Math 249B. Tits systems

1. Introduction

Let (G, T) be a split connected reductive group over a field k, and $\Phi = \Phi(G, T)$. Fix a positive system of roots $\Phi^+ \subset \Phi$, and let B be the unique Borel k-subgroup of G containing T such that $\Phi(B, T) = \Phi^+$. Let $W = W(G, T)(k) = N_{G(k)}(T)/T(k)$, and for each $w \in W$ let $n_w \in N_{G(k)}(T)$ be a representative of W. For each $a \in \Phi$ we let $r_a \in W$ be the associated involution, and we let Δ denote the base of Φ^+. The Bruhat cell $C(w) = Bn_wB$ depends only on w, not n_w, and for the closed subset

$$\Phi'_w = \Phi^+ \bigcap w(-\Phi^+) \subset \Phi^+$$

we proved that the multiplication map

$$U_{\Phi'_w} n_w \times B \to C(w)$$

is an isomorphism of k-schemes. We also proved the Bruhat decomposition: $W \to B(k)\backslash G(k)/B(k)$ is bijective.

The purpose of this handout is to show that $(G(k), T(k), B(k), \{r_a\}_{a \in \Delta})$ satisfies the axioms of a *Tits system* (the definition of which we will give), from which many wonderful group-theoretic consequences follow. For example, we will obtain the simplicity of $G(k)/Z_G(k)$ for any *split* absolutely simple and simply connected semisimple k-group G away from three cases: $\mathrm{SL}_2(\mathbf{F}_2) = S_3$, $\mathrm{SL}_2(\mathbf{F}_3)/\mathbf{F}_3^\times = A_4$, and $\mathrm{Sp}_4(\mathbf{F}_2) = S_6$.

2. Tits systems and applications

Inspired by the structural properties of groups of the form $G(k)$ for a connected reductive group G over a field k, Tits discovered a remarkably useful concept (developed in §1.2ff in Chapter IV of Bourbaki):

Definition 2.1. Let H be an abstract group. A *Tits system* for H is a triple (B, N, S) consisting of subgroups $B, N \subset H$ and a subset $S \subset N/(N \bigcap B)$ such that:

1. B and N generate H, and $N \bigcap B$ is normal in N,
2. the elements of S have order 2 in $W := N/(N \bigcap B)$ and generate W,
3. for each $w \in W$ and $s \in S$, $C(s)C(w) \subset C(w) \cup C(sw)$ where $C(w') := Bn_{w'}B$ for any $n_{w'} \in N$ representing an element $w' \in W$ (visibly $C(w')$ depends only on w', not on the choice of $n_{w'}$, and this axiom can be equivalently written as $n_sBn_wB \subset Bn_wB \cup Bn_sn_wB$),
4. for all $s \in S$, n_s does not normalize B (equivalently, since $s^2 = 1$ in W, $n_sBn_s \not\subset B$).

These axioms imply that $W \to B\backslash H/B$ is bijective and that (W, S) is a Coxeter group (see Bourbaki Chapter IV: §2.3, Theorem 1 and §2.4 Theorem 2 respectively), but this implication will not be used in our work with split reductive groups. For a connected reductive k-group $\mathscr{G}$ equipped with a split maximal k-torus $\mathscr{T}$ and a Borel k-subgroup $\mathscr{B}$ containing $\mathscr{T}$, let $H = \mathscr{G}(k)$ and define $B = \mathscr{B}(k)$, $N = N_{\mathscr{G}(k)}(\mathscr{T})$, and $S = \{r_a\}_{a \in \Delta}$ (with Δ the base of $\Phi(\mathscr{G}, \mathscr{T})$). Clearly $W = W(\mathscr{G}, \mathscr{T})(k)$, and the general theory of root systems ensures that (W, S) a Coxeter group. We have already shown that $W \to B\backslash H/B$ is bijective for such examples, and this will be used to prove:

Proposition 2.2. *The triple (B, N, S) is a Tits system for H.*

Proof. The first axiom consists of two parts: (i) $\mathscr{G}(k)$ is generated by $\mathscr{B}(k)$ and $N_{\mathscr{G}(k)}(\mathscr{T})$, and (ii) $N_{\mathscr{G}(k)}(\mathscr{T}) \bigcap \mathscr{B}(k)$ is normal in $N_{\mathscr{G}(k)}(\mathscr{T})$. The normality is immediate from the fact that $N_{\mathscr{G}}(\mathscr{T}) \bigcap \mathscr{B} = \mathscr{T}$ (as $W(\mathscr{G}, \mathscr{T})(k)$ acts simply transitively on the set of Borel k-subgroups containing $\mathscr{T}$). The assertion in (i) follows from the established Bruhat decomposition $\mathscr{G}(k) = \coprod \mathscr{B}(k) n_w \mathscr{B}(k)$ on k-points.

The second axiom for Tits systems expresses the fact that the Weyl group $W(\mathscr{G}, \mathscr{T})(k)$ is generated by the reflections in the simple positive roots (relative to the positive system of roots $\Phi(\mathscr{B}, \mathscr{T})$). The identification of $W(\mathscr{G}, \mathscr{T})(k)$ with $W(\Phi(\mathscr{G}, \mathscr{T}))$ reduce this to a general property of Weyl groups of root systems.

The third axiom is a special case of the inclusion

$$G_a B n_w B \subset B n_w B \cup B n_a n_w B$$

proved in the handout on the geometric Bruhat decomposition (see (3) there), since $G_a(k)$ contains a representative for $r_a \in W$. The fourth axiom follows from Chevalley's result that Borel subgroups are self-normalizing, since $\mathscr{B} \bigcap N_{\mathscr{G}}(\mathscr{T}) = \mathscr{T}$. ■

It is a general fact that for any Tits system (H, B, N, S), the set $S \subset W$ is uniquely determined by the pair of subgroups (B, N) in H (see Corollary to Theorem 3 in §2.5 of Bourbaki Chapter IV). For this reason, often one focuses attention on the pair (B, N), and such a pair in H for which the axioms of a Tits system are satisfied (for some, necessarily unique, subset $S \subset W$) is called a *BN-pair* in H.

Remark 2.3. A subgroup of H is called *parabolic* (relative to a BN-pair for H) if it contains a conjugate of B. There is a general parameterization of conjugacy classes of parabolic subgroups of H (labeled by subsets of S), inspired by the case of parabolic subgroups of split connected reductive groups that we shall discuss (and recovering the result for algebraic groups when the ground field is separably closed). See Bourbaki Chapter IV, §2.6 Proposition 4 for a precise formulation and proof of this result.

Inspired by the examples arising from split connected reductive groups, $B \bigcap N$ is usually denoted as T, and a BN-pair is called *split* when $B = T \ltimes U$ for a normal subgroup U of B. For instance, if $(\mathscr{G}, \mathscr{T})$ is a split connected reductive group as above then the associated BN-pair $(\mathscr{B}(k), N_{\mathscr{G}(k)}(\mathscr{T}))$ in $\mathscr{G}(k)$ is split by taking $U = U_{\Phi^+}(k)$ for $\Phi^+ = \Phi(\mathscr{B}, \mathscr{T})$.

Theorem 2.4. *Let (H, B, N) be a split BN-pair using $U \subset N$, and assume that U is solvable. Let $Z = \bigcap_{h \in H} hBh^{-1}$. Let H^+ be the normal subgroup of H generated by the conjugates of U. Assume that $W \neq 1$ (equivalently, $S \neq \emptyset$) and that the Coxeter graph associated to the Coxeter system (W, S) is connected.*

If H^+ is its own commutator subgroup and $Z \bigcap U = 1$ then $Z \bigcap H^+$ is the center of H^+ and the quotient $H^+/(Z \bigcap H^+)$ is either simple non-abelian or trivial.

Remark 2.5. The notion of Coxeter graph mentioned in this theorem is defined in §1.9 of Chapter IV of Bourbaki, and in the examples arising from split connected reductive groups $\mathscr{G}$ this is precisely the underlying graph of the Dynkin diagram (so it is connected if and only if $\mathscr{D}(\mathscr{G})$ is absolutely simple). The hypothesis $Z \bigcap U = 1$ holds in such examples because $Z = Z_{\mathscr{G}}(k)$.

