MATH 249B NOTES: ALTERATIONS

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1. Resolution of singularities

1.1. **First definition.**

We first introduce the notion of resolution of singularities, which we will later refine and modify.

Definition 1.1.1. Let X be a reduced (locally) noetherian scheme. A *resolution of singularities* of X is a map $f\colon X' \to X$ such that

(1) X' is regular,
(2) f is proper,
(3) f is birational: there exists a dense open $U' \subset X'$ and $U \subset X$ such that f restricts to an isomorphism
$$F|_{U'}\colon U' \simeq U.$$

Exercise 1.1. Show that f is birational if and only if there exists a dense open $U \subset X$ such that $f^{-1}(U) \subset X'$ is dense open and $f^{-1}(U) \simeq U$. [Hint: shrink U in the definition by removing $f(X' - U')$.]

There are stronger notions of resolution of singularities. To motivate them, let us first mention that for "nice" X (where nice means "excellent"), the regular set $\mathrm{Reg}(X) \subset X$, defined as $\{x \in X\colon \mathscr{O}_x \text{ regular}\}$, is Zariski open in X. If X is finite type over an algebraically closed field, this can be seen concretely in terms of Jacobians.

For resolutions of singularities, one could further ask for the open subscheme U in (1.1) to be the *full* regular locus, *and* $(X' - f^{-1}(U))_{\mathrm{red}} \subset X'$ to have nice geometric structure, namely to be a *strict normal crossings divisor* (to be defined soon).

Remark 1.2. At the end of the day, we'll be most interested in varieties over an algebraically closed fields. However, in the middle of arguments one sometimes passes out of this realm (such as to work over completions of local rings), so it's useful to have a more general framework. Also, de Jong's method in [deJ] adapts to apply to a *relative* situation over *discrete valuation rings* (so, for schemes which are not over a field); this is important for arithmetic applications and is addressed in the final part of his original paper (and for reasons of time we will say nothing about that in this course, but we will treat some intermediate steps in wide enough generality to be applicable in the adaptation to working over discrete valuation rings).

1.2. **Strict normal crossings divisors.**

Definition 1.2.1. Let S be a regular scheme. A *strict normal crossings divisor* (sncd) in S is a closed subscheme D defined by an invertible ideal $\mathscr{I}_D \subset \mathscr{O}_S$ (=: effective Cartier divisor) such that

(1) D is reduced,
(2) the (reduced) irreducible components $\{D_i\}_{i\in I}$ of D are regular,

(3) (most important) for any finite subset $J \subset I$,

$$D_j = \bigcap_{j \in J} D_j$$

is regular of codimension $\#J$ at all points; i.e. for all $\xi \in D_J$

$$\dim \mathscr{O}_{D_J,\xi} = \dim \mathscr{O}_{S,\xi} - \#J.$$

Exercise 1.3. Show that the definition is equivalent to saying that for all $\xi \in D$ and the set $\{D_i\}_{i \in I_\xi}$ of D_i's through ξ, we have

$$\mathscr{I}_{D_i,\xi} = t_i \mathscr{O}_{S,\xi}$$

for such i with $\{t_i\}_{i \in I_\xi}$ part of a regular system of parameters of $\mathscr{O}_{S,\xi}$.

One unfortunate feature of the definition sncd is that it is not "very local"; i.e., not local for the étale or "analytic" topologies. This is due to the basic but fundamental fact that irreducibility is not étale-local (or more concretely, a completion of a noetherian local domain need not be a domain):

Example 1.4. Let $D = \{y^2 = x^2(x+1)\} \subset \mathbf{A}^2_k$, for char $k \neq 2$. This integral divisor fails to be a sncd because it is non-regular at the origin. However, "very locally" at the origin (namely, upon passing to the finite étale cover near $(0,0)$ defined by $u^2 = 1 + x$) it *looks* like the intersection of two transversely intersecting regular divisors $y = \pm ux$, which is what an sncd looks like at its non-regular points. Another way to make this precise is to pass to completions: Spec $\widehat{\mathscr{O}}_{\mathbf{A}^2,0} \supset$ Spec $\widehat{\mathscr{O}}_{D,0}$ is a sncd (since the completion contains $\sqrt{1+x}$, by Hensel's Lemma).

Definition 1.2.2. A reduced Cartier divisor D in a regular scheme S is a *normal crossings divisor* (ncd) if for all $\xi \in D$, there exists an étale neighborhood $S' \to S$ of ξ such that such that $(S', D' := S' \times_S D)$ is an sncd.

Remark 1.5. For S excellent, $D \subset S$ is an ncd if and only if for all $\xi \in D$ the closed subscheme Spec $\widehat{\mathscr{O}}_{D,\xi} \subset$ Spec $\widehat{\mathscr{O}}_{S,\xi}$ is an ncd. This is not obvious; it uses Artin-Popescu approximation (which we will discuss later); the hard part is to show that being an sncd after completion at a point ensures the existence of an étale neighborhood (much less drastic than completion!) over which D becomes an sncd. This is useful in practice because completions are often a more convenient framework to perform concrete calculations (e.g., the nature of étale maps simplifies a lot after passing to completions).

To get back to a sncd from a ncd at the end of the day, we will need to know the relationship between the two notions.

Exercise 1.6. Show that a normal crossings divisor is a strict normal crossing divisor precisely when its irreducible components D_i (given the reduced structure) are regular.

1.3. **Hironaka's Theorem.** Hironaka proved that in characteristic 0, varieties have the best imaginable resolutions of singularities. Here is a sample version of his result (other variants are also available, in analytic settings and for "embedded resolution", etc.):

Theorem 1.3.1 (Hironaka)**.** Let X be reduced, separated of finite type over a field k of characteristic 0. There exists a resolution of singularities $X' \to X$ such that

(1) f is an isomorphism over $U := \mathrm{Reg}(X) = X^{\mathrm{sm}} \subset X$.
(2) $(X' - f^{-1}(U))_{\mathrm{red}} \subset X'$ is ncd.

Remark 1.7. What Hironaka proved was much stronger. For example, he showed that f can be built as a composition of blow-ups along smooth closed subschemes (we will discuss blow-ups in detail later).

It is a wide open question whether this holds in characteristic $p > 0$. de Jong's alterations provide a substitute notion that works in most applications, in any characteristic (and even in some relative situations).

1.4. **Alterations.** For most applications, one can weaken the birationality aspect of resolution of singularities as follows.

Definition 1.4.1. Suppose X is an integral noetherian scheme. An *alteration* of X is a map $f\colon X' \to X$ such that

(1) X' is integral and regular,
(2) f is proper and dominant,
(3) $[k(X')\colon k(X)] < \infty$.

Exercise 1.8. Show that the third condition is equivalent to the existence of a dense open $U \subset X$ such that $f^{-1}(U) \to U$ is finite (and flat, after shrinking again); note that $f^{-1}(U)$ is automatically dense in X' since it is a non-empty open subset (by dominance of f) and X' is assumed to be irreducible (even integral).

deJong's main theorem is a weakened form of Hironaka's Theorem, with alterations in place of resolution of singularities, in which one loses all control over the open subscheme U over which the morphism is actually finite (in particular, we cannot guarantee that it contains any particular closed point in the regular locus):

Theorem 1.4.2. Let X be a variety over a field k; i.e., an integral separated scheme of finite type over k. For $Z \subset X$ any proper closed subset, there exists an alteration $f\colon X' \to X$ and an open immersion

$$J\colon X' \hookrightarrow \overline{X}'$$

such that

(1) $\overline{X}'$ is regular and projective,
(2) $\underbrace{f^{-1}(Z) \cup (\overline{X}' - j(X')))_{\mathrm{red}}}_{\text{complement of } f^{-1}(X-Z)} \subset \overline{X}'$ is sncd.

Remark 1.9. In the statement of the theorem, why not simply take Z to be empty? The answer is that the theorem will be proved by an inductive means, and we really *need* the flexibility of a closed subscheme Z for the inductive step (even if in practice we may often only care about the case of empty Z). Informally, the role of Z is inspired by Hironaka's result on embedded resolution of singularities (which we did not define).

Remark 1.10. If k is perfect, the proof will give that we can arrange for the extension $k(X')/k(X)$ to be separable (so $f : X' \to X$ is finite étale over some dense open in X). The proof doesn't control the degree of this field extension, but Gabber later made an improvement arranging the degree *not* to be divisible by any desired finite set of primes away from $\mathrm{char}(k)$. In later work, deJong proved an equivariant version of his result for the action by finite groups. But we stress again that the method *cannot* control U: it doesn't even guarantee finiteness of f over any particular closed point of X.

1.5. **Applications.** We now give some applications of alterations (where they can be used as a substitute for resolution of singularities), and also an example where de Jong's Theorem appears to be insufficient to replace Hironaka's Theorem.

1.5.1. *Grauert-Remmert Theorem.* Let X be a normal $\mathbf{C}$-scheme, locally of finite type. Given a surjective finite étale map $E \to X$, the analytification is a finite-degree covering space $E^{\mathrm{an}} \to X^{\mathrm{an}}$. This defines a functor

$$\mathrm{F\acute{e}t}(X) \to \mathrm{F\acute{e}t}(X^{\mathrm{an}})$$

between finite étale X-schemes and analytic spaces finite étale over X^{an} (which is categorically "the same" as finite-degree proper local homeomorphisms to $X(\mathbf{C})$); these finite étale maps may not be surjective.

The theorem of Grauert and Remmert is that this functor is an equivalence of categories. This is amazing because it holds even when X is not proper! The idea of the proof is to use Hironaka to reduce to the case where X is the complement of a normal crossings divisor in a smooth projective variety. In the projective case, we can use GAGA. The details are carried over [SGA1, XII].

Exercise 1.11. Adapt Grothendieck's argument in [SGA1] to use just alterations, rather than resolution of singularities.

1.5.2. *Artin comparison theorem.* Let X be a separated, finite type scheme over $\mathbf{C}$, and $\mathscr{F}$ a constructible abelian sheaf on $X_{\text{ét}}$, or a constructible ℓ-adic sheaf.

Theorem 1.5.1. The natural pullback map

$$\mathrm{H}^*(X_{\text{ét}}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^*(X^{\text{an}}, \mathscr{F}^{\text{an}}) \tag{1.1}$$

is an isomorphism.

Remark 1.12. This is also true, and much easier to prove, for cohomology with compact supports. Why is that case easier to prove? The reason is that cohomology with compact supports is much better behaved with respect to stratification by locally closed sets due to the excision sequence involving H_c^*'s throughout; in contrast, the excision sequence for ordinary cohomology involves an anomalous term $\mathrm{H}_Z^*(X, \cdot)$ for cohomology with supports along a closed set, and that messes up the induction. The upshot of the more convenient excision sequence for H_c^* is that it permits a reasonably rapid reduction to the case of curves.

The proof of (1.1) uses resolution of singularities in dimension $\leq \dim X$. For an analogue with non-Archimedean étale cohomology allowing positive characteristic, an entirely different proof was developed by Berkovich that adapts almost verbatim to the complex-analytic setting above for *constructible* abelian $\mathscr{F}$. So for this much, resolution of singularities is not needed. However, it *is needed* to bootstrap to the ℓ-adic case (that, oddly enough, does not seem to be documented in the literature); this is handled by a beautiful argument of Deligne that can also be carried out using alterations.

