimensional member.

First suppose $m \notin \nu(F)$, so F is in the $G(k)$-orbit of a unique F_I with $m \in I$. By Lemma 3.12(1), if also $m-1 \notin \nu(F)$ then the $G(k)$-conjugacy class of P_F corresponds to the subset of Δ complementary to $\nu(F)$ (this complementary subset contains m and $m-1$), whereas if $m-1 \in \nu(F)$ then P_F corresponds to the subset of $\{1, \dots, m-2\}$ complementary to $\nu(F) - \{m-1\}$. Hence, in these cases we can reconstruct the numerical invariants of F from the $G(k)$-conjugacy class of P_F. But the $G(k)$-orbit of F is characterized by the numerical invariants when $m \notin \nu(F)$, so the cases with $m \notin \nu(F)$ are settled.

Now consider good isotropic flags F with $m \in \nu(F)$, so $m-1 \notin \nu(F)$. In these cases the numerical invariants $\nu(F)$ characterize the $\mathrm{O}_{2m}(k)$-orbit of F that consists of two $G(k)$-orbits. For $J = \nu(F) - \{m\} \subset \{1, \dots, m-2\}$ and its complement J' inside $\{1, \dots, m-2\}$, Lemma 3.12 shows that the subset of Δ corresponding to the $G(k)$-conjugacy class of P_F is either $J' \cup \{m-1\}$ or $J' \cup \{m\}$ depending respectively on whether F is in the $G(k)$-orbit of $F_{J\cup\{m-1\}}$ or in the other $G(k)$-orbit in its $\mathrm{O}_{2m}(k)$-orbit. In these cases the $\mathrm{O}_{2m}(k)$-conjugacy class of P_F consists of two $G(k)$-conjugacy classes, and we have shown that the subset of Δ corresponding to the $G(k)$-conjugacy class of P_F again detects the $G(k)$-orbit of F (it depends on which of $m-1$ or m is in the associated subset of Δ). The proof of Theorem 3.9 is finally done! ■

Remark 3.13. Let's see what the preceding analysis tells us about the maximal (proper) parabolic k-subgroups of G. Away from the case $n = 2r \geq 4$ such subgroups correspond exactly to minimal (non-empty) isotropic flags, which is to say nonzero isotropic subspaces. In the split case for odd n the subset of Δ complementary to the vertex a_i corresponds to the stabilizer of i-dimensional isotropic subspaces.

Now suppose $n = 2r \geq 4$, so G is split. The case-checking with Lemma 3.12 near the end of the proof of Theorem 3.9 shows that if $n \geq 6$ then an isotropic flag F consisting of a single maximal isotropic subspace the subset of Δ associated to P_F is the complement I of exactly one of the vertices a_{r-1} or a_r (the "short legs" in the picture for the D_r-diagram with $r \geq 4$, and the two endpoints in the diagram for $\mathrm{D}_3 = \mathrm{A}_3$ when $r = 3$); this also holds when $n = 4$ by direct inspection.

The reader may wonder why this is not incompatible with triality when $r = 4$; i.e., the existence of order-3 automorphisms of the D_4-diagram. Doesn't such extra symmetry prevent the possibility of distinguishing 2 of the 3 extremal vertices in that case? There is no inconsistency because the order-3 diagram automorphisms lift (via a pinning) to order-3 automorphisms of Spin_8 that permute the 3 order-2 subgroups of the center transitively and so *do not* descend to automorphisms of $G = \mathrm{SO}_8$. In other words, working with SO_8 already picks out a preferred order-2 subgroup of the center of Spin_8, and that breaks the symmetry among the 3 extremal vertices in the D_4-diagram for the purpose of analyzing questions concerning SO_8!

