Math 249B. Unirationality

1. Introduction

This handout aims to prove a result that is very useful for solving problems with connected reductive groups over infinite fields:

Theorem 1.1. *Let G be a smooth connected affine group over a field K. If K is perfect or G is reductive then G is unirational over K (i.e., admits a dominant K-morphism from a dense open subset of an affine space over K).*

The importance of this theorem is the consequence that over an *infinite* ground field, the set of rational points of a connected reductive group is always Zariski-dense. That is a very powerful tool for relating abstract group theory of the set of rational points to the structure of the algebraic group (e.g., checking normality of a closed K-subgroup scheme by using conjugation by K-rational points in settings which are sensitive to ground field extension, such as with maximal K-split tori or minimal parabolic K-subgroups).

Remark 1.2. In the absence of reductivity and perfectness, it is *not* true that the set $G(K)$ of rational points is necessarily Zariski-dense. The classic counterexample of a positive-dimensional G for which $G(K)$ is even *finite* is due to Rosenlicht: if $K = k(t)$ for a field k of characteristic $p > 0$ and $G \subset \mathbf{G}_{\mathrm{a}}^2$ is the subgroup defined by $y^q = x - tx^q$ for a p-power $q > 2$ then $G(K) = \{(0,0)\}$ if $p > 2$ whereas $G(K) = \{(0,0), (1/t, 0)\}$ if $p = 2$. (Allowing $q = p = 2$ gives a smooth affine conic, hence infinitely many K-points.) Rather generally, if K is *any* imperfect field of characteristic $p > 0$ and $q > 2$ is a p-power then for any $a \in K - K^p$ the K-group $\{y^q = x - ax^q\}$ is not unirational over K (though its locus of K-points may be Zariski-dense, such as if $K = K_s$). See Examples 11.3.1 and 11.3.2 in "Pseudo-reductive Groups" for more details.

2. Proof of Theorem 1.1

First we show that the general case over perfect fields reduces to the reductive case. Over perfect K the unipotent radical $\mathscr{R}_u(G_{\overline{K}})$ descends to a normal unipotent smooth connected K-subgroup $U \subset G$ due to Galois descent, and G/U is reductive. Since K is perfect, we know that U is K-split; i.e., has a composition series

$$\{1 = U_0 \subset U_1 \subset \cdots \subset U_n = U\}$$

over K whose successive quotients U_i/U_{i-1} are K-isomorphic to $\mathbf{G}_{\mathrm{a}}$. Thus, a U-torsor over *any* field extension of K (such as the function field $K(G/U)$) vanishes by successive applications of additive Hilbert 90, so the U-torsor $q : G \to G/U$ admits a rational point on its generic fiber. This $K(G/U)$-point "spreads out" to a section s of q over a dense open Ω in G/U, so there is an open immersion $U \times \Omega \to G$ defined by $(u, \omega) \mapsto u \cdot s(\omega)$. Thus, if G/U (and hence Ω) is unirational over K then the unirationality of G reduces to that of U. But the same torsor method applied to a composition series of U via induction on $\dim(U)$ shows that U has a dense open subset that is open in an affine space. (Stronger cohomological methods show that U itself is K-isomorphic to an affine space, but we do not need that here.)

Now it remains to treat the reductive case over any field, though we will treat characteristic zero by a separate argument from positive characteristic (and finite fields will also be treated by a special argument). The idea of the proof in the reductive case is to show that G is "generated by K-tori", which is to say that there is a finite set of K-tori in G that generate G as a K-group, or equivalently a finite set of K-tori $T_1, \dots, T_r \subset G$ such that the multiplication map of K-schemes

$$T_1 \times \cdots \times T_r \to G$$

is dominant. The reason that such dominance suffices is to due:

Lemma 2.1. *Every torus T over a field K is unirational over K.*

Proof. Let $\Gamma = \mathrm{Gal}(K_s/K)$, so the category of K-tori is anti-equivalent to the category of Γ-lattices (i.e., finite free $\mathbf{Z}$-modules equipped with a discrete continuous left Γ-action); the Γ-lattice associated to T is the character group $\mathrm{X}(T_{K_s})$.