1.5.3. *Deligne's theory of mixed Hodge structures.* There is a notion of "proper hypercovering", which is an abstraction of Cech coverings and allows other types of maps than "open embeddings" to be used in the study of sufficiently topological cohomological investigations. In particular, due to the proper base change theorem in étale cohomology, it is fruitful to work with hypercoverings whose constituents are *proper smooth varieties* with suitable "sncd boundary", and such hypercovers can be built by systematic use of de Jong's theorem instead of Hironaka's Theorem.

1.5.4. *A non-application.* Here is an example in which de Jong's alterations do *not* appear to suffice in place of Hironaka's Theorem. Let X be a smooth scheme over $\mathbf{C}$, separated of finite type. We can form the *algebraic de Rham complex*

$$\Omega^\bullet_{X_{\mathbf{C}}} = (\mathscr{O}_X \xrightarrow{d} \Omega^1_{X/\mathbf{C}} \xrightarrow{d} \Omega^2_{X/\mathbf{C}} \xrightarrow{d} \dots)$$

Grothendieck defined the algebraic de Rham cohomology to be the hypercohomology of this complex:

$$\mathrm{H}^i_{\mathrm{dR}}(X/\mathbf{C}) = \mathbf{H}^i(\Omega^\bullet_{X/\mathbf{C}}).$$

There's a natural "analytification" $\Omega^\bullet_{X^{\mathrm{an}}/\mathbf{C}}$ (being careful about the fact that the d-maps are not $\mathscr{O}_X$-linear!), so one has a map

$$\theta^i_X : \mathrm{H}^i_{\mathrm{dR}}(X/\mathbf{C}) = \mathbf{H}^i(\Omega^\bullet_{X/\mathbf{C}}) \to \mathbf{H}^i(\Omega^\bullet_{X^{\mathrm{an}}/\mathbf{C}}).$$

By the $\overline{\partial}$-Poincaré Lemma, the latter is isomorphic to $\mathbf{H}^i(X(\mathbf{C}), \underline{\mathbf{C}})$. In the proper case, GAGA combined with a spectral sequence for hypercohomology implies that θ^i_X is an isomorphism.

Grothendieck proved in IHES 29 that θ^i_X is an isomorphism even without properness. A striking consequence is that if X is affine, then $\mathrm{H}^i(X(\mathbf{C}), \mathbf{C}) \simeq \mathbf{H}^i(\Omega^\bullet_{X/\mathbf{C}}) = \mathrm{H}^i(\Omega^\bullet_{A/\mathbf{C}})$ (the first isomorphism by Grothendieck's result, and the second by acyclicity of coherent sheaves on affines). That is, topological $\mathbf{C}$-valued cohomology of $X(\mathbf{C})$ for smooth affine X can be computed entirely in terms of algebraic differential forms!

How did Grothendieck achieve this? The idea is to bootstrap from the proper case by first forming a compactification $X \hookrightarrow \overline{X}$ and then using a strong enough form of Hironaka's theorem (applied to $\overline{X}$) to find a resolution of singularities $\overline{X}' \to \overline{X}$ such that if X' is the pre-image of X then not only is $X' \to X$ an isomorphism but also $\overline{X}' - X'$ is a normal crossings divisor. In other words, we have a Cartesian diagram

$$\begin{array}{ccc} X' & \hookrightarrow & \overline{X'} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \overline{X} \end{array}$$

whose left side is an isomorphism that thereby identifies X as the complement of a normal crossings divisor in the smooth proper $\overline{X}'$. Grothendieck related the de Rham cohomology of $X = X'$ to the hypercohomology of the deRham complex on $\overline{X'}^{\mathrm{an}}$ modified to permit controlled poles along that normal crossings divisor.

The *key point* is that one knows that X is untouched by the resolution of singularities, being in the smooth locus of $\overline{X}$. This is what breaks down for alterations. In de Jong's version, one cannot control where the map restricts to a finite morphism, let alone an étale map. This is bad because formation of differential forms interacts badly with respect to a map that is not étale, so one can't relate $\Omega^\bullet_{X/\mathbf{C}}$ to $\Omega^\bullet_{\overline{X}'/\mathbf{C}}$ in any useful way.

Remark 1.13. Bhatt proved a result that gets around this issue in the lci case, establishing a version of Grothendieck's isomorphism beyond the proper case with lci singularities by working with the so-called infinitesimal site (in lieu of the deRham complex, to which it is closely related only in the smooth case) and using more advanced homological tools than were available in the 1960's.

1.6. **Regular vs smooth.** We want to emphasize the technical distinction between regular and smooth. Let S be a locally finite type scheme over a field k.

Definition 1.6.1. The scheme S is *regular* if all the local rings $\mathscr{O}_{S,s}$ are regular. (It is sufficient to verify this at *closed* points $s \in S$, in view of Serre's theorem that the localization at a prime ideal for any regular local ring is again regular.)

The scheme S is *smooth* over k if $S_{\overline{k}}$ is regular. There are several equivalent formulations of this "geometric" condition:

- $S_{k'}$ is regular for all finite extensions k'/k,
- $S_{k'}$ is regular for all extensions k'/k,
- $S_{k'}$ is regular for one perfect extension k'/k,
- $S \to \operatorname{Spec} k$ satisfies the infinitesimal criterion for smoothness.

Smoothness over a field k always implies regularity, but the converse is false (if k is not perfect). For instance, smoothness is preserved by ground field extension, but regularity is not:

Exercise 1.14. Let k be an imperfect field of characteristic $p > 0$, and $a \in k - k^p$. Pick $m > 1$ such that $p \nmid m$ (e.g., $m = 2$ when p is odd). Check that $\{y^m = x^p - a\} \subset \mathbf{A}^2_k$ is Dedekind, but obviously $C_{\overline{k}} = \{y^m = z^p\}$ (for $z := x - a^{1/p}$) is not regular at $(0, 0)$.

Example 1.15. Let $K/k(x)$ be a finite extension. Consider the diagram

$$\begin{array}{ccc} K & \supset & A \\ {\scriptstyle \text{finite}}\Big| & & \Big|{\scriptstyle \text{integral closure}} \\ K(x) & \supset & k[x] \end{array}$$

Then Spec A is dense open in a unique projective k-curve C with function field K. It can happen that C is geometrically irreducible over k yet $C_{\overline{k}}$ is nowhere reduced, *even if* $k \subset K$ is relatively algebraically closed. (We demand this final condition on how k sits inside K to avoid the lame example in which k admits a non-trivial purely inseparable extension inside K.)

Exercise 1.16 (MacLane)**.** Take $k = \mathbf{F}_p(s,t)$, Spec $A = \{sx^p + ty^p = 1\} \subset \mathbf{A}^2$. Show that A is Dedekind, and $k \subset \text{Frac}(A)$ is relatively algebraically closed. Evidently, after adjoining the pth roots of s and t the curve Spec A becomes *everywhere* non-reduced.

In de Jong's theorem with imperfect k one *cannot* expect to arrange that the k-variety X has an alteration $X' \to X$ with X' *generically smooth* over k. To see what is an obstruction to this, suppose a smooth alteration exist, so we have between dense opens a finite flat surjection

$$\begin{array}{ccc} U' & \supset & X \\ {\scriptstyle \text{finite flat}}\downarrow & & \\ U & \supset & X' \end{array}$$

Regularity descends across faithfully flat maps [Mat2, 23.7(i)]. So if U' were k-smooth (as we could arrange if X' were generically k-smooth) then $U'_{\overline{k}}$ were regular, so $U_{\overline{k}}$ would be regular, so X would be generically smooth; equivalently, $k(X)/k$ would be *separable* in the sense of field theory (i.e., $k(X) \otimes_k \overline{k}$ is reduced for all finite extensions k'/k). But this latter property can fail even if $k \subset k(X)$ is relatively algebraically closed, as illustrated with MacLane's example above. Thus, regularity without generic smoothness is the best we can hope for when k is imperfect.

Remark 1.17. The regularity will be achieved in de Jong's proof by building an alteration X' that is smooth *over* a finite extension (typically inseparable) of k. That is, even if k is algebraically closed in $k(X)$, we do *not* arrange that k is also algebraically closed in $k(X')$.

2. Preparations for the proof of de Jong's Theorem

2.1. **Basic outline.** Our object of interest is a variety X over k. By a trivial application of Chow's Lemma, we may assume that X is quasi-projective. Also, we will see that it suffices to replace k with $\overline{k}$; eventually we will chase coefficients in $\overline{k}$ to make the desired construction over k from one over $\overline{k}$.

The method involves induction on $d = \dim X$. Using normalization, the cases $d \leq 1$ are easy. In general, with $d \geq 2$, we'll blow up X appropriately so that it is "fibered in curves", meaning that there is a surjective map

$$\begin{CD} X \\ @VV{f}V \\ \mathbf{P}_k^{d-1} \end{CD}$$

With all geometric fibers connected of dimension 1 (but possibly with nasty singularities or non-reduced), and such that the generic fiber X_η is *smooth*. Next we will use properties of the moduli stack $\overline{\mathscr{M}}_{g,n}$ and the Semistable Reduction Theorem for curves (which will be discussed shortly), to produce an alteration $Y \to \mathbf{P}_k^{d-1}$ and also an alteration $\mathscr{X} \twoheadrightarrow X$ fitting into a commutative diagram

$$\begin{CD} \mathscr{X} @>{\text{alteration}}>> X \\ @V{f'}VV @VV{f}V \\ Y @>>{\text{alteration}}> \mathbf{P}_k^{d-1} \end{CD}$$

where f' is flat with semistable geometrically connected fibers of dimension 1 (and smooth generic fiber). By induction, we can find a regular alteration of $Y' \to Y$, so $X' := \mathscr{X} \times_Y Y'$ (which remains integral) is an alteration of X that is semistable over a *regular* Y' (with smooth generic fiber). Then, in this nice situation, we resolve X' by hand (using carefully-chosen blow-ups).

Remark 2.1. Non-trivial extensions of function fields arise in the method (i.e., alterations rather than birational maps) due to the role of field extensions in the Semistable Reduction Theorem as discussed below.

2.2. **Semistable curves.** We first discuss the notion of semistable curves over a field k; the version over more general base schemes will be crucial later on, at which time we will revisit the topic from a wider point of view.

Theorem 2.2.1. Let X be a finite type scheme over k of pure dimension 1. For closed points $x \in X$, the following are equivalent:

(1) As $\overline{k}$-algebras, $\widehat{\mathscr{O}}_{X_{\overline{k}},\overline{x}}$ is isomorphic to $\overline{k}[\![t]\!]$ or $\overline{k}[\![u,v]\!]/(uv)$ for some $\overline{k}$-point $\overline{x}$ over x,

(2) x lies in the open k-smooth locus X^{sm} or there is an étale neighborhood $(X', x') \to (X, x)$ that is also étale of the crossing of the coordinate axes in the plane over k:

$$\begin{array}{ccc} & (X', x') & \\ \overset{\text{ét}}{\swarrow} & & \overset{\text{ét}}{\searrow} \\ (X, x) & & (\{uv = 0\} \subset \mathbf{A}^2_k, (0,0)) \end{array}$$

Definition 2.2.2. We say that X is *semistable* at x if it satisfies the above equivalent conditions, and is *semistable* if it is so at all closed points.