The isotropic flags consisting of a single nonzero isotropic subspace of dimension $d \leq m - 2$ constitute a single $G(k)$-orbit and their associated $G(k)$-conjugacy class of parabolic stabilizers corresponds to the subset of Δ given by the complement of the vertex a_d. In this way we obtain all maximal proper subsets of Δ (i.e., complements of a single point), and hence we conclude that for $r \geq 2$ the maximal parabolic k-subgroups of SO_{2r} correspond exactly to the stabilizers of nonzero isotropic subspaces of dimension $d \neq r - 1$ (reasonable since any isotropic subspace W of dimension $r - 1$ has the same stabilizer as a 2-step flag obtained with either of the two isotropic r-dimensional subspaces containing W). Each fixed d corresponds to exactly one $\mathrm{SO}_{2r}(k)$-conjugacy class of such parabolic k-subgroups except that the single $\mathrm{O}_{2r}(k)$-orbit of r-dimensional isotropic subspaces corresponds to *two* such conjugacy classes (one per $\mathrm{SO}_{2r}(k)$-orbit of maximal isotropic subspaces, distinguished by the parity of the dimension with which the maximal isotropic subspace meets the "standard" one spanned by $e_1, \dots, e_r$ in the coordinatization used in the proof of Theorem 3.9).

Example 3.14. As in the symplectic case, the maximal (proper) parabolic k-subgroups correspond to the minimal non-empty flags, which is to say the stabilizers of nonzero isotropic subspaces. This corresponds to subsets of Δ complementary to a single vertex.

In the case of split q, or more specifically $G = \mathrm{SO}_n$, we have described these above. Do any of them have an abelian unipotent radical? Since $\mathrm{SO}_3 \simeq \mathrm{PGL}_2$, $\mathrm{SO}_4 = \mathrm{SL}_2 \times^{\mu_2} \mathrm{SL}_2$, and $\mathrm{SO}_6 \simeq \mathrm{SL}_4/\mu_2$, for $n = 3, 4, 6$ all maximal parabolic subgroups have abelian unipotent radical. Thus, assume $n = 5$ or $n \geq 7$, so either $n = 2m + 1$ with $m \geq 2$ or $n = 2m$ with $m \geq 4$. These cases can be analyzed similarly to the symplectic case in Example 2.2, except we use the Bourbaki tables for types B_m ($m \geq 2$) and D_m ($m \geq 4$).

More specifically, by inspecting the coefficients of the highest positive root to rule out many possibilities, we see that for type B_m ($m \geq 2$) there is exactly one conjugacy class of parabolics with abelian unipotent radical, corresponding to the complement of the extremal long root in the diagram. (This is consistent with the case of $\mathrm{B}_1 = \mathrm{A}_1$ and especially with $\mathrm{C}_2 = \mathrm{B}_2$.) These are the stabilizers of isotropic lines.

Likewise, for type D_m ($m \geq 4$) there are exactly three such conjugacy classes, corresponding to the complement of any of the 3 extremal roots in the diagram. (This is consistent with the case of $\mathrm{D}_3 = \mathrm{A}_3$ for which all 3 conjugacy classes of maximal parabolic subgroups have abelian unipotent radical, and the case of $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$ for which the unique conjugacy class of maximal parabolic subgroups has abelian unipotent radical.) These are the stabilizers of either an isotropic line or of a maximal isotropic subspace (the latter giving rise to two $G(k)$-conjugacy classes, one for each of the two $G(k)$-orbits of maximal isotropic subspaces). It is easy to check directly for even n that the stabilizer of a maximal isotropic subspace $W \subset V$ has abelian unipotent radical: this unipotent radical corresponds to the vector space of linear maps $W \to V/W = W^*$ that are B_q-alternating.

Remark 3.15. Let $S \subset G = \mathrm{SO}(q)$ be a maximal split k-torus. Certainly ${}_k\Phi = \Phi(G,S)$ is of type D_r when $n = 2r \geq 4$ (as then q is split, so G is split), and otherwise we claim it is of type B_r (regardless of the parity of n).