Theorem 2.4 is a special case of the Corollary to Theorem 5 in §2.7 of Chapter IV of Bourbaki (in view of Remark 2 after that Corollary). To apply this theorem to $H = \mathscr{G}(k)$ for split connected semisimple k-groups $\mathscr{G}$ that are absolutely simple and simply connected (with k any field), we have to prove that the nontrivial subgroup $\mathscr{G}(k)^+$ is its own commutator subgroup. Note that by Proposition 2.5 of the handout on the geometric Bruhat decomposition we have $\mathscr{G}(k)^+ = \mathscr{G}(k)$, so

Theorem 2.4 implies that if the nontrivial group $\mathscr{G}(k)$ is perfect then its center is $Z_{\mathscr{G}}(k)$ and the nontrivial quotient $\mathscr{G}(k)/Z_{\mathscr{G}}(k)$ is a non-abelian simple group.

But when is the nontrivial group $\mathscr{G}(k)$ perfect? There are a few cases in which $\mathscr{G}(k)$ has a nontrivial abelian quotient (and so is not perfect), namely the solvable groups $\mathrm{SL}_2(\mathbf{F}_2)$ and $\mathrm{SL}_2(\mathbf{F}_3)$ as well as the group $\mathrm{Sp}_4(\mathbf{F}_2) \simeq S_6$ (see the list of accidental isomorphisms among finite groups early in C.6 in my Luminy SGA3 notes). These turn out to be the only counterexamples when working with *split* $\mathscr{G}$:

Proposition 2.6. *The group $\mathscr{G}(k)$ is perfect except for $\mathscr{G} = \mathrm{SL}_2$ over $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathscr{G} = \mathrm{Sp}_4$ over $\mathbf{F}_2$.*

Proof. The exceptional cases have been explained already, so we now show that in all other cases $\mathscr{G}(k)$ is perfect. By Proposition 2.5 of the handout on the geometric Bruhat decomposition, it is the same to work with $\mathscr{G}(k)^+$. It suffices to show that $U_a(k)$ is contained in the commutator subgroup of $\mathscr{G}(k)^+$ for each root a.

Any pair of opposite root groups $U_{\pm c}$ in the split simply connected $\mathscr{G}$ generate an SL_2 in which $U_{\pm c}$ are the standard unipotent subgroups $U^{\pm}$. It is classical that the subgroups $U^{\pm}(k)$ generate $\mathrm{SL}_2(k)$ for any k, and that $\mathrm{SL}_2(k)$ is its own commutator subgroup when $k \neq \mathbf{F}_2, \mathbf{F}_3$, so we are done if $|k| > 3$.

Only the cases $|k| \leq 3$ remain, so we may and do assume that we are not in type A_1. That is, the Dynkin diagram has at least 2 vertices. Hence, for any root c there is an adjacent root c', and if the diagram is has two root lengths then we can choose c and c' to have distinct lengths. Since the Weyl group acts transitively on the roots of a common length, to show that $U_a(k)$ lies in the commutator subgroup of $\mathscr{G}(k)^+$ for all roots a, it suffices to treat one root of each length. The root groups $U_c, U_{c'}$ lie in the connected semisimple subgroup $\mathscr{D}(Z_G(T_{c,c'}))$ that is split and *simply connected* (!) of rank 2, where $T_{c,c'} = (\ker c \cap \ker c')^0_{\mathrm{red}} \subset T$. In other words, this group is of type A_2, $\mathrm{B}_2 = \mathrm{C}_2$, or G_2, which is to say (by the Isomorphism Theorem over k!) $\mathscr{G}$ is either SL_3, Sp_4, or G_2. It is therefore sufficient (though not necessary) to treat just these cases, though over $\mathbf{F}_2$ that won't work: we need a finer analysis over $\mathbf{F}_2$ to handle the groups which have just been reduced to type B_2 (as the desired perfectness conclusion is false over $\mathbf{F}_2$ for type B_2).