Remark 2.2. Condition (2) in the preceding theorem implies that $k(x)/k$ is separable, due to (X', x') being an étale neighborhood over k for both (X, x) and of the origin in $\{uv = 0\}$ (ensuring that as extensions of k, $k(x')$ is finite separable over both $k(x)$ and k). This property of the residue field is not automatic for isolated non-smooth points on geometrically integral curves over imperfect fields in general. For instance, consider the curve $C = \{v^2 = u^p - a\}$ with $a \in k - k^p$ for imperfect k of characteristic $p > 0$. This becomes $v^2 = z^p$ over $\overline{k}$", and at the unique non-smooth point $x \in C$ the residue field is $k(a^{1/p})$ over k.

Remark 2.3. The proof that (1) implies (2) in Theorem 2.2.1 is serious: it uses Artin approximation. A reference is [FK, Ch. III, §2], where they investigate the structure of "relative ordinary double point singularities". This elucidates the special feature of this geometric singularity in contrast with the example at the end of Remark 2.2: it is formally cut out by a non-degenerate quadratic form.

At the end of [SGA7$_{\mathrm{I}}$, Exp. I] there is given Deligne's elegant proof of:

Theorem 2.2.3 (Semistable Reduction for abelian varieties)**.** Let A be an abelian variety over a field $K = \mathrm{Frac}(R)$, where R is a discrete valuation ring. There exists a finite separable extension K'/K such that for the R-finite integral closure R' of R in K' (a semi-local Dedekind domain), the Néron model of $A_{K'}$ has special fibers over R' with identity component an extension of an abelian variety by a torus (so there is no occurrence of $\mathbf{G}_a$ as a subgroup of those geometric special fibers).

Remark 2.4. Although it is not stated explicitly, the proof shows that one can take any K' such that $A_{K'}[\ell]$ is K'-split for $\ell \in R^\times$ that is an odd prime or 4.

Applying this to the Jacobians of curves and doing more hard work (using results of Raynaud relating Néron models of Jacobians to relative Picard schemes), in [DM] Deligne and Mumford proved:

Theorem 2.2.4 (Semistable Reduction for curves)**.** For K, R as above and X a smooth proper geometrically connected curve over K, there exist K', R' as above such that $X_{K'} = \mathscr{X}'_{K'}$ for $\mathscr{X}'$ a proper flat R'-scheme with semistable geometrically connected fibers.

Where does this $\mathscr{X}'$ come from (when $\mathrm{Pic}^0_{X_{K'}/K'}$ has semi-abelian reduction)?

(1) The first step is to analyzes the "minimal regular proper model" for $X_{K'}$. But why does *any* regular proper flat model exist for this generic fiber? In the 1960's Shafarevich and Lichtenbaum independently showed (via inspiration from the theory of minimal surfaces over $\mathbf{C}$) that *if* there exists a regular proper flat model, then there is a minimal one that is moreover unique in a precise sense when X has positive genus. This is already a non-trivial achievement, but it rests on the existence of *some* regular proper flat model, and that lies deeper.

To get started with making a regular proper flat model, now working over K and R for ease of notation, pick a closed immersion $X \hookrightarrow \mathbf{P}^n_K$ over K. Viewing this inside $\mathbf{P}^n_R$, let $\mathscr{X}$ be the schematic closure; this is a proper flat model over R with $\mathscr{X}_K = X$. Pass to the normalization $\widetilde{\mathscr{X}}$; this is R-finite over $\mathscr{X}$ (due to some auxiliary considerations with excellence over $\widehat{R}$). It could have horrible singularities in its special fiber, but then we can use a form of resolution of singularities for excellent surfaces: a hard theorem of Lipman over $\widehat{R}$ (which is excellent) and a trick of Hironaka says that the process of "normalize, then blow up the finitely many non-regular points, and repeat" terminates.

For a reference on the proof of this resolution result, see Artin's article [A] on Lipman's proof and Chinburg's article [Ch] for the Lichtenbaum-Shafarevich result on minimal models. We need to infer information about the special fiber $\mathscr{X}_0$ from information about the Néron model, at least when the Néron model has semistable reduction; this is what Deligne and Mumford do in [DM], building on work of Raynaud that relates such Néron models to relative Picard schemes.

Example 2.5. Suppose $g = 1$ and $X(K) \neq \emptyset$, so X is an elliptic curve. It turns out (but is far from obvious) that the relative smooth locus $\mathscr{X}^{\mathrm{sm}}$ coincides with $\mathrm{Néron}(X)$. There are two cases with non-smooth special fiber:

- Additive reduction: $\mathscr{X}_{\overline{k}}$ has a cusp,
- Multiplicative reduction: $\mathscr{X}_{\overline{k}}$ has a node.

The second of these two cases is the "semistable reduction" case (when the special fiber isn't entirely smooth), and if R is complete one can always arrange that $[K' \colon K] \mid 24$ in the semistable reduction theorem for such X (with $X(K)$ non-empty!).

2.3. **Excellence.** We need to understand a technical condition called "excellence". Its main purpose for us will be to transfer information both ways between a noetherian local ring and its completion. The good news, as we shall see, its that basically everything we care about is excellent (but that miracle is not to be taken lightly, since its proof is not trivial).

A reference for what follows is [Mat1, Ch. 13] and [EGA, IV_2 §5-§7] (especially §7.8ff. in loc. cit.). First, we make a definition for a class of rings whose dimension theory exhibits the familiar features for schemes of finite type over fields:

Definition 2.3.1. A noetherian ring A is *catenary* if for all primes $\mathfrak{p} \subset \mathfrak{p}'$, all maximal chains of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{p}'$$

have the same length n. We say that A is *universally catenary* if all finite type A-algebra A' are catenary.

Example 2.6. By [Mat2, 17.4], quotients of Cohen-Macaulay rings are catenary. In fact, any such ring is automatically universally catenary as well. This is because if $B \twoheadrightarrow A$ is a surjection with B Cohen-Macaulay, then $A[X_1, \ldots, X_n]$ is a quotient of $B[X_1, \ldots, X_n]$, which is Cohen-Macaulay.

This implies that basically everything we care about is universally catenary, since basically every ring we care about is a quotient of a regular ring (which is Cohen-Macaulay). Note in particular that by the Cohen Structure Theorem, *every* complete local noetherian ring is a quotient of a regular ring, and hence is universally catenary. (There do exist non-catenary noetherian rings, but they are rather hard to construct. It is a non-trivial theorem that Dedekind domains are universally catenary, so every finitely generated ring over $\mathbf{Z}$ or a discrete valuation ring is catenary.)

2.3.1. *G-rings.* We now introduce the most important condition, since the motivation for excellence is to codify a notion that ensures properties transfer well between a ring and its completion.

Definition 2.3.2. A noetherian ring A is a *G-ring* if for all primes $\mathfrak{p} \subset A$, the (faithfully flat) map

$$\operatorname{Spec} \widehat{A}_{\mathfrak{p}} \to \operatorname{Spec} A_{\mathfrak{p}}$$

has "geometrically regular" fibers. This means that for all $\mathfrak{q} \in \operatorname{Spec} A_{\mathfrak{p}}$, the fiber algebra $\kappa(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \widehat{A}_{\mathfrak{p}}$ is geometrically regular over $\kappa(\mathfrak{q})$; i.e., this (visibly noetherian!) $\kappa(\mathfrak{q})$-algebra (which is almost never finite type over $\kappa(\mathfrak{q})$) is regular and remains so after any finite extension on $\kappa(\mathfrak{q})$. (If this fiber algebra were finite type over $\kappa(\mathfrak{q})$, then geometric regularity would just be one of the equivalent definitions of $\kappa(\mathfrak{q})$-smoothness.)

For many interesting properties of local noetherian rings, they can be transferred up and down through a faithfully flat local map if all fiber algebras satisfy the given property (and regularity subsumes essentially all properties of interest such as: normality, CM, reducedness, etc.).

2.3.2. *Excellent rings.*

Definition 2.3.3. A Noetherian ring A is *excellent* if it satisfies the following three conditions:

(1) A is universally catenary.
(2) A is a G-ring.
(3) For all finitely generated A-algebras A', the locus $\mathrm{Reg}(A') \subset \mathrm{Spec}\ A'$ of regular points is open.

The most important condition for our purposes is (2).

Remark 2.7. There are some more robust equivalent versions of (3) given in [EGA, IV_2 6.12.4] and [Mat1, §32B, Thm. 73].

Remark 2.8. The combination of (2) and (3) is called *quasi-excellence*. Since in practice every ring is universally catenary, there is no real difference in practice.

Example 2.9. From the definitions, if $A = R$ is a discrete valuation ring with fraction field K then A is a G-ring if and only if $K' \otimes_K \widehat{K}$ is reduced for all finite extensions K'/K (which is one of the equivalent characterizations of the typically huge non-algebraic extension $\widehat{K}/K$ being separable in the sense of field theory). This is of course automatic in characteristic 0.

Exercise 2.10. Check the the preceding criterion for discrete valuation rings does hold for the local ring $R = \mathscr{O}_{C,c}$ at a closed point c on a regular curve C over a field. This is not a trivial matter, in view of the next example.

Example 2.11. There are discrete valuation rings that are *not* G-rings (in characteristic $p > 0$, obviously). See [BLR, §3.6, Ex. 11]. for such examples.

Example 2.12. All fields and all Dedekind domains with generic characteristic 0 are G-rings.

The following result summarizes some highlights from [EGA, IV_2, §7.9].

Theorem 2.3.4. Grothendieck-Nagata]

(1) Every complete local noetherian ring and Dedekind domain with fraction field of characteristic 0 (e.g. $\mathbf{Z}$) is excellent.

(2) Excellence is inherited by any localization, and any finitely generated algebra.

(3) If A is excellent and reduced then the normalization $A \to \widetilde{A}$ is finite.

(4) Assume A is excellent. If $\mathbf{P}$ is any of a long list of "nice" homological conditions on local noetherian rings, and $\mathbf{P}(A)$ denotes the set of points $\mathfrak{p} \in \operatorname{Spec} A$ such that $A_{\mathfrak{p}}$ satisfies $\mathbf{P}$ then

- $\mathbf{P}(A)$ is open,
- for any ideal $I \subset A$, and $\widehat{A}$ the I-adic completion, the natural map $f\colon \operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$ satisfies $f^{-1}(\mathbf{P}(A)) = \mathbf{P}(\widehat{A})$.

In particular, the last assertion implies that if A is excellent local then upon taking I to be $\mathfrak{m}_A$ we have that A satisfies $\mathbf{P}$ if and only if $\widehat{A}$ does!

Definition 2.3.5. We say that a local noetherian scheme X is *excellent* if it has an affine open cover $X = \{\operatorname{Spec} A_\alpha\}$ with each A_α excellent. (This easily implies that every affine open $\operatorname{Spec} A \subset X$ has A excellent.)

The theorem implies that every algebra finitely generated over a field or Dedekind domain, or localizations thereof, is excellent. This covers most of the rings we ultimately care about (but the excellence of certain completions will be technically important later on as well).