To understand this, recall that there is an orthogonal decomposition

$$V = H_1 \perp \cdots \perp H_r \perp V_0$$

for r hyperbolic planes H_j and an *anisotropic* non-degenerate quadratic space V_0. Letting $\{L_j, L'_j\}$ be the two isotropic lines in H_j, we can make $(t_1, \dots, t_r) \in \mathrm{GL}_1^r$ act on V via the trivial action on V_0, and acting on H_j through scaling by t_j on L_j and t_j^{-1} on L'_j. This identifies GL_1^r as a split torus inside $\mathrm{GL}(V)$ that leaves q invariant, so it lands inside $\mathrm{O}(q)$ and hence inside $\mathrm{SO}(q) = G$. As such, this is an r-dimensional split torus, so every S arises in this way for a suitable choice of such hyperbolic planes. (Explicitly, S is equal to $\prod \mathrm{SO}(H_j) \subset \mathrm{SO}(q) = G$.)

Let $q_0 = q|_{V_0}$, $Q = q|_W$ for $W := H_1 \perp \cdots \perp H_r$, and $T_0 \subset \mathrm{SO}(q_0)$ a maximal k-torus, so by dimension reasons the k-torus

$$S \times T_0 \subset \mathrm{SO}(Q) \times \mathrm{SO}(q_0) \subset G$$

is a maximal k-torus. The split case has already been settled in the handout on root systems for classical groups, so we now assume $1 \leq r < [n/2]$ (so $n \geq 4$).

Pick bases of $H_1, \dots, H_r$ lying in the two isotropic lines of each of these hyperbolic planes, and arrange label these basis vectors as $e_1, \dots, e_{2r}$ with H_j spanned by e_j and $e_{2r-+1-j}$ ($1 \leq j \leq r$). Since now $V_0 \neq 0$ (as $r < n/2$), we can pick a line ℓ in V_0 to get the subspace $W \perp \ell$ on which q is conformal to the split form in $2r+1$ variables. Choose a basis $\{e_{2r+1}, \dots, e_n\}$ of V_0 extending $\ell = ke_{2r+1}$. Using the trivial action on these *additional* basis vectors, we get a copy of $\mathrm{SO}_{2r+1} = \mathrm{SO}(W \perp \ell)$ inside G containing S as a split maximal torus. Explicitly, if we denote an element of $\mathrm{End}(V) = \mathrm{Mat}_n(k)$ as a block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A \in \mathrm{End}(W)$, $B \in \mathrm{Hom}(W, V_0)$, $C \in \mathrm{Hom}(V_0, W)$, and $D \in \mathrm{End}(V_0)$, then S corresponds to blocks

$$\begin{pmatrix} A & 0 \\ 0 & 1_{n-2r} \end{pmatrix}$$

with $A \subset \mathrm{GL}_{2r}$ consisting of diagonal elements

$$\mathrm{diag}(t_1, \ldots, t_r, t_r^{-1}, \ldots, t_1^{-1}).$$

It is an elementary matter to calculate S-conjugation against $\mathfrak{gl}(V)$ to read off that the upper-left $(2r+1) \times (2r+1)$ block provides the set of nontrivial S-weights as for the root system $\Phi(\mathrm{SO}(W \perp \ell), S)$ of type B_r, the lower-right $(n-2r) \times (n-2r)$ block has trivial S-action, and the S-weights occurring in the entries for B and C are exactly what is seen along their edges adjacent to A. Hence, ${}_k\Phi = \Phi(\mathrm{SO}(W \perp \ell), S)$ is of type B_r whenever $r < [n/2]$ (regardless of the parity of n) and

$$Z_G(S) = S \times \mathrm{SO}(q_0)$$

with (V_0, q_0) non-degenerate of dimension $n - 2r$. Note that the root spaces for the relative roots occurring outside the upper-left $r \times r$ block become very large as $n - 2r$ grows.