Let K'/K be a finite Galois subextension of K_s that splits T, so $\mathrm{X}(T_{K_s})$ is a $\mathrm{Gal}(K'/K)$-lattice. The category of $\mathrm{Gal}(K'/K)$-representations on finite-dimensional $\mathbf{Q}$-vector spaces is semisimple, and more specifically $\mathrm{X}(T_{K_s})_{\mathbf{Q}}$ is a subrepresentation of a finite direct sum of copies of the regular representation $\mathbf{Q}[\mathrm{Gal}(K'/K)]$. Scaling by a sufficiently divisible nonzero integer, we thereby identify $\mathrm{X}(T_{K_s})$ as a subrepresentation of a finite direct sum of copies of $\mathbf{Z}[\mathrm{Gal}(K'/K)]$ equipped with its natural Γ-action.

The Γ-lattice $\mathbf{Z}[\mathrm{Gal}(K'/K)]$ corresponds to the Weil restriction torus $\mathrm{R}_{K'/K}(\mathrm{GL}_1)$ (check!), so an inclusion

$$\mathrm{X}(T_{K_s}) \hookrightarrow \mathbf{Z}[\mathrm{Gal}(K'/K)]^{\oplus r}$$

corresponds to a surjective map of K-tori $\mathrm{R}_{K'/K}(\mathrm{GL}_1)^r \to T$. The unirationality of tori over K is thereby reduced to the special case of tori of the form $\mathrm{R}_{K'/K}(\mathrm{GL}_1)$ for finite separable extensions K'/K. By thinking functorially, we see that $\mathrm{R}_{K'/K}(\mathrm{GL}_1)$ is the open non-vanishing locus of the norm morphism $\mathrm{N}_{K'/K} : \mathrm{R}_{K'/K}(\mathbf{A}^1_{K'}) \to \mathbf{A}^1_K$). ■

Beware that we *cannot* expect to establish dominance of a map $T_1 \times \cdots \times T_r \to G$ by proving surjectivity on tangent spaces at $(e, \dots, e)$ (which would ensure smoothness, and hence openness, at $(e, \dots, e)$), since such surjectivity can fail: for $G = \mathrm{SL}_2$ in characteristic 2 the diagonal maximal torus D has Lie equal equal to $\mathrm{Lie}(D[2])$ where $D[2] = \mu_2$ is the *center* of G, so conjugation on D has no effect on this Lie algebra! That is, all maximal tori in SL_2 have the *same* Lie algebra. Observe that this problem does not arise for PGL_2.

As in the proof of Grothendieck's theorem on (geometrically) maximal tori, we will use Zariski-density arguments with rational points of Lie algebras, so the case of finite ground fields has to be treated separately. Thus, we first dispose of the case of finite fields:

Lemma 2.2. *Any connected reductive group G over a finite field k is unirational over k.*

Proof. We abandon the attempt to show that G is generated by maximal k-tori, and instead proceed in another way. (In fact G *is* generated by its maximal k-tori, but Gabber's proof of this fact over finite fields in the proof of Proposition A.2.11 of "Pseudo-reductive groups" [2nd ed.] is rather delicate and so we omit it.) By Corollary 1.5 of the handout on Lang's Theorem we proved there exists a Borel k-subgroup $B \subset G$. Choose a maximal k-torus $T \subset B$. Over $\overline{k}$ there is a unique Borel subgroup of $G_{\overline{k}}$ containing $T_{\overline{k}}$ that is "opposite" to $B_{\overline{k}}$ (i.e., its intersection with $B_{\overline{k}}$ is precisely $T_{\overline{k}}$, or equivalently its Lie algebra supports precisely the set of roots $-\Phi(B_{\overline{k}}, T_{\overline{k}})$ inside $\Phi(G_{\overline{k}}, T_{\overline{k}})$). The uniqueness over $\overline{k}$ implies via Galois descent that this opposite Borel subgroup descends to a Borel k-subgroup $B' \subset G$ containing T with $B' \cap B = T$.