Remark 2.13. A very interesting further class of cases are local rings on complex analytic spaces. Rather generally, for regular $\mathbf{Q}$-algebras, there is a "Jacobian criterion" for excellence [Mat1, Thm. 101] which holds for the local ring $\mathscr{O}^{\mathrm{an}}_{\mathbf{C}^n,0}$ (whose noetherianity is a serious fact from the theory of several complex variables, and then regularity is seen by computing the completion). Thus, passing to quotients of such rings yields that local rings on all complex analytic spaces are excellent! This implies that if X is a locally finite type $\mathbf{C}$-scheme then for $x \in X(\mathbf{C})$, the local ring $\mathscr{O}_{X,x}$ has $\mathbf{P}$ if and only if $\mathscr{O}_{X^{\mathrm{an}},x}$ has $\mathbf{P}$, because the two local rings are excellent and have the *same completion*. This is used all the time in the study of analytification of algebraic $\mathbf{C}$-schemes.

2.3.3. *Relationship between resolution of singularities and the G-ring property.* We won't use the following dicussion, but it is good for awareness as to why excellence (or really the G-ring property) is intimately tied up with any general result towards resolution of singularities for a wide class of schemes.

Theorem 2.3.6. Suppose X is a locally noetherian scheme. Assume the following:

for all Zariski-open $U \subset X$ and finite $Y \to U$ with Y integral,
Y admits a resolution of singularities.

Then X is quasi-excellent. In particular, all the local rings $\mathscr{O}_{X,x}$ are G-rings.

Remark 2.14. The statement of this result in [EGA, IV_2, 7.9.5] omits the open subset U, but that makes the hypotheses appear to be not local enough for the proof to work (especially since we are considering locally noetherian Y and not just noetherian Y, so extending a finite scheme over an open subset to a finite scheme over the entirety of X is not so clear).

Let's discuss the essential and most striking part of the proof: that the local rings $\mathscr{O}_{X,x}$ must be G-rings. The problem is Zariski-local due to the local nature of the hypotheses, so we may assume that $X = \mathrm{Spec}\ A$ is affine. Pick $\mathfrak{p} \in \mathrm{Spec}\ A$ and $\mathfrak{q} \subset \mathfrak{p}$ in A (so $\mathfrak{q}$ corresponds to a point in $\mathrm{Spec}\ A_{\mathfrak{p}}$, which we denote by the same name). We want to show that $\kappa(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \widehat{A}_{\mathfrak{p}}$ is geometrically regular over $\kappa(\mathfrak{q})$. Replacing A with $(A/\mathfrak{q})_{\mathfrak{p}}$, we may assume that A is a local domain (by inspecting that we may spread out finite algebras over this localization to finite algebras over some basic affine open around $\mathfrak{p}$ to access the hypothesis concerning resolutuon for integral schemes finite over an open subset of X). Let $K = \mathrm{Frac}(A)$. We want to show that for all finite extensions K'/K, the noetherian ring $K' \otimes_A \widehat{A} = K' \otimes_K (K \otimes_A \widehat{A})$ is regular.

First we reduce to just showing the statement for $K' = K$. We can find an A-finite $A' \subset K'$ such that $K' = \mathrm{Frac}(A')$, so

$$K' \otimes_A \widehat{A} = K' \otimes_{A'} (A' \otimes_A \widehat{A}).$$

Now, $A' \otimes_A \widehat{A} \simeq \prod_{\mathfrak{m}'} \widehat{A'}_{\mathfrak{m}'}$, where $A'_{\mathfrak{m}}$ has fraction field K', so we can rename $A'_{\mathfrak{m}'}$ as A and thus reduce to showing that $K \otimes_A \widehat{A}$ is regular. (The A-scheme $\mathrm{Spec}\ A'$ plays the role of Y in the hypothesis, at least after some "spreading out" from our current situation over a local ring.)

Now we handle the case $K' = K$. Let $Z \to \mathrm{Spec}\ A = X$ be a resolution of singularities. Consider the base change

$$\begin{array}{ccc} Z' & \xrightarrow{h'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\ \widehat{A} & \xrightarrow[h]{} & \mathrm{Spec}\ A \end{array}$$

We want the generic fiber of h to be regular. The map f is proper birational, hence an isomorphism over a non-empty open set in $\mathrm{Spec}\ A$, so the same holds for f' over $\mathrm{Spec}\ \widehat{A}$.This implies that the local rings of $K \otimes_A \widehat{A}$

occur already as local rings on Z', so it is enough to show that Z' is a regular scheme. It may seem surprising that Z' is regular since we have not assumed anything particularly nice about the fiber algebras for h.

Now for the brilliant step: since $\widehat{A}$ is excellent (by Theorem 2.3.4, all complete local noetherian rings are excellent!), the regular locus $\mathrm{Reg}(Z') \subset Z'$ is open. But Z' is $\widehat{A}$-proper, so to show that $\mathrm{Reg}(Z') = Z'$ it suffices to show that the regular locus contains the special fiber of f'. Denote the special fibers by Z_0 and Z'_0, so it is enough to show that the completed local rings of Z' at points of Z'_0 are regular. But these completed local rings are limits of the rings of functions on the infinitesimal special fibers. For $z' \in Z'_0$ we have $\widehat{\mathscr{O}}_{Z,h'(z')} \simeq \widehat{\mathscr{O}}_{Z',z'}$ because f, f' have the *same* infinitesimal special fibers, as $Z' = Z \otimes_A \widehat{A}$. Thus, the regularity of Z (by design) saves the day.

2.4. **Strict transforms.** For later purposes, we introduce one final general definition before we get started With the proof of deJong's theorem. Let $f\colon X \to S$ be a finite type separated morphism, with X and S noetherian and integral. Let $\psi\colon S' \to S$ be a proper birational morphism (so S' is also integral). Consider the fiber square

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ S' & \xrightarrow{\psi} & S \end{array}$$

Let $\eta \in S$ be the generic point of S, and $\eta' \in S'$ the generic point of S'.

Definition 2.4.1. The *strict transform* X' of $X \to S$ with respect to ψ is the schematic closure of $X_\eta \times_\eta \eta' = X_\eta$ in $X \times_S S'$:

$$\begin{array}{ccccc} X' & \longrightarrow & X \times_S S' & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow f \\ & & S' & \xrightarrow{\psi} & S \end{array}$$

Remark 2.15. In many situations later on there will be an evident closed S'-flat subscheme $\mathscr{X}' \subset X \times_S S'$ with full generic fiber; in such cases we claim $\mathscr{X}' = X'$. The reason is that because S' is integral, S'-flatness implies that $\mathscr{X}'$ is the schematic closure of its own generic fiber $\mathscr{X}'_{\eta'}$.

3. Preliminary reduction steps

We are now going to start the proof.

3.1. **Statement of the main theorem.**

Theorem 3.1.1 (de Jong, 4.1)**.** Let X be a variety over a field k (i.e. an integral, separated, finite type k-scheme). Let $Z \subsetneq X$ be a proper closed subset. Then there exists

$$\begin{array}{ccc} X' & \overset{j}{\hookrightarrow} & \overline{X'} \\ {\scriptstyle\varphi}\downarrow & & \\ X & & \end{array}$$

With j a dense open immersion and φ an alteration such that

- $\overline{X}'$ is a regular projective k-scheme,
- $\underbrace{(\varphi^{-1}(Z) \cup (\overline{X}' - j(X')))_{\mathrm{red}}}_{\text{complement of } \varphi^{-1}(X-Z)} \subset \overline{X}'$ is sncd in $\overline{X}'$.

If in addition k is perfect then we can arrange φ to be generically étale.

Remark 3.1. We view $(\varphi^{-1}(Z) \cup (\overline{X}' - j(X')))_{\mathrm{red}}$ as "$\partial_{\overline{X_1}}(X_1 - \varphi^{-1}(Z))$", and will use the latter notation in the future.

Remark 3.2. We do *not* control the algebraic closure of k in $k(X')$, even if X is geometrically integral over k. We'll build X' as smooth over some finite extension k'/k, over which we have little control. In fact, we'll mostly work over algebraically closed k, and then spread things out to get the desired result over k.

We'll induct on $d = \dim X \geq 0$ (with k fixed).

If $d = 0$, then there is nothing to do. [In this case $X = \operatorname{Spec} k'$ for a finite extension k'/k, and $Z = \emptyset$.]

3.2. **Reduction to algebraically closed base field.** Let's grant the full result in dimension d over $\overline{k}$, and deduce it over k. Consider $(X_{\overline{k}})_{\mathrm{red}}$. This has pure dimension d, because X does.

Pick an irreducible component $X' \subset (X_{\overline{k}})_{\mathrm{red}}$, with the reduced scheme structure. It is easy to check that $\pi \colon X' \twoheadrightarrow X$ is surjective, so $Z' = \pi^{-1}(Z) \subsetneq X'$. By the assumed Theorem 3.1.1 in dimension d over $\overline{k}$, we have a generically étale alteration

$$\begin{array}{ccc} X_1' & \overset{j_1'}{\hookrightarrow} & \overline{X}_1' \\ {\scriptstyle\varphi_1'}\downarrow & & \\ X' & & \end{array}$$

where $\overline{X}_1'$ is $\overline{k}$-smooth and projective, and $\partial_{\overline{X}_1'}(X_1' - Z_1')$ is an sncd in $\overline{X}_1'$ with $Z_1' := (\varphi_1')^{-1}(Z') \subset X_1'$.

Remark 3.3. Recall that a sncd is a union of $\overline{k}$-smooth irreducible hypersurfaces such that all finite intersections are $\overline{k}$-smooth with the "expected" pure dimension (or equivalently, "expected" pure codimension).

Now we want to bring this down to some finite extension of k. By "standard" direct limit formalism [EGA, IV_3 §8, §9, §11, ...] since $\overline{k} = \varinjlim_\alpha k_\alpha$ for $[k_\alpha : k] < \infty$, we can descend the entire situation over $\overline{k}$ down to some finite extension K/k inside $\overline{k}$: there exists a finite type K-scheme X'' such that

- $X'' \otimes_K \overline{k} = X'$ inside $(X_{\overline{k}})_{\text{red}}$ with $X'' \subset (X_K)_{\text{red}}$, and $(X_K)_{\text{red}} \otimes_K \overline{k} = (X_{\overline{k}})_{\text{red}}$.

  Remark 3.4. There is a technical but important point here. Extension of scalars does not preserve reducedness in general. However, since the nilradical for a noetherian ring is finitely generated, a nilradical formed over $\overline{k}$ has generators on an affine open all defined over some *finite* extension K/k inside $\overline{k}$. Thus, after killing the nilradical over such a K we see that further extension of scalars from K to $\overline{k}$ preserves reducedness. This is important, because after guaranteeing that condition we may need to further extend K to obtain the additonal properties below.

- There exists

$$\begin{array}{ccc} X_1'' & \overset{j_1''}{\hookrightarrow} & \overline{X}_1'' \\ {\scriptstyle \varphi_1'}\downarrow & & \\ X'' & & \end{array}$$

  with j_1'' an open embedding, φ_1'' a generically étale alteration, and X_1'' projective and *smooth* over K, and $\partial(X_1'' - Z_1'')$ is a sncd in $\overline{X}_1''$.

