Math 249B. Effect of replacing Y with an alteration

We have put ourselves in the following geometric setup over an algebraically closed field k. We have a map $f : X \to Y$ between projective varieties with $\dim X = d \geq 2$, $Z \subset X$ is a non-empty closed subset that is the support of a Cartier divisor (so Z had pure dimension $d-1$), and:

- (v) X is normal,
- (vi),(a-c) f has all fibers geometrically connected of pure dimension 1 (so $\dim Y = d - 1 > 0$) with the Zariski-open subset $\mathrm{sm}(X/Y) \subset X$ fiberwise dense over Y (i.e., meets each X_y in a dense open subset), and there is a dense open $U \subset Y$ such that $X_U \to U$ is smooth,
- (vi)(d) $f : Z \to Y$ is finite and generically étale,
- (vi)(e) for all geometric points $\overline{y}$ of Y, the relative smooth locus $\mathrm{sm}(X/Y)$ meets $Z_{\overline{y}}$ in at least 3 points on every irreducible component of $X_{\overline{y}}$.

Keep in mind that we do *not* make any assumptions about normality or especially smoothness for Y (even though in the original context it was a projective space), because we are going to need to replace Y with various alterations later on.

We wish to address the effect of "replacing" Y with another projective variety Y' that is a (generically étale) alteration of Y. We shall need to make such changes in Y at many stages in subsequent arguments and so must be clear at the outset about how this affects X and Z and the above running hypotheses. Though (v) was important to arrive at the present setup (especially with U as above), that has served its purpose and it won't be a problem to give up (v) later on. Thus, our aim is really to explain how all of the above properties apart from (v) are preserved under a suitable replacement process when given a generically étale alteration $\psi : Y' \to Y$ and trying to replace Y with Y'.

Let $X' = (X \times_Y Y')_{\mathrm{red}}$, and $Z' = (Z \times_Y Y')_{\mathrm{red}}$. We claim that $f' : X' \to Y'$ and $Z' \subset X'$ satisfy all of the above conditions except possibly that (v). Since $X_U \to U$ is smooth with geometrically connected fibers of dimension 1, for the dense open $U' = \psi^{-1}(U) \subset Y'$ clearly $X'_{U'} = X \times_Y U' = X_U \times_U U'$ is U'-smooth with geometrically connected fibers of dimension 1, so for the projective variety Y' of dimension $d-1$ we see that $X' \to Y'$ is smooth over the dense open $U' \subset Y'$ with geometric fibers over U' that are smooth and connected of dimension 1. In particular, once we show that the reduced X' is irreducible (hence integral) it follows that X' has dimension d and dominant onto X, so the preimage Z' of Z in X' is the support of a Cartier divisor (hence is of pure dimension $d-1$). So we now prove:

Proposition 0.1. *The reduced projective k-scheme X' is irreducible.*

Proof. The Zariski-dense open $\Omega = \mathrm{sm}(X/Y) \subset X$ is fiberwise-dense over Y, so likewise for $\Omega' := \Omega \times_Y Y'$ inside $X \times_Y Y'$ over Y'. In particular, Ω' is reduced, so this is also an open subscheme of $X' := (X \times_Y Y')_{\mathrm{red}}$ that is fiberwise-dense over Y'. It follows that any irreducible component of X' must meet Ω', so it suffices to show that Ω' is irreducible.

Since $\Omega \to Y$ is flat, so is $\Omega' \to Y'$. Thus, all generic points of Ω' lie over the generic point η' of Y'. But the alteration $Y' \to Y$ has fiber η' over the generic point η of Y, so $\Omega'_{\eta'} = \Omega_\eta \times_\eta \eta'$. It therefore suffices to show that Ω_η is geometrically irreducible over η. But Ω_η is dense open in the η-scheme X_η that is a smooth and geometrically connected curve, so the desired geometric irreducibility follows (since smooth connected schemes over fields are irreducible). ■

Since the open subscheme $\mathrm{sm}(X/Y) \times_Y Y' \subset X \times_Y Y'$ is Y'-smooth (hence reduced) and fiberwise-dense over Y', it is also an open subscheme of $X' = (X \times_Y Y')_{\mathrm{red}}$ that is Y'-smooth and fiberwise-dense over Y'. Thus, it is contained in the Zariski-open subset $\mathrm{sm}(X'/Y') \subset X'$, so $\mathrm{sm}(X'/Y')$

is also fiberwise-dense inside X' over Y'. It is now clear that (vi)(a-c) holds for $X' \to Y'$ (using $U' = \psi^{-1}(U)$ as introduced above).

To show that Z' with pure dimension $d-1$ is finite and generically étale over Y', we note that finiteness is equivalent to quasi-finiteness (as Z' and Y' are k-proper), and that in turn is a topological property easily seen since $Z' = (Z \times_Y Y')_{\mathrm{red}}$. The generic étaleness for Z' over Y' is therefore equivalent to η'-étaleness of the generic fiber $Z'_{\eta'}$. This generic fiber is a localization of Z', and localization commutes with the formation of nilradicals, so $Z'_{\eta'}$ is the underlying reduced scheme of $(Z \times_Y Y')_{\eta'} = Z_\eta \times_\eta \eta'$ (recalll that $Y'_\eta = \eta'$ as *schemes*, since $Y' \to Y$ is a generically finite dominant map between varieties). But Z_η is η-étale by (vi)(d), so $Z_\eta \times_\eta \eta'$ is η'-étale and therefore reduced. In other words, $Z'_{\eta'} = Z_\eta \times_\eta \eta'$, so η'-étaleness of $Z'_{\eta'}$ follows from η-étaleness of Z_η. In other words, (vi)(d) holds for $Z' \to Y'$.

Finally, (vi)(e) holds for $(X' \to Y', Z')$ because we have seen that $\mathrm{sm}(X/Y) \times_Y Y' \subset \mathrm{sm}(X'/Y')$ and for any geometric point $\overline{y}'$ of Y' over a geometric point $\overline{y}$ of Y the closed subset $Z'_{\overline{y}'} \subset X'_{\overline{y}'}$ has the same underlying space as $Z_{\overline{y}} \times_{\overline{y}} \overline{y}' \subset X_{\overline{y}} \times_{\overline{y}} \overline{y}'$.