By another application of Galois descent for the perfect field k, the k-unipotent radicals $U := \mathscr{R}_{u,k}(B)$ and $U' := \mathscr{R}_{u,k}(B')$ descend the unipotent radicals of $B_{\overline{k}}$ and $B'_{\overline{k}}$ respectively. Thus, the "open cell" structure for $(G_{\overline{k}}, T_{\overline{k}}, B_{\overline{k}})$ implies that the multiplication map

$$U' \times T \times U \to G$$

is an open immersion, so it suffices to show that each of T, U, and U' are unirational over k. The case of T is handled by Lemma 2.1, so it suffices to show that any unipotent smooth connected affine k-group U is unirational over k. This unirationality has been explained at the start of this section for *split* unipotent smooth connected groups over any field, and the split condition is automatic when the ground field is perfect (such as a finite field). ■

Now we may and do assume that K is infinite. We give two proofs, depending on the characteristic of K. The arguments are similar, but technically not quite the same.

Case 1: characteristic 0. First we assume K has characteristic 0, and shall proceed by induction on $\dim(G)$ without a reductivity hypothesis. As we have already noted, we may assume both that G is reductive (so the characteristic 0 hypothesis has not been of much use yet) and that all lower-dimensional smooth connected K-subgroups are generated by K-tori. Consider the central isogeny $\pi : G \to G^{\mathrm{ad}} := G/Z_G$ with G^{ad} having *trivial* center. As for any central quotient map between connected reductive groups, the formation of images and preimages defines a bijective correspondence between the sets of maximal K-tori of G (all of which contain Z_G) and of G^{ad}. Hence, if we can find maximal K-tori $T'_1, \dots, T'_r$ of G^{ad} that generate G^{ad} then G is generated by their maximal K-torus preimages $T_i = \pi^{-1}(T'_i)$. Consequently, it is harmless to assume that G is semisimple and nontrivial with trivial center due to the very useful:

Lemma 2.3. *Let $f' : G'' \to G'$ and $f : G' \to G$ be central quotient maps between connected reductive groups over a field k. The composite quotient map $f \circ f' : G'' \to G$ also has central kernel. In particular, G/Z_G has trivial center.*

The conclusion is false without centrality; e.g., if $U \subset \mathrm{SL}_3$ is the unipotent radical of a Borel subgroup then U contains a central subgroup $Z := \mathbf{G}_{\mathrm{a}}$ given by the upper-right matrix entry, and $U/Z = \mathbf{G}_{\mathrm{a}}^2$, so we have central quotient maps $U \to U/Z$ and $U/Z \to \mathbf{G}_{\mathrm{a}}$ whose composition has non-central kernel.

Proof. We may and do assume $k = \overline{k}$. Since $G''/Z_{G''}$ is a central quotient of G', it suffices to show that the central quotient G of G' dominates it. Provided that $G''/Z_{G''}$ has trivial center, it follows that the central kernel of $f : G' \to G$ dies in the quotient $G''/Z_{G''}$ of G', so the desired domination would follow.

We may rename G'' as G and reduce to checking that G/Z_G has trivial center. Let $T \subset G$ be a maximal torus, so T/Z_G is a maximal torus in G/Z_G. The center of G/Z_G is contained in T/Z_G and hence is killed by all roots in $\Phi(G/Z_G, T/Z_G)$. Thus, it suffices to show that the kernels of all such roots have trivial intersection. We have shown in class (see Proposition

3.4.1 and Remark 3.4.3) that the central quotient map $G \to G/Z_G$ identifies the root systems for T and T/Z_G respectively, so it suffices to show that the intersection of the kernels of the roots in $\Phi := \Phi(G,T)$ coincides with Z_G.

