Math 249B. Torus centralizer of a coset space

1. Introduction

Our main aim is to establish the following fact, the proof of which is somewhat tricky:

Lemma 1.1. *Assume $k = \overline{k}$, and let S be a k-torus in a smooth connected affine k-group G. The Borel subgroups of $Z_G(S)$ are precisely the subgroups $Z_B(S) = B \cap Z_G(S)$ (scheme-theoretic intersection, as always) for Borel subgroups B of G which contain S.*

The Borel property for $Z_B(S)$ rests crucially on the fact that S is a torus. That is, if $H \subset B$ is merely a smooth connected subgroup then typically $Z_B(H)$ is not a Borel subgroup of $Z_G(H)$, even when both centralizers are smooth and connected. For example, if $G = \mathrm{GL}_3$ and B is the standard upper-triangular Borel subgroup and $H \subset \mathscr{R}_u(B)$ is the subgroup that fixes e_1 and e_2 and sends e_3 into $\mathrm{span}(e_2, e_3)$ then $Z_G(H)$ is easily computed to be smooth, connected, and *solvable* of dimension 5 whereas $Z_B(H)$ is smooth and connected of dimension 4.

Proof. We proved in the previous course (using dynamic methods) that the schematic centralizer $Z_G(S)$ is smooth and connected. This contains a Borel subgroup B', and we claim that $S \subset B'$. Indeed, S lies in *some* Borel subgroup of $Z_G(S)$, all Borel subgroups in a smooth connected affine group over $k = \overline{k}$ are conjugate, and S is *central* in $Z_G(S)$, so indeed $S \subseteq B'$. In turn, B' is contained in a Borel subgroup B of G (via the characterization of Borel subgroups as maximal smooth connected solvable subgroups, rather than the "minimal parabolic" viewpoint). But $Z_B(S)$ is a smooth connected subgroup of B, so it is solvable, yet it lies in $Z_G(S)$. The inclusion $B' \subseteq Z_B(S) = B \cap Z_G(S)$ is therefore an equality by maximality of B' in $Z_G(S)$. Thus, we have found a Borel subgroup B in G containing S such that $Z_B(S)$ is equal to an arbitrarily chosen Borel subgroup B' of $Z_G(S)$. This proves that all Borel subgroups of $Z_G(S)$ have the asserted form.

Conversely, we wish to show that if B is a Borel subgroup of G containing S then the smooth connected solvable subgroup $Z_G(S) \cap B = Z_B(S)$ is a Borel subgroup of $Z_G(S)$. This is the more interesting direction. Since $Z_B(S)$ is smooth and connected, and it inherits solvability from B, it suffices to prove that $Z_G(S)/Z_B(S)$ is complete. Since $S \subseteq B$, the S-conjugation on G preserves B and so induces an action on the complete coset space G/B. By HW8, Exercise 3 of the previous course, the scheme-theoretic fixed locus $(G/B)^S$ is *smooth*. But this fixed locus is obviously closed in G/B, so it is complete. There is an evident map $Z_G(S)/Z_B(S) \to (G/B)^S$ which factors through the (irreducible) connected component of the identity of the target (since $Z_G(S)$ is connected), and we will show that it is an isomorphism onto this component. That will provide the desired completeness for $Z_G(S)/Z_B(S)$. The next result provides this completeness from that of $(G/B)^S$. ■

Proposition 1.2. *If H is a smooth closed subgroup of G (not necessarily connected or solvable) that is normalized by a torus $S \subset G$, then under the resulting left multiplication action on $(G/H)^S$ by $Z_G(S)$ all orbit maps $Z_G(S) \to (G/H)^S$ through points $g_0 \in (G/H)^S(k)$ are smooth. In particular, the orbits are open and hence coincide with the connected components of $(G/H)^S$.*

More specifically, the natural map of smooth varieties

$$f : Z_G(S)/Z_H(S) \to (G/H)^S$$

(induced by the orbit map through $1 \bmod H$*, with* $\mathrm{Stab}_{Z_G(S)}(1 \bmod H) = Z_H(S)$*) is an isomorphism onto the identity component of the target.*

Although H may not contain S, the scheme-theoretic centralizer $Z_H(S)$ for the S-action on H still makes sense and is *smooth*: we simply apply the usual centralizer theory to S viewed as the second factor of the semi-direct product $H \rtimes S$ and observe that $Z_{H \rtimes S}(1 \times S) = Z_H(S) \times S$ as functors on k-algebras. The reader is referred to Proposition 11.15 in Borel's book for a direct proof of Proposition 1.2 in the special case that $H = B$; allowing more general H clarifies the key properties of tori that underlie the proof, and this generalization will be used a lot in what follows.

Before proving Proposition 1.2, we consider an example that illustrates some striking "non-homogeneity" of $(G/H)^S$.