  Now $X'' \subset (X_K)_{\text{red}}$ is an irreducible component, and $X_K \twoheadrightarrow X$ is surjective, so $X'' \hookrightarrow (X_K)_{\text{red}} \to X$ is finite and generically flat:

$$\begin{array}{ccc} X'' & \hookrightarrow & (X_K)_{\text{red}} \\ & \searrow & \downarrow \\ & & X \end{array}$$

Therefore, the composite map

$$\begin{array}{c} X_1'' \\ \downarrow \varphi_1' \\ X'' \\ \downarrow \\ X \end{array}$$

is an alteration that one sees settles the problem for X. Note that by design, X_1'' is smooth over the finite extension K/k over which we had very little control.

If k is perfect then K/k is separable, so X_K is reduced with $X_K \to X$ finite étale, so $X'' \to X$ is generically étale because $X_K \to X$ is (since some dense open in X'' is also open in X_K). The upshot is that if k is perfect, then $X_1'' \to X$ is generically étale.

3.3. **Reduction to projective case.** By Chow's Lemma (see [EGA, II, §5] for a version that applies to separated schemes of finite type over a noetherian base, not requiring any properness hypotheses), there exists a modification

$$\pi\colon X' \to X$$

with X' integral and *quasi-projective*, so after replacing (X, Z) with $(X', Z' = \pi^{-1}(Z))$ we pass to the case where X is quasi-projective. Now we can form a compactification $X \hookrightarrow \overline{X}$ with $\overline{X}$ projective. Let $\overline{Z} = \overline{X} - (X - Z) = (\text{closure of } Z) \cup (\overline{X} - X)$; this is also what we've been calling $\partial_{\overline{X}}(X - Z)$. So $\overline{Z} \subsetneq \overline{X}$ is proper closed, with $\overline{Z} \cap X = Z$.

Suppose we have the solution for the projective $\overline{X}$; i.e., an alteration

$$\begin{array}{c} \overline{X}_1 \\ \downarrow \varphi_1 \\ \overline{X} \end{array}$$

with $\overline{X}_1$ smooth, and $\varphi_1^{-1}(\overline{Z}) \subset \overline{X}_1$ a sncd. Look at $X_1 = \varphi_1^{-1}(X) \hookrightarrow \overline{X}_1$: we claim that

$$\begin{array}{ccc} X_1 & \hookrightarrow & \overline{X}_1 \\ \downarrow \varphi_1|_{X_1} & & \\ X & & \end{array}$$

is the desired alteration. The alteration property and regularity of $\overline{X}_1$ are guaranteed by construction. We have to check what happens over Z. For this, note that we rigged $\overline{Z}$ (which was not just the closure of Z, but also included the boundary of X in $\overline{X}$) so that

$$\partial_{\overline{X}_1}(X_1 - \varphi_1^{-1}(Z)) = \partial(\overline{X}_1 - \varphi_1^{-1}(\overline{Z}))$$

so it is an sncd by construction.

3.4. **Reduction to blowup.** If $Z \neq \emptyset$, then we can pass to $\mathrm{Bl}_Z(X)$ so that Z is the underlying space of a Cartier divisor. Notice that this condition is preserved by pre-image with respect to any alteration on X. So we may assume that Z is the underlying space of a Cartier divisor (i.e., Zariski-locally is the zero locus of a single nonzero element of a domain).

3.5. **Reduction to the normalization.** We can pass to the (finite birational!) normalization $\widetilde{X} \to X$, so X is normal. This completes the case $d = 1$, because a normal curve is regular and any finite set of closed points in such a curve *is* a sncd (!).

3.6. **Increasing Z.** We now show that, *under our running assumptions*, we can increase Z without loss of generality. What this means is:

Lemma 3.5. *If $\varphi\colon X' \to X$ is an alteration and $Z \subset \widetilde{Z} \subsetneq X$ such that $\varphi^{-1}(\widetilde{Z}) \subset X'$ is an sncd, then $\varphi^{-1}(Z) \subset X'$ is also sncd.*

Proof. Since Z is Cartier, $\varphi^{-1}(Z)$ is also Cartier, and lies inside the sncd $\varphi^{-1}(\widetilde{Z})$. So for dimension reasons, $\varphi^{-1}(Z)$ is a union of irreducible components of the sncd $\varphi^{-1}(\widetilde{Z})$. But it is easy to see that the union of irreducible components of an sncd is still an sncd. ☐

Remark 3.6. Any alteration $\varphi\colon X' \to X$ must be finite over some open set $U \subset X$ with $\mathrm{codim}_X(X - U) \geq 2$. This is an exercise using the valuative criterion of properness; see the handout for details. (The point is to prove finiteness at the codimension-1 points.)

4. Construction of good curve fibrations

Now we want to realize X as a "curve fibration" over $\mathbf{P}^{d-1}$ at the cost of some mild blowup.

4.1. **Warmup.** As a warm-up case, we consider $X = \mathbf{P}^d$. For $p \in \mathbf{P}^d$, we have

$$\mathrm{Bl}_p(\mathbf{P}^d) = \{(q, \ell) \in \mathbf{P}^d \times \mathbf{P}^{d-1} \mid q \in \ell\}$$

Viewing $\mathbf{P}^{d-1} = \{ \text{ lines } \ell \ni p\}$. This has two "fibrations."

$$\begin{array}{ccc} & \mathrm{Bl}_p(\mathbf{P}^d) & \\ \swarrow^{\pi} & & \searrow^{\varphi} \\ \mathbf{P}^{d-1} & & \mathbf{P}^d \end{array}$$

Let's think about what these maps look like.

- The map φ is an isomorphism over $\mathbf{P}^d - p$, and $\varphi^{-1}(p) = \mathbf{P}^{d-1}$.
- The map π is a $\mathbf{P}^1$-bundle for the Zariski topology, e.g. $\pi^{-1}(\ell)$ is the line ℓ viewed as a subscheme of $\mathbf{P}^d$.

Consider the closed set $Z \subset \mathbf{P}^d$ missing p, so $\varphi^{-1}(Z) \simeq Z$. We want to think about how this sits over the $\mathbf{P}^{d-1}$.

- If $\dim Z < d-1$ (which is the case we'll most often be interested in), does $\varphi^{-1}(Z)$ map birationally onto its image under π, for "generic" p?
- If $\dim Z = d-1$, is $\varphi^{-1}(Z) \to \mathbf{P}^{d-1}$ generically étale (again, for generic p)?

Since $\pi^{-1}(\{\ell\}) = \ell \subset \mathbf{P}^d$ as *schemes*, the $\{\ell\}$-fiber of

$$\pi|_{\varphi^{-1}(Z)} \colon \varphi^{-1}(Z) \to \mathbf{P}^{d-1}$$

is $\varphi^{-1}(Z) \cap \ell$, which is isomorphic to $Z \cap \ell$ as schemes.

Note that $Z \cap \ell \subsetneq \ell$ is finite. By a suitable Bertini Theorem (cf. Jouanolou), if Z is pure dimension $d-1$ and reduced, so that $Z^{\text{sing}} \subset \mathbf{P}^d$ has codimension at least 2, then there exists a dense open locus of lines $\ell \subset \mathbf{P}^d$ such that $Z \cap \ell = Z^{\text{sm}} \cap \ell$ is smooth of dimension 0, hence étale. To construct a generically étale projection π, first pick such an ℓ_0, and then $p \in \ell_0 - (\ell_0 \cap Z)$. This ensures that

$$\pi \colon \varphi_p^{-1}(Z) \to \mathbf{P}^{d-1}$$

Has étale fiber at $\{\ell_0\}$. But then, what about flatness? This may come for free at points, if one knows smoothness of the source and target...

4.2. **The technical lemma.** That was just a warm-up example to illustrate the relevant issues. In general, we want a similar procedure using X in place of $\mathbf{P}^d$ - we want to blow up X at a *suitable* finite subset avoiding $Z \subsetneq X$.

Lemma 4.1 (de Jong, 4.11)**.** *Consider pairs (X, Z) under our running assumptions, except that X is not necessarily normal. There exists a finite subset $S \subset$*

$X^{\mathrm{sm}}(k) - Z(k)$ such that for

$$\begin{array}{ccc} & X' = \mathrm{Bl}_S(X) & \\ {}^{f}\swarrow & & \searrow^{\varphi} \\ \mathbf{P}^{d-1} & & X \end{array}$$

The fibers of f are pure dimension 1 and generically smooth, and

$$F \colon \varphi^{-1}(Z) \to \mathbf{P}^{d-1}$$

is generically étale and finite.

Furthermore, if X is normal then we can arrange that there exists a dense open $U \subset \mathbf{P}^{d-1}$ such that $f^{-1}(U) \to U$ is smooth with geometrically connected fibers.

Remark 4.2. Why haven't we assumed that X is normal? We will want to apply this Lemma to X which are not the same as in Theorem 3.1.1.

The idea is to perform a projective version of Noether normalization to find a finite map $X \xrightarrow{h} \mathbf{P}^d$ which is "nice" with respect to Z, and take $S = h^{-1}(p)$ where p is "nice" with respect to $h(Z) \subset \mathbf{P}^d$ (via the warm-up §4.1). "Nice" should mean, for instance, that

- $h \colon Z \to h(Z)$ is birational onto its image.
- $h^{-1}(p)$ is smooth (so that the blowup of X along $h^{-1}(p)$ is still normal).

This will requiring composing "projections from points" for $X \hookrightarrow \mathbf{P}^N_k$.

4.3. **Review on projections from points.** For $p \in \mathbf{P}^N$, let

$$\pi_p \colon \mathbf{P}^N \to \mathbf{P}^{N-1}$$

denote the projection map. For a fixed hyperplane $H = \mathbf{P}^{N-1} \subset \mathbf{P}^N$ (with $p \notin H$), we can visualize $\pi_p(q)$ as the (unique!) intersection point of the line $\overline{qp}$ with H. See the visualization in Figure 1.

Desired property. For "generic" $p \notin X$, we want that

- If $d = N - 1$, then $\pi_p \colon X \to \mathbf{P}^{N-1}$ is generically étale, finite, and Z maps birationally onto its image.
- If $d < N - 1$, then $\pi_p \colon X \to \mathbf{P}^{N-1}$ is birational onto its image, and likewise for Z.

Observe that $(\pi_p|_X)^{-1}([\ell]) = \ell \cap X$ as a *scheme* inside X, since $\pi_p^{-1}([\ell]) = \ell - \{p\}$ inside $\mathbf{P}^N - \{p\}$. This is how one can control the étaleness of a fiber.

The subtlety is in ensuring that the choice of genericity is retained for compositions of such processes, which is not obvious because the situation for each step depends on the one before.