Let $M := \bigcap_{a\in\Phi}(\ker a)$, so clearly $Z_G \subset M$. To prove the reverse inclusion, note that since $M \subset T$ we know that $Z_G(M)$ is *smooth* with Lie algebra $\mathfrak{g}^M$. But the definition of M shows immediately that $\mathfrak{g}^M = \mathfrak{g}$, so $Z_G(M)$ has full Lie algebra yet is also smooth, so $Z_G(M) = G$. Thus, $M \subset Z_G$. ∎

It follows that $G_{\overline{K}}$ contains (i) a non-central GL_1 and (ii) *no* nontrivial central subgroup scheme over $\overline{K}$. It was precisely under such geometric hypotheses on a general smooth connected affine group over an infinite field K that we showed (via Zariski-density considerations in the Lie algebra over K, and a lot of extra work in positive characteristic) in the proof of Grothendieck's theorem on the existence of geometrically maximal K-tori that $\mathfrak{g}$ contains a semisimple element X that is non-central (i.e., $\mathrm{ad}_{\mathfrak{g}}(X) \neq 0$). The non-central non-nilpotent locus in $\mathfrak{g}$ is Zariski-open and non-empty, hence Zariski-dense (as K is infinite), so for any proper K-subspace $V \subset \mathfrak{g}$ there exists a non-central non-nilpotent $X \in \mathfrak{g} - V$. So far this is characteristic-free (and the non-nilpotence of X will not be used in characteristic 0).

Since $\mathrm{char}(K) = 0$, the K-subgroup $Z_G(X)^0$ is smooth and connected with Lie algebra equal to the *proper* subspace $\mathfrak{z}_{\mathfrak{g}}(X) \subset \mathfrak{g}$ which contains X and so is not contained in V. In other words, we have found a lower-dimensional smooth connected K-subgroup $H \subset G$ whose Lie algebra does not contain a specified proper K-subspace of $\mathfrak{g}$. Applying this procedure several times, we arrive at a finite set of smooth connected proper K-subgroups $H_1, \dots, H_n \subset G$ whose K-group structure is unclear but whose Lie algebras span G. Hence, the multiplication map

$$H_1 \times \cdots \times H_n \to G$$

is surjective on tangent spaces at the identity points, so it is smooth there. Thus, this map is dominant, so the unirationality of the lower-dimensional H_i's over K implies the unirationality of G over K. This settles the argument in characteristic 0.

Case 2: positive characteristic. Now assume $\mathrm{char}(K) = p > 0$, with G reductive if K is not perfect. We will again use the Lie algebra to dig out suitable lower-dimensional smooth connected K-subgroups of G for applying dimension induction. Exactly as above, we may always arrange (without an initial reductivity hypothesis for perfect K) that G is semisimple with trivial center, and that the problem is solved in all lower-dimensional case (with a reductivity hypothesis if K is not perfect).

The infinitude of K once again ensures that for any proper K-subspace V of $\mathfrak{g}$ there exists a non-central non-nilpotent $X \in \mathfrak{g} - V$. The Jordan decomposition $X_{\mathrm{s}} + X_{\mathrm{n}}$ of X in $\mathfrak{g}_{\overline{K}}$ may not be K-rational, but for a sufficiently large p-power q the Jordan decomposition $X_{\mathrm{s}}^{[q]} + X_{\mathrm{n}}^{[q]}$ of $X^{[q]}$ is K-rational. Note that the semisimple X_{s} is nonzero (since X is *non-nilpotent*), so $X_{\mathrm{s}}^{[q]} \neq 0$ by semisimplicity. Taking q large enough ensures that $X_{\mathrm{n}}^{[q]} = 0$, so $X^{[q]}$ is nonzero and semisimple. Consideration of the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}_N(K)$ arising from a K-subgroup inclusion $G \hookrightarrow \mathrm{GL}_N$ shows that $[X, X_{\mathrm{s}}^{[q]}] = 0$, so X lies in the Lie-theoretic centralizer $\mathfrak{z}_{\mathfrak{g}}(X^{[q]})$.

As in the proof of Grothendieck's theorem on geometrically maximal tori, the $\overline{K}$-span of the pairwise commuting semisimple elements $X^{[qp^i]}$ ($i \geq 0$) has a $\overline{K}$-basis consisting of nonzero semisimple elements X_i satisfying $X_i^{[p]} = X_i$. Such X_i are necessarily K-rational (since $X_i = X_i^{[p^a]}$ for all $a \geq 0$). Each $\mathfrak{z}_{\mathfrak{g}}(X_i)$ contains X and so is not contained in the initial choice of proper K-subspace V of $\mathfrak{g}$. Applying this construction several times, we can find a finite set of nonzero (semisimple) elements $Y_1, \dots, Y_m \in \mathfrak{g}$ such that $Y_i^{[p]} = Y_i$ for all i and the Lie-centralizers $\mathfrak{z}_{\mathfrak{g}}(Y_i)$ span $\mathfrak{g}$.