Example 1.3. Consider $G = \mathrm{SL}_3$ with diagonal maximal torus T and "root group" $U \simeq \mathbf{G}_a$ consisting of matrices of the form

$$u(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

The group U is normalized by T, with $t = \mathrm{diag}(t_1, t_2, t_3)$ ($\prod t_i = 1$) satisfying $tu(x)t^{-1} = u((t_2/t_3)x)$. Let $H = T \ltimes U$ inside G. This contains the 1-dimensional tori

$$S = \{\mathrm{diag}(t, t, t^{-2})\}, \;\; S' = \{\mathrm{diag}(t^{-2}, t, t)\}$$

which are visibly conjugate in G using the determinant-1 matrix π_0 that swaps the first and third standard basis vectors and negates the second one.

Note that S' is *central* in H (as it conjugates trivially on U) whereas S is not. Hence, $Z_H(S') = H$ is of dimension 3 whereas $Z_H(S)$ must have strictly smaller dimension and so coincides with its 2-dimensional subtorus T (due to its a-priori connectedness, or by computation). Clearly $Z_G(S) \simeq \mathrm{GL}_2$ has dimension 4 (as does $Z_G(S')$, due to the conjugacy of S and S' in G, or by direct computation). Thus, in $(G/H)^S$ the connected component of $1 \bmod H$ has dimension 2 whereas the connected component of $\pi_0 \bmod H$ has dimension 1. More explicitly, these connected components are respectively given by $Z_G(S)/Z_H(S) = \mathrm{GL}_2/\mathrm{diag} = \mathbf{P}^1 \times \mathbf{P}^1 - \Delta$ (which is affine either via ampleness of Δ or by identifying Δ as a hyperplane section under the Segre embedding into $\mathbf{P}^3$) and $Z_G(S')/Z_H(S') = \mathrm{GL}_2/B' = \mathbf{P}^1$. In particular, one of the connected components of $(G/H)^S$ is affine and other is complete!

2. Proof of Proposition 1.2

First note that the S-orbits in G/H are connected, and distinct orbits are disjoint, so once openness is proved for all orbits it follows that there are just finitely many orbits and these are the connected components of $(G/H)^S$. As a preliminary step, we reduce to the case that H contains S by using a semidirect product trick. Namely, for $G' = G \rtimes S$, $H' = H \rtimes S$, and $S' = 1 \times S$, we have

$$Z_{G'}(S') = Z_G(S) \times S, \;\; Z_{H'}(S') = Z_H(S) \times S, \;\; (G'/H')^{S'} = (G/H)^S$$

and the orbit map of $Z_{G'}(S')$ through a point $(g_0, s) \bmod H' \in (G'/H')^{S'}(k)$ is thereby identified with the orbit map of $Z_G(S)$ through $g_0 \bmod H \in (G/H)^S(k)$. Hence, we can work with (G', H', S') instead, so now $S \subset H$.

Next, we reduce to the study of the orbit map through $1 \bmod H$. For a point $g_0 \in G(k)$, the coset g_0H viewed as a point of $(G/H)(k)$ is S-fixed precisely when the commutator $(g_0^{-1}sg_0)s$ lies in H for all $s \in S(k)$. Since now $S \subset H$, it is the same to say $g_0^{-1}Sg_0 \subset H$. For any such g_0, the $Z_G(S)$-stabilizer of g_0H is $g_0Z_H(g_0^{-1}Sg_0)g_0^{-1}$. The left multiplication map $G/H \simeq G/H$ defined by $x \mapsto g_0^{-1}x$ intertwines the S-conjugation action on the source with $g_0^{-1}Sg_0$-action on the target (via the isomorphism $S \simeq g_0^{-1}Sg_0$ defined by $t \mapsto g_0^{-1}tg_0$). Under the resulting isomorphism $\phi : (G/H)^S \simeq (G/H)^{g_0^{-1}Sg_0}$ between fixed spaces for these subtori of H, the point $g_0 \bmod H$ is carried to 1 and the orbit map $Z_G(S) \to (G/H)^S$ through g_0H is intertwined with the orbit map

$Z_G(g_0^{-1}Sg_0) \to (G/H)^{g_0^{-1}Sg_0}$ through 1 mod H. Hence, at the cost of passing to $g_0^{-1}Sg_0$ in place of S, it suffices to study the orbit map through 1.

Finally, it remains to prove that the natural map $Z_G(S) \to (G/H)^S$ is a smooth morphism. This amounts to surjectivity of the induced map between tangent spaces at all k-points of the source (since source and target are both smooth over $k = \overline{k}$). By equivariance for the left multiplication action of $Z_G(S)$ on both sides, homogeneity considerations on the source reduce the surjectivity to the case of tangent spaces at the identity. This tangent map is the natural map $\mathfrak{g}^S \to (\mathfrak{g}/\mathfrak{h})^S$ (see HW7, Exercise 4(ii) of the previous course), for which surjectivity follows from the complete reducibility of linear representations of tori (such as S).