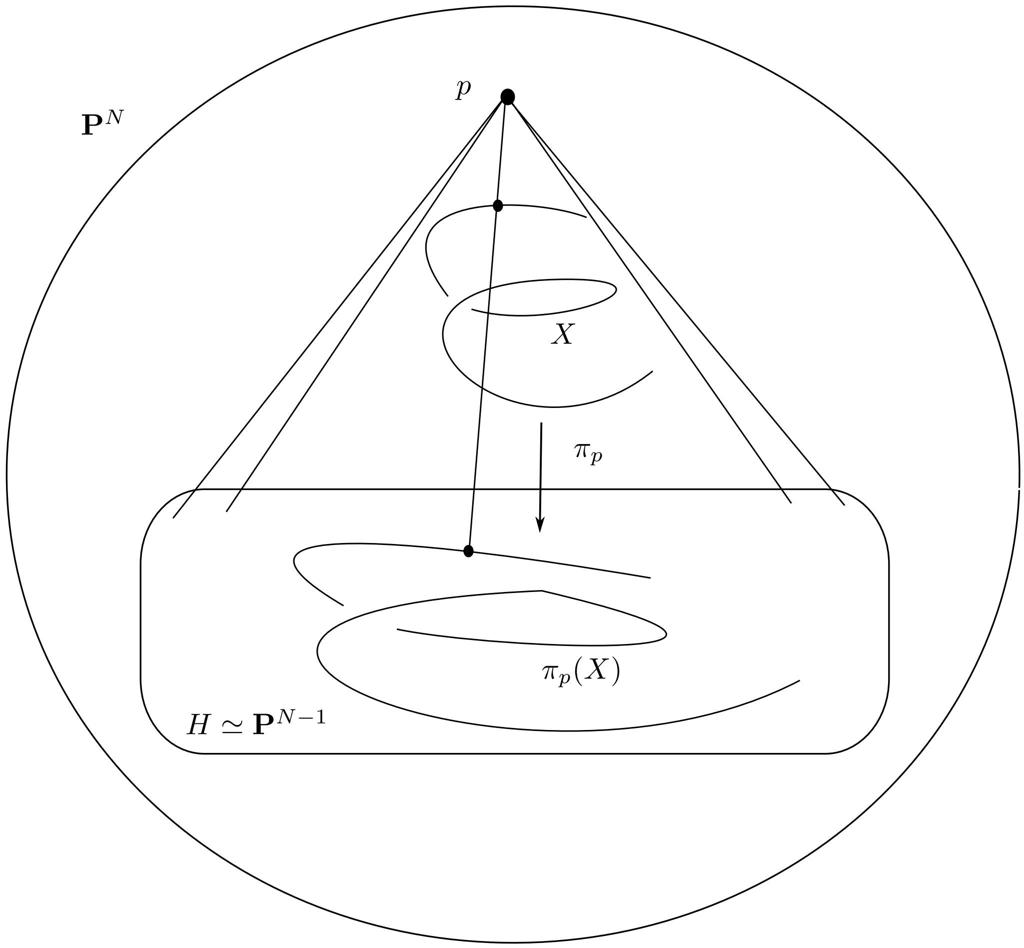


FIGURE 1. A picture of the projection of a curve X from a point p onto a hyperplane H.

5. PROJECTION... WHAT'S THE POINT?

Projection from a point can be described in coordinate-free terms, as we now explain. As always, we follow Grothendieck's convention that $\mathbf{P}(V) = \text{Proj}(\text{Sym}(V))$, so this represents the functor of (isomorphism classes of) line-bundle quotients of V; it is contravariant in V (which will be convenient later). Suppose $\dim V \geq 2$ and let $p \in \mathbf{P}(V)$ be a k-point, so $p = \mathbf{P}(V/W) \in \mathbf{P}(V)$ and the open subscheme $\mathbf{P}(V) - \{p\}$ classifies line bundle quotients $\lambda : V_S \twoheadrightarrow \mathscr{L}$ over k-schemes S such that $\lambda_s|_{W_s} \neq 0$ for all $s \in S$, which is to say λ carries W_S onto $\mathscr{L}$ too. Thus, functorially we have

$$\pi_p : \mathbf{P}(V) - \mathbf{P}(V/W) \to \mathbf{P}(W)$$
$$\lambda \mapsto \lambda|_W.$$

The important point to observe is how the target depends on W, and that it is not naturally "inside" $\mathbf{P}(V)$ (keep in mind that $\mathbf{P}(V)$ is contravariant in V).

Example 5.1. For $V = k^{N+1}$, $p = [1, 0, \ldots, 0]$, and $W = \{x_0 = 0\}$ we have naturally $\mathbf{P}(V) = \mathbf{P}^N_k$, $\mathbf{P}(W) = \mathbf{P}^{N-1}_k$, and $\pi_p : \mathbf{P}^N_k - \{p\} \to \mathbf{P}^{N-1}_k$ is given by the habitual formula

$$[x_0, \ldots, x_N] \mapsto [x_1, \ldots, x_N].$$

Returning to our setup of interest, we have

$$X \subset \mathbf{P}^N_k - \{p\} \to \mathbf{P}^{N-1}_k = \{\ell \ni p\},$$

And

$$\left(\pi_p|_X\right)^{-1}(\{\ell\}) = \ell \cap X$$

As *schemes* (not just as sets); this will be very important in some subsequent calculations.

Proposition 5.2 ([deJ, Prop. 2.11]). *Let k be a field and $X \subset \mathbf{P}^N_k$ a closed subscheme that is generically smooth of pure dimension $d \leq N - 1$. There exists a dense open $U \subset \mathbf{P}^N_k - X$ so that for all finite separable extensions k'/k and points $p \in U(k')$, the projection*

$$\pi_p \colon X_{k'} \to \mathbf{P}^{N-1}_{k'}$$

satisfies

(α) *π_p is birational onto its image if $d < N - 1$,*
(β) *π_p is generically étale onto $\mathbf{P}^{N-1}_{k'}$ if $d = N - 1$.*

Remark 5.3. We do not assume X is irreducible so that we may apply this result to Z later on. Hence, for (α) we cannot treat the irreducible components separately because we also need that distinct components have distinct images. For (β) we can reduce to the irreducible case, but not for (α).

Proof. To start, let's reduce to the case k is separably closed. Assume we know the statement over k_s, for some dense open $U' \subset \mathbf{P}^N_{k_s}$. Let's now deduce the general case. For any $p \in \left(\mathbf{P}^N_k - X\right)(k')$, whether or not π_p satisfies (α) (respectively (β)) is the same for $\pi_p \otimes_{k'} k_s$ (using a choice of embedding $k' \to k_s$ over k). Thus, it suffices to find a non-empty open $U \subset \mathbf{P}^N_k - X$ such that $U_{k_S} \subset U'$. Certainly $U' = V_{k_s}$ for some dense open

$V \subset \mathbf{P}^N_K - X_K$, for some finite Galois K/k inside k_s, by standard limit arguments (as k_s is the direct limit of finite Galois extensions of k). Hence, the intersection

$$\bigcap_{\gamma \in \mathrm{Gal}(K/k)} \gamma^*(V) \subset \mathbf{P}^N_K - X_K = \left(\mathbf{P}^N_k - X\right)_K$$

is a $\mathrm{Gal}(K/k)$-stable non-empty open subscheme. This descends to an open subscheme $U \subset \mathbf{P}^N_k - X$ that does the job. Hence, without loss of generality, now $k = k_s$.

We first treat (β), so $d = N - 1$. Note that the non-smooth locus $X^{\mathrm{sing}} := X - X^{\mathrm{sm}} \subset \mathbf{P}^N$ has codimension at least 2 in $\mathbf{P}^N$. We want to find lines that "cut X nicely." Since $X^{\mathrm{sing}} \subset \mathbf{P}^N_k$ has codimension at least 2, it misses most lines $\ell \subset \mathbf{P}^N_k$. Let $V = X^{\mathrm{sm}} = \coprod V_j$ where V_j are the open subsets where the smooth loci of irreducible components of X avoid the other irreducible components of X. For most lines $\ell \subset \mathbf{P}^N_k$ we have

$$\ell \cap X = \ell \cap V = \coprod_j \left(\ell \cap V_j\right)$$

We'll now need the following setup for a Bertini theorem, following [Jou]. Let F be a field and $Z \subset \mathbf{P}^N_F$ a locally closed subscheme of pure dimension $d \leq N - 1$. (It will be useful to permit quasi-projective Z, not just projective Z.) For a choice of $r \leq d$, let $\mathbf{G} := \mathrm{Gr}(r, N)$ be the Grassmannian of codimension-r subspaces of $\mathbf{P}^N$. Consider the universal subspace

$$\begin{array}{ccc} \mathscr{V} & \longrightarrow & \mathbf{P}^N \times \mathbf{G} \supset Z_{\mathbf{G}} := Z \times \mathbf{G} \\ & \searrow \qquad \swarrow & \\ & \mathbf{G} & \end{array} \tag{5.1}$$

Example 5.4. For example, if F'/F is a field and $L \in \mathbf{G}(F')$, the map on L-fibers is given by

$$\begin{array}{ccc} \mathscr{V}_L = L & \longrightarrow & \mathbf{P}^N_{F'} \\ & \searrow \qquad \swarrow & \\ & \{L\} = \mathrm{Spec}\, F'. & \end{array} \tag{5.2}$$

Thus, passing to fibers over the F'-point $\{L\}$ for the diagram

$$\begin{array}{ccc} \mathscr{V} \cap Z_{\mathbf{G}} & \longrightarrow & Z_{\mathbf{G}} \\ & \searrow \qquad \swarrow & \\ & \mathbf{G} & \end{array} \tag{5.3}$$

recovers the closed subscheme $L \cap Z_{F'} \hookrightarrow Z_{F'}$.

The following "Bertini theorem" combines several results proved in the beautiful book [Jou].

Theorem 5.5 (Bertini theorem)**.** *Consider all F'/F and $L \in \mathbf{G}(F')$ as above.*

(i) If Z is smooth then there is a dense open subscheme $\Omega_1 \subset \mathbf{G}$ such that $\{L\} \in \Omega_1(F')$ if and only if $L \cap Z_{F'}$ is smooth of pure dimension $d - r$.
(ii) If Z is geometrically reduced then there is a dense open subscheme $\Omega_2 \subset \mathbf{G}$ such that $\{L\} \in \Omega_2(F')$ if and only if $L \cap Z_{F'}$ is geometrically reduced of pure dimension $d - r$.
(iii) If Z is geometrically irreducible and $r \leq d - 1$, then there is a dense open subscheme $\Omega_3 \subset \mathbf{G}$ such that $\{L\} \in \Omega_3(F')$ if and only if $L \cap Z_{F'}$ is geometrically irreducible of pure dimension $d - r$.

Remark 5.6. Observe that the first two conditions are local on Z, but the third is not.

Remark 5.7. If $r > d$ then there exists a dense open $\Omega \subset \mathbf{G}$ so that for all $L \in \Omega(F')$ we have $Z_{F'} \cap L = \emptyset$.

We'll now use Theorem 5.5(ii) in the case $r = d = N - 1$. That is, we will slice by lines $\ell \subset \mathbf{P}^N$. Again, $\mathbf{G}$ is the Grassmannian of all lines $\ell \subset \mathbf{P}^N$ and $k = k_s$, so any dense open subset of a non-empty smooth k-scheme has lots of k-points. By Theorem 5.5 we get a line $\ell \subset \mathbf{P}^N_k$ (with $k = k_s$) so that (with $V = X^{\mathrm{sm}}$)

$$\ell \cap X = \ell \cap V = \coprod_j (\ell \cap V_j),$$

with all $\ell \cap V_j$ non-empty and étale over k. In particular, the intersection is finite.