We shall prove that $\mathfrak{z}_{\mathfrak{g}}(Y_i) = \mathrm{Lie}(H_i)$ for smooth connected *reductive* K-subgroups H_i of G with *non-trivial* scheme-theoretic center Z_{H_i}. Since $Z_G = 1$ by design (as G has been arranged to be semisimple of adjoint type), it would follow that $H_i \neq G$, so $\dim H_i < \dim G$ for all i and hence we could conclude by dimension induction. To find such H_i's, rather generally if $X \in \mathfrak{g}$ is a nonzero (semisimple) element satisfying $X^{[p]} = X$ then we claim that $\mathfrak{z}_{\mathfrak{g}}(X) = \mathrm{Lie}(H)$ for a connected reductive K-subgroup H with $Z_H \neq 1$.

In the proof of Grothendieck's theorem on geometrically maximal tori we saw that any such X spans the tangent line to a K-subgroup μ that is the image of a K-subgroup inclusion $\mu_p \hookrightarrow G$. The schematic centralizer $Z_G(\mu)$ is *smooth*, and hence of smaller dimension than G since the schematic center Z_G is trivial whereas $Z_G(\mu)$ has schematic center that contains $\mu \neq 1$. Moreover, clearly $X \in \mathrm{Lie}(Z_G(\mu)) = \mathrm{Lie}(Z_G(\mu)^0)$. We may therefore define $H = Z_G(\mu)^0$ provided that the evident inclusion

$$\mathrm{Lie}(Z_G(\mu)) \subset \mathfrak{z}_{\mathfrak{g}}(X)$$

is an equality and $Z_G(\mu)^0$ is smooth.

Let $\mathfrak{m} = \mathrm{Lie}(\mu) \subset \mathfrak{g}$ and define the K-subgroup $\mathrm{Fix}(\mathfrak{m}) \subset \mathrm{GL}(\mathfrak{g})$ to be the subgroup of linear automorphisms of $\mathfrak{g}$ that restrict to the identity on $\mathfrak{m}$. Thus, we have a Cartesian square of K-groups

$$\begin{array}{ccc} Z_G(\mu) & \longrightarrow & \mathrm{Fix}(\mathfrak{m}) \\ \downarrow & & \downarrow \\ G & \xrightarrow[\mathrm{Ad}_G]{} & \mathrm{GL}(\mathfrak{g}) \end{array}$$

because we can verify the Cartesian property on R-valued points for any k-algebra R: this follows from the fact that the natural map

$$\mathrm{Hom}_{R\text{-gp}}(\mu_R, G_R) \to \mathrm{Hom}_{p\text{-Lie}}(\mathrm{Lie}(\mu)_R, \mathfrak{g}_R)$$

is bijective. (Here we use in an essential way that $\mu \simeq \mu_p$.) Passing to the Cartesian square of Lie algebras, we conclude that $\mathrm{Lie}(Z_G(\mu)) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{m})$ since $\mathrm{d}(\mathrm{Ad}_G)(e) = \mathrm{ad}_{\mathfrak{g}}$.

The above Cartesian square of K-groups implies that the schematic centralizer $Z_G(X)$ of $X \in \mathfrak{g}$ for the adjoint action of G on $\mathfrak{g}$ coincides with $Z_G(\mu)$, and in 13.19 of Borel's "Linear algebraic groups" he shows rather generally that $Z_G(X)^0$ is reductive for any nonzero semisimple $X \in \mathfrak{g}$ (over a field of any characteristic). Since our approach to the theory of linear algebraic groups avoids the $Z_G(X)$-construction, we now give a direct proof of what we need (and a bit more):

Proposition 2.4. *Let G be a smooth connected affine group over a field K of characteristic $p > 0$, and let $\mu \subset G$ be a connected K-subgroup scheme of multiplicative type (e.g., μ_p).*