For SL_3, a direct calculation with the standard base $\Delta = \{a, b\}$ and standard root group parameterizations $u_c : \mathbf{G}_{\mathrm{a}} \simeq U_c$ (up to a sign) gives the commutation relation

$$(u_a(x), u_b(y)) = u_{a+b}(\pm xy),$$

so $U_{a+b}(k)$ lies in the commutator subgroup of $\mathscr{G}(k)^+$. But all roots have the same length, and hence are conjugate under the action of the Weyl group, so all $U_c(k)$ lies in the commutator subgroup of $\mathscr{G}(k)^+$.

For G_2, an inspection of the picture of the root system shows that the roots of a common length (long or short) constitute a root system of type A_2. More specifically, for any root a there is another root a' such that $\{a, a'\}$ is the base for a root system of type A_2, and hence the split simply connected semisimple subgroup $\mathscr{D}(Z_G(T_{a,a'}))$ with root system $\Phi \bigcap (\mathbf{Q}a + \mathbf{Q}a')$ of type A_3 is an SL_3 containing U_a as one of its root groups (relative to the intersection with $\mathscr{T}$). Thus, the settled case of A_2 implies that $\mathscr{G}(k)^+$ is its own commutator subgroup.

The only $\mathscr{G}$ that can give rise to type B_2 by the above reduction step are the ones with a double bond in the Dynkin diagram (as the multiplicity of an edge is defined in terms of a pair of a root with a coroot, not by a non-canonical Euclidean structure!). By the classification of connected Dynkin diagram, the only such possible diagrams are types B and C (in rank $n \geq 2$) and type F_4. By inspection, every root in the Dynkin diagram of F_4 is linked to another of the same length, so we can reduce F_4 to the settled case of A_2 as we did for G_2.

It remains types B and C in rank $n \geq 3$. Since $n-1 \geq 2$, we can use the settled A_2-case to express the elements of $U_c(k)$ as commutators for all roots c of the length given by the leftmost $n-1$ vertices in the diagram (i.e., long roots for type B and short roots for type C). So the remaining problem for types B and C is to handle $U_c(k)$ for some root c of the *other* root-length (namely, short roots for type B and long roots for type C).

Consider the simply connected subgroup Sp_4 of type $\mathrm{B}_2 = \mathrm{C}_2$ associated to the pair of simple positive roots with distinct lengths that are adjacent in the diagram, say with a short and b long. In Example 5.3.9 of the course notes we defined explicit $u_c : \mathbf{G}_{\mathrm{a}} \simeq U_c$ for Sp_4 and roots c lying in a suitable Φ^+ for the split diagonal torus, and we obtained

$$(u_a(x), u_b(y)) = u_{2a+b}(x^2y)u_{a+b}(-xy), \;\; (u_a(x), u_{a+b}(y)) = u_{2a+b}(-2xy).$$

Focus on the first relation with $x = 1$. The left side is a commutator, and the right side is a product of terms from the respective root groups U_{2a+b} and U_{a+b} with $2a+b$ long and $a+b$ short, so one of the two terms on the right side is already known to be in the commutator subgroup. Hence, so is the other term, so we have handled the root length missed by the A_2-arguments used above for types B and C in rank ≥ 3. ■

The perfectness conclusion above allows us to apply the final assertion in Theorem 2.4 to obtain by an entirely uniform method (up to some special case with the finite fields of size 2 or 3) the classical simplicity results for matrix groups over all fields (especially significant for finite fields):

Corollary 2.7. *Let G be a split connected semisimple group over a field k such that it is absolutely simple and simply connected. If $k = \mathbf{F}_3$ then assume $G \not\simeq \mathrm{SL}_2$, and if $k = \mathbf{F}_2$ then assume $G \not\simeq \mathrm{SL}_2$, Sp_4. The center of $G(k)$ is $Z_G(k)$, and $G(k)/Z_G(k)$ is a non-abelian simple group.*

We emphasize the power of root systems to give a uniform approach to such a simplicity result that was classically proved by extensive case-by-case procedures.