Fix such a line ℓ_0. Choose any point $p \in (\ell_0 - \ell_0 \cap X)(k)$ with $p \notin \ell_0 \cap X$, so for the map

$$\pi_p \colon X \to \mathbf{P}^{N-1}_k = \{\ell \ni p\}$$

we see that $\pi_p^{-1}(\{\ell_0\})$ is étale and meets X inside X^{sm} touching every irreducible component of X. By openness on the source for the quasi-finite locus of a map of finite type between noetherian schemes (i.e., openness of the locus of points isolated in their fibers), we conclude that π_p is generically quasi-finite and hence dominant (and even surjective, due to properness).

Since π_p is a dominant map, π_p is flat on some dense open W. We claim that the open locus where W meet the open complement of the support of the coherent sheaf $\Omega^1_{X/\mathbf{P}^{N-1}}$ to touch each irreducible component of X; at such points π_p is étale (flat and unramified). Thus, it suffices to show that the complement of the support of $\Omega^1_{X/\mathbf{P}^{N-1}}$ is dense in X.

Consider $x_0 \in \pi_p^{-1}(\{\ell_0\})$; there is such a point x_0 on each irreducible component of X. Since the formation of Ω^1 commutes with base change, the fiber $\Omega^1_{X/\mathbf{P}^{N-1}}(x_0)$ of the coherent sheaf $\Omega^1_{X/\mathbf{P}^{N-1}}$ at the point x_0 is the same as the fiber at x_0 for Ω^1 on the π_p-fiber through x_0. But we have used Bertini's theorem to ensure that this fiber scheme is étale, so we conclude that $\Omega^1_{X/\mathbf{P}^{N-1}}(x_0) = 0$. By Nakayama's Lemma, it follows that $\Omega^1_{X/\mathbf{P}^{N-1}}$ vanishes near x_0.

To complete (β), it only remains to show the locus of points p we have been using sweeps out at least a dense open locus in $\mathbf{P}^N$. We shall do this by analyzing an incidence correspondence:

$$\begin{array}{ccc} & \left\{(x,\ell) \in \mathbf{P}^N \times \mathbf{G} : x \in \ell\right\} & \\ \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} \\ \mathbf{P}^N & & \mathbf{G} = \left\{\ell \subset \mathbf{P}^N\right\}. \end{array} \tag{5.4}$$

Observe that pr_1 is a Zariski $\mathbf{P}^{N-1}$-bundle over $\mathbf{P}^N$, and pr_2 is a Zariski $\mathbf{P}^1$-bundle over $\mathbf{G}$. Letting $\Omega \subset \mathbf{G}$ be the non-empty open locus from the above application of part (ii) in Bertini's theorem, the set of p we're looking for is

$$\mathrm{pr}_1\left(\mathrm{pr}_2^{-1}(\Omega)\right) \cap \left(\mathbf{P}^N - X\right).$$

These two sets are both non-empty opens because pr_1 is an open map. This completes the treatment of (β).

Now we move on to (α), so $d \le N-2$.

Goal 5.8. We seek a dense open $U \subset \mathbf{P}^N - X$ so that for all $p \in U(k)$ the map

$$\pi_p : X \to \mathbf{P}^{N-1}$$

is birational onto its image.

Without loss of generality we may assume X is reduced, since it is generically reduced, and the higher codimension non-reduced locus has no impact on the birationality assertion. We first seek a good line for slicing each irreducible component of X, where the lines will be found inside a generic codimension-d linear subspaces $W \subset \mathbf{P}^N$ (so $\dim W = N - d \ge 2$) that meets X nicely.

To be precise, let $\{X_j\}$ be the irreducible components of X with reduced closed subscheme structure. We have $\dim X_j = d$ (recall the assumption that X was pure dimensional). Let

$$V_j := X_j^{\mathrm{sm}} - \left(\cup_{i \neq j} X_j^{\mathrm{sm}} \cap X_i\right),$$

a smooth dense open in X_j. We seek lines $\ell_j \subset \mathbf{P}_k^N$ so that

$$\ell_j \cap X_j = \ell_j \cap V_j = \operatorname{Spec} k.$$

Let $\mathbf{G} = \operatorname{Gr}(d, N)$ be the Grassmannian classifying linear subspaces of $\mathbf{P}^N$ with codimension d. By Bertini's theorem (i.e., Theorem 5.5, where Z is only required to be quasi-projective rather than projective) applied to each X_j and V_j, we can find $W \in \mathbf{G}(k)$ so that for every j we have

(1) $W \cap X_j$ is finite and non-empty,
(2) $W \cap (X_j - V_j) = \varnothing$ (so $W \cap X_j = W \cap V_j$),
(3) $W \cap V_j$ is étale (so a disjoint union of finitely many k-points since $k = k_s$).

Since each $W \cap X_j = W \cap V_j$ is a finite non-empty set of k-points, we may pick one such k-point q_j for each j.

Recall W has dimension $N - d \geq 2$. Pick a line $\ell_j \subset W$ through q_j missing the rest of the finitely many points where W meets $X = \cup X_j$. Pick a k-point $p_j \in \ell_j - (\ell_j \cap X_j)$ and consider the map

$$\pi_{p_j} : X_j \to \mathbf{P}^{N-1} = \{\ell \ni p_j\}.$$

This map has fiber over $\{\ell\}$ equal to the scheme $\ell \cap X_j$ that is finite since it is closed in the line ℓ but misses the point $p_j \in \ell - (\ell \cap X_j)$. We have as schemes that

$$\begin{aligned} \pi_{p_j}^{-1}(\{\ell_j\}) &= X_j \cap \ell_j \\ &= (X_j \cap W) \cap \ell_j \\ &= q_j \simeq \operatorname{Spec} k. \end{aligned}$$

The upshot is that

$$\pi_{p_j} : X_j \to \pi_{p_j}(X_j)$$

is a finite surjection between varieties such that its fiber scheme over $\{\ell_j\}$ is a single k-point. The final part of the next lemma completes the proof of Proposition 5.2. $\square$

Lemma 5.9. *With notation as above,*

(1) *π_{p_j} is birational onto its image,*
(2) *For each j, there exists a non-empty open $U_j \subset \mathbf{P}^N_k - X_j$ so that all points in $U_j(k)$ arise as such a p_j.*
(3) *For a suitable dense open $U \subset \cap_j U_j \subset \mathbf{P}^N - X$ and all $p \in U(k)$, the images $\pi_p(X_j)$ are pairwise disjoint and hence*
$$\pi_p \colon X \to \pi_p(X)$$
is birational.

Proof. Let's prove each in order.

(1) It suffices to show more generally that for any finite map $f \colon X \to Y$ to a noetherian scheme Y and any pair of points $y, \eta \in Y$ with y in the closure of η (e.g., η could be the generic point if Y is irreducible), the fiber scheme X_y and X_η that are finite over the respective residue fields $k(y)$ and $k(\eta)$ satisfy
$$\deg_{k(y)} X_y \geq \deg_{k(\eta)} X_\eta$$
where $\deg_k Z := \dim_k k[Z]$ for any finite scheme Z over a field k. Indeed, once this is proved, it follows that the non-empty finite fiber of $\pi_{p_j} : X_j \to \pi_{p_j}(X_j)$ has degree 1 (since we found a k-point of the target whose fiber-degree is 1), and that expresses exactly birationality of this map between varieties.

We may assume $y \neq \eta$, so (by arguments with the Krull-Akizuki theorem, or with blow-ups) there exists a discrete valuation ring R and a map
$$\operatorname{Spec} R \to Y$$
carrying the generic point to η and the special point to y (but with possibly gigantic residue field extension over each, especially at y). The formation of the degree of a finite scheme over a field is unaffected by extension of the field, so we may apply base change along such a map to arrange that $Y = \operatorname{Spec} R$.

Now $X = \operatorname{Spec} A$ with R-finite A. If K denotes the fraction field of R and k denotes its residue field then our assertion is $\dim_K(A_K) \leq \dim_k(A_k)$, which holds with A replaced by any finitely generated R-module.

(2) Recall $\dim W \geq 2$, so we may pick $p \in W$ not on any line joining two points of the finite étale k-scheme $W \cap X = \coprod (W \cap V_j)$. For $q_j \in (W \cap V_j)(k)$, the line $\lambda_j := \overline{pq_j}$ meets X only in q_j, since we

ensured p is not on the line joining any two distinct points of $W \cap X$. The finite map

$$\pi_p \colon X \to \mathbf{P}^{N-1} = \{\ell \ni p\}$$

therefore satisfies

$$\pi_p^{-1}\left(\{\lambda_j\}\right) = \{q_j\} \notin X_i$$

for all $i \neq j$. Hence, this p not only works as a common choice of p_j for all j but moreover $\pi_p(X_j)$ changes as we vary j, so $\pi_p : X \to \pi_p(X)$ is birational.

One can use an incidence correspondence argument similar to the one employed for lines in the treatment of (β), but now working with $\mathrm{Gr}(d, N)$, to obtain that the possibilities for such p exhaust at least a dense open locus in $\mathbf{P}^N_k - X$.

(3) This follows from the discussion in the previous part.

□

Now we return to our original setting of interest over $k = \overline{k}$ with $Z \subsetneq X \subset \mathbf{P}^N_k$ where X is a closed subvariety of dimension $d < N$ with $d \geq 2$ and Z is the support of a Cartier divisor (and so has pure dimension $d-1$) equipped with the reduced structure. We now apply Proposition 5.2 $N-d$ times to both X and Z to arrive at the following:

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{P}^d \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\pi} & \pi(Z) \end{array} \tag{5.5}$$

where the top map is finite and generically étale and the bottom map is birational.

Let $\Omega \subset \mathbf{P}^d - \pi(Z)$ be a dense open such that $\pi^{-1}(\Omega) \to \Omega$ is étale. Choose $\xi \in \Omega(k)$ generic with respect to $\pi(Z)$ in the sense of our warm-up example; that is, the finite projection

$$\mathrm{pr}_\xi \colon \pi(Z) \to \mathbf{P}^{d-1} = \{\ell \ni \xi\}$$

is generically étale.

We aim to use $X' := \mathrm{Bl}_S(X)$ for $S := \pi^{-1}(\xi)$. Because blow-up commutes with flat base change, we will be able to study this situation using our model calculation for $\mathbf{P}^d$. The idea is that X' is fibered over the space $\mathbf{P}^{d-1}$ of lines ℓ through ξ, with ℓ-fiber equal to $\pi^{-1}(\ell)$.

5.1. **Completing the proof of Lemma 4.1.** We have composed $N-d$ projections from sufficiently generic choices of points, and each choice of point depended on the previous choices. The total collection of choices made corresponds to a collection of $N-d$ independent points in $\mathbf{P}^N$, or more specifically an $(N-d-1)$-plane in $\mathbf{P}^N$ disjoint from X away from which we are projecting. We'd like to understand how generic that linear subspace is (in the corresponding Grassmannian). We'll come back to this, but first let's recall the genericity conditions imposed on the choice of

$$\xi \in \mathbf{P}^d - \pi(Z)$$

above:

(1) $\xi \in \Omega$, with Ω a dense open in $\mathbf{P}^d$ for which

$$\pi^{-1}(\Omega) \xrightarrow{\pi} \Omega$$

is étale,

(2) ξ is "generic" with respect to $\pi(Z)$, meaning that the natural map

$$\mathrm{Bl}_\xi(\mathbf{P}^d) \to \mathbf{P}^{d-1} = \{\ell \ni \xi\}$$

is generically étale when restricted to

$$\pi(Z) \subset \mathbf{P}^d - \xi \subset \mathrm{Bl}_\xi(\mathbf{P}^d).$$

Let $S := \pi^{-1}(\xi)$. By the above assumptions, $S \subset X^{\mathrm{sm}}$ since $\pi : X \to \mathbf{P}^d$ has smooth target and is étale over ξ. Therefore, $\mathrm{Bl}_S(X)$ is normal if X is because $\mathrm{Bl}_S(X)$ is a gluing of $X - S$ and $\mathrm{Bl}_S(X^{\mathrm{sm}})$ along $X^{\mathrm{sm}} - S$. Here we are using that the blow-up $\mathrm{Bl}_p(W)$ of a smooth k-scheme W at a k-point p is smooth as well. There are two easy ways to see this:

(1) The completed local rings on $\mathrm{Bl}_p(W)$ at points of the exceptional divisor are identified with ones on $\mathrm{Bl}_0(\mathbf{A}^n)$ and hence are all regular.

(2) By shrinking W around p (as we may do), we can arrange that there is an étale map

$$\begin{aligned} f\colon W &\to \mathbf{A}^n \\ p &\mapsto \{0\} \end{aligned}$$

for which p is the entire fiber over 0. Then by compatibility of blow-up with flat base change we have a fiber square

$$\begin{array}{ccc} \mathrm{Bl}_p(W) & \longrightarrow & \mathrm{Bl}_0(\mathbf{A}^n) \\ \downarrow & & \downarrow \\ W & \longrightarrow & \mathbf{A}^n \end{array} \tag{5.6}$$

and $\mathrm{Bl}_0(\mathbf{A}^n)$ is smooth by inspection.

In view of our explicit description of blow-up of projective space at a point via an incidence relation, in the fiber square

$$\begin{array}{ccc} X' & \longrightarrow & \mathrm{Bl}_{\xi}\,\mathbf{P}^d \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & \mathbf{P}^d. \end{array} \tag{5.7}$$

we have

$$X' = \left\{ (x, \ell) \in X \times \mathbf{P}^{d-1} : x \in \pi^{-1}(\ell) \right\}$$

where $\mathbf{P}^{d-1}$ is the space of lines through ξ. Also, π is flat and even étale over $\Omega \ni \xi$, so by compatibility of blow up with flat base change we have $X' = \mathrm{Bl}_S(X)$. Thus, $\mathrm{Bl}_S(X)$ now has an incidence-relation description similar to that of $\mathrm{Bl}_{\xi}(\mathbf{P}^d)$; this will be very useful below.

Let's now study the second projection $f := \mathrm{pr}_2$, with pr_2 as in

$$\begin{array}{ccccc} & & X' & & \\ & \swarrow_{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ X & & & & \mathbf{P}^{d-1}. \end{array} \tag{5.8}$$

Note that on

$$Z \subset X - S \subset \mathrm{Bl}_S(X) = X'$$

we have that $f|_Z$ is generically étale and finite onto $\mathbf{P}^{d-1}$ because of our choice of ξ (generic with respect to $\pi(Z)$!).

We make the following observations, in sequence:

(1) For $\{\ell\} \in \mathbf{P}^{d-1}$ we have

$$f^{-1}\left(\{\ell\}\right) = \pi^{-1}(\ell)$$

as schemes. Remember that $\pi^{-1}(\ell)$ is 1-dimensional since π is a finite surjection. In fact, $f^{-1}(\{\ell\})$ has dimension ≥ 1 at all points (i.e., no isolated points) due to the fact that $\ell \subset \mathbf{P}^d$ is locally defined by $d-1$ equations. Thus, $\pi^{-1}(\ell) \subset X$ is pure dimension 1 everywhere.

(2) Each $\pi^{-1}(\ell)$ is generically smooth. Indeed, $\pi^{-1}(\ell) \to \ell$ carries each irreducible component of $\pi^{-1}(\ell)$ surjectively onto ℓ for dimension reasons because it is proper and π is finite. Yet, this is a (scheme theoretic) base change of the map $\pi : X \to \mathbf{P}^d$ that is étale over an open neighborhood Ω of ξ, so $\pi^{-1}(\ell) \to \ell$ is étale over $\ell \cap \Omega$. Thus, $\pi^{-1}(\ell)$ is smooth on the dense open $\pi^{-1}\left(\ell \cap \Omega\right)$.

In fact, not only are the fibers of f generically smooth, but we claim that f is a smooth map at k-points that are smooth in their fibers. This is a prototype for a property we will need in some other contexts later (bootstrapping from a fibral property to a relative property without flatness assumptions), and will follow from the next lemma that is useful but weaker variant of [deJ, Lemma 2.8].

Lemma 5.10 (Weak [deJ, Lemma 2.8]). *Let $h : A \to B$ be a local map between complete local noetherian rings such that*

(1) A is a domain of dimension δ
(2) $\dim B = \delta + r$
(3) $B/\mathfrak{m}_A B \simeq k_A[\![t_1, \ldots, t_r]\!]$ as k_A-algebras.

Then $B \simeq A[\![T_1, \ldots, T_r]\!]$ as A-algebras.

The crucial feature is that we do not assume h to be flat (but we require A to be a domain, as occurs when we form completed local rings on a smooth scheme over a field such as $\mathbf{P}^{d-1}$).

Proof. Pick $T_1, \ldots, T_r \in \mathfrak{m}_B$ lifting $t_1, \ldots, t_r \in B/\mathfrak{m}_A B$. We have a map of A-algebras

$$\begin{aligned} A[\![X_1, \ldots, X_r]\!] &\xrightarrow{\phi} B \\ X_i &\mapsto T_i. \end{aligned}$$

Modulo $\mathfrak{m}_A$ this is just the map

$$\begin{aligned} k_A[\![X_1, \ldots, X_r]\!] &\xrightarrow{\phi} k_A[\![t_1, \ldots, t_n]\!], \\ X_i &\mapsto t_i \end{aligned}$$

that is an isomorphism, so by successive approximation and $\mathfrak{m}_A$-adic completeness (and separatedness) we see that ϕ is surjective.

Now, we claim that in fact ϕ is an isomorphism. This holds because the source of ϕ is a *domain* with dimension $\delta + r$ and B (which we don't know yet to be a domain or not!) has dimension $\delta + r$, and the quotient of a noetherian local domain modulo any nonzero ideal has strictly smaller dimension. □

Remark 5.11. More generally Lemma 5.10 shows that if $f : X \to Y$ is a map of finite type between noetherian schemes and x is a rational point in the smooth locus of its fiber $X_{f(x)}$ such that the "dimension formula holds at x" (l.e., $\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{Y,f(x)} + \dim \mathscr{O}_{X_{f(x)},x}$) then x is in the smooth locus for f provided that the completed local ring on Y at $f(x)$ is a domain (such as occurs when Y is regular at $f(x)$, or normal and excellent).

By applying the preceding lemma to the map of completions

$$\mathscr{O}^{\wedge}_{\mathbf{P}^{d-1},f(x)} \to \mathscr{O}^{\wedge}_{X',x'}.$$

for x' any k-point of X' smooth in its fiber $X'_{f(x')}$ (as is applicable because completed local rings of the smooth $\mathbf{P}^{d-1}$ at k-points are formal power series rings over k in $d-1$ parameters), we conclude that f is smooth at any k-point x' that is a smooth point in its fiber $X'_{f(x')}$.

To complete the proof of Lemma 4.1, it remains to prove the next result.

Lemma 5.12. *If X is normal then for a suitable choice of ξ:*

1. *there exists a dense open $U \subset \mathbf{P}^{d-1}$ so that $f^{-1}(U) \to U$ is smooth and*
2. *$f\colon X' \to \mathbf{P}^{d-1}$ is its own Stein factorization (so it has geometrically connected fibers, by* [EGA, III, 4.3.3]*).*

Let us first prove (1). The relative smooth locus $\mathrm{sm}(X'/\mathbf{P}^{d-1}) \subset X'$ is open, so by properness of $f\colon X' \to \mathbf{P}^{d-1}$ it is enough for this open set to contain one fiber X'_y. By Lemma 5.10, this is equivalent to X'_y being smooth, so we just need to find some $\ell \subset \mathbf{P}^d$ through ξ so that $\pi^{-1}(\ell) \subset X$ is smooth.

Recall that π is constructed by projecting from $N-d+1$ independent points in $\mathbf{P}^N$, and ξ corresponds to another point. Overall, this corresponds to an $(N-d+1)$-plane $\Lambda \subset \mathbf{P}^N_k$.

Since X is normal, we know $X - X^{\mathrm{sm}} \subset X$ has dimension at most $d-2$. But $N-d+1 = N-(d-1)$. Therefore, general such linear spaces $\Lambda \subset \mathbf{P}^N$ of codimension $d-1$ meet X entirely in X^{sm}. Thus, as long as the possible planes $\Lambda \subset \mathbf{P}^N$ encoding all of our choices from iterated Bertini theorems really sweep out at least a dense open set in the appropriate Grassmannian we are done. That such genericity does hold for Λ is addressed in a handout on "iterated genericity".

5.2. **A Stein factorization.** It remains (for the proof of Lemma 4.1) to establish part (2) in Lemma 5.12. Recall that after passing to $\mathrm{Bl}_S(X)$ (renamed as X and still normal) we were in the following situation:

1. all $X_{\overline{y}}$ are pure of dimension 1
2. there exists a dense open $U \subset Y$ so that $f^{-1}(U) \to U$ is smooth
3. $\mathrm{sm}(X/Y) \cap X_{\overline{y}} \subset X_{\overline{y}}$ is dense for all geometric points $\overline{y} \in Y$.

Recall that by design of f, such $X_{\overline{y}}$ are controlled by Bertini methods, at least over a dense open in Y. These $X_{\overline{y}}$ are $X \cap W$ for $W \subset \mathbf{P}^N_k$ sufficiently general linear spaces of codimension $d-1$. So, $W \cap X = W \cap X^{\mathrm{sm}}$ is irreducible and smooth of dimension 1 (since Bertini applies to the smooth quasi-projective X^{sm} of dimension d). The upshot is that there exist many $y_0 \in U(k)$ so that

X_{y_0} is geometrically irreducible and smooth. We will use the existence of this geometrically connected fiber y_0 to settle our Stein factorization problem:

Lemma 5.13. *The map $f\colon X \to Y$ is its own Stein factorization (i.e., $\mathscr{O}_Y \simeq f_*\mathscr{O}_X$), and in particular all geometric fibers of f are connected.*

Proof. We have $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is an injection as f is a surjection of varieties. Note that Y is normal, so this inclusion of coherent sheaves of domains is an equality if and only if it is an equality over the generic point of Y. In fact, Letting T be the localization of Y at y_0, we'll show

$$\mathscr{O}_{Y,y_0} \to (f_*\mathscr{O}_X)_{y_0} = H^0(X_T, \mathscr{O}_{X_T})$$

is an isomorphism. This is a special case of the next result. □

Proposition 5.14