(1) *There exists a maximal K-torus of G containing μ; equivalently, the maximal tori of the smooth connected subgroup $Z_G(\mu)^0$ are maximal in G.*
(2) *If G is reductive then the identity component $Z_G(\mu)^0$ is reductive.*

Proof. Without loss of generality K is algebraically closed. To prove (1) it suffices to find *some* torus S of G containing μ, as then any maximal torus of G containing S also centralizes μ (since tori are commutative). We shall prove (1) using induction on $\dim(G)$. The centralizer $H = Z_G(\mu)$ is smooth and $\mu \subset H^0$ is a nontrivial subgroup of multiplicative type, so H^0 is not unipotent. Hence, a maximal torus T of H^0 is not trivial, so $Z_{H^0}(T)$ is a smooth connected group.

If $Z_{H^0}(T) \neq H^0$ then by dimension induction some torus of $Z_{H^0}(T)$ contains μ, so (1) would be proved. Thus, we just need to consider the possibility that $Z_{H^0}(T) = H^0$, which is to say that H^0 has a central maximal torus, so (by conjugacy) H^0 has T as its unique maximal torus. In this situation we claim that $\mu \subset T$. The quotient H^0/T is a smooth connected affine group containing no nontrivial tori, so it must be unipotent. Hence, the composite map $\mu \to H^0 \to H^0/T$ has to be trivial since μ is of multiplicative type, so

$$\mu \subset \ker(H^0 \to H^0/T) = T.$$

Turning to the proof of (2), by (1) we may choose a maximal torus T of G containing μ. Assuming that the unipotent radical U of $Z_G(\mu)^0$ is nontrivial, we seek a contradiction. Since U is stable under T-conjugation, $\mathrm{Lie}(U)$ is a T-stable subspace of $\mathfrak{g}$. This subspace supports only nontrivial T-weights since $\mathfrak{g}^T = T$ by the reductivity of G. Hence, the nonzero $\mathrm{Lie}(U)$ is a direct sum of some of the root spaces $\mathfrak{g}_a$ for $a \in \Phi(G,T)$. Choose such an a and let $T_a = (\ker a)^0_{\mathrm{red}}$ be the codimension-1 subtorus killed by a.

The centralizer $H := Z_G(T_a)$ is a connected reductive subgroup of G with semisimple rank 1, and it meets U nontrivially since $\mathrm{Lie}(U) \cap \mathrm{Lie}(H) = \mathrm{Lie}(U)^{T_a} \supset \mathfrak{g}_a$. But $U \cap H$ is the centralizer for the T_a-action on U, so it inherits smoothness and connectedness from U. We conclude that $Z_H(\mu) = H \cap Z_G(\mu)$ contains the nontrivial $U \cap H$ in its unipotent radical. Thus, to reach a contradiction we may replace G with H to reduce to the case of semisimple rank 1.

We may assume that the subgroup μ is non-central in G (or else there is nothing to do), so μ has nontrivial image μ' in $G/Z_G = \mathrm{PGL}_2$. The nontrivial unipotent radical of $Z_G(\mu)^0$ has nontrivial image U' in PGL_2. But $Z_G(\mu)^0$ contains a maximal torus of G, so its image T' in PGL_2 is a maximal torus that must normalize U'. By dimension considerations, $B' := T' \ltimes U'$ is a Borel subgroup of PGL_2, and this forces $Z_G(\mu)^0$ to map onto B' (as it cannot map onto the reductive PGL_2, due to the normality of U' in its image). Thus, B' is the centralizer of μ' in PGL_2.

Applying a conjugation to PGL_2 brings $T' = \mathrm{GL}_1$ to the diagonal torus D and brings U' to the upper unipotent subgroup U^+. By inspection, the action of $D = \mathrm{GL}_1$ on $U^+ = \mathbf{G}_\mathrm{a}$ inside PGL_2 is given by ordinary scaling (possibly composed with inversion, depending on which identification $D = \mathrm{GL}_1$ is chosen), so a nontrivial subgroup scheme of D cannot centralize U^+. This is a contradiction, since $\mu' \neq 1$. ■