3. Kneser-Tits conjecture

Let G be a connected semisimple group over a field k. Assume moreover that G is *k-isotropic*: it contains $\mathbf{G}_m$ as a k-subgroup. (In class we saw that this is equivalent to the existence of a proper parabolic k-subgroup since Z_G is finite; it is also equivalent to the existence of $\mathbf{G}_{\mathrm{a}}$ as a k-subgroup, but that equivalence lies much deeper.) For example, since any connected reductive group over a finite field contains a Borel k-subgroup if k is finite (by Lang's theorem), if the semisimple G is nontrivial then necessarily G is k-isotropic.

The Borel–Tits structure theory for isotropic semisimple groups G over any field k is the focus of the final two weeks of the course. The fundamental results in the theory are summarized in:

Theorem 3.1 (Borel–Tits)**.** *All maximal k-split k-tori S in G are $G(k)$-conjugate, the set $\Phi(G, S)$ of nontrivial S-weights on $\mathrm{Lie}(G)$ is a (possibly non-reduced) root system in $\mathrm{X}(S)_{\mathbf{Q}}$, and the minimal parabolic k-subgroups P in G are $G(k)$-conjugate.*

Every such P contains some S, and every S lies in some P, and the assignment $P \mapsto \Phi(P, S)$ is a bijection from the set of minimal parabolic k-subgroups $P \supseteq S$ onto the set of positive systems of roots in $\Phi(G, S)$.

The étale k-group $W(G, S) := N_G(S)/Z_G(S)$ is constant, $N_G(S)(k)/Z_G(S)(k) \to W(G, S)(k)$ is an isomorphism, and naturally $W(G, S)(k) = W(\Phi(G, S))$.

The common dimension of the maximal k-split tori in G is called the *k-rank*. Note that the surjectivity of $N_G(S)(k)/Z_G(S)(k) \to W(G, S)(k)$ is quite remarkable, since it cannot be proved cohomologically: $\mathrm{H}^1(k, Z_G(S))$ can be nontrivial (when k is not finite), as we will illustrate with explicit examples later. A big challenge in proving these results (especially with finite k) is that we cannot use extension of the ground field as readily as in the split case (since such an operation is not compatible with the formation of S in general).

For every $a \in \Phi(G, S)$, the dynamic method provides root groups $U_a \subset G$ which admit a unique characterization similarly to the split case but there are some deviations: $\dim U_a$ can be rather large, if a is multipliable in $\Phi(G, S)$ then $\mathrm{Lie}(U_a) = \mathfrak{g}_a \oplus \mathfrak{g}_{2a}$, and in this multipliable case U_a can be non-commutative. (The commutative root groups are always vector groups.) In view of these results, we get a canonical normal subgroup $G(k)^+$ in $G(k)$: the subgroup generated by the $G(k)$-conjugates of $U_a(k)$ for $a \in \Phi(G, S)$ (or even just $a \in \Phi(G, S)^+$). By using a version of the theory of the open cell and description of $G(k)$-conjugacy classes of parabolic k-subgroups of G, $G(k)^+$ has a more "invariant" description: it is the subgroup of $G(k)$ generated by the subgroups $U(k)$ for the (k-split!) k-unipotent radicals $U = \mathscr{R}_{u,k}(P)$ of the parabolic k-subgroups P of G.

The theory of Tits systems can be used to establish the simplicity of the quotient of $G(k)^+$ modulo its center when G is simply connected and absolutely simple and $|k| > 3$, and so it is natural to ask for such G whether or not $G(k)^+ = G(k)$. For example, if G is k-split (and simply connected and absolutely simple) then this always holds, by Proposition 2.5 of the handout on the geometric Bruhat decomposition.

The *Kneser-Tits conjecture* is that $G(k)^+ = G(k)$ for any simply connected absolutely simple k-isotropic G over any field k. This was settled by Steinberg when G is quasi-split over k (i.e. contains a Borel k-subgroup), which includes the case of finite k. Over non-archimedean local fields it was settled by Platonov in characteristic 0 using the classification theory of semisimple groups (and later proved in a classification-free and characteristic-free manner by Prasad–Ragunathan). Platonov also found a counterexample over $k = \mathbf{Q}(x, y)$ of type A. The problem was settled affirmatively in general over all global fields only recently by Gille (who handled the thorniest case of certain forms of E_6), building on the work of many others (Garibaldi, Prasad–Ragunathan, Tits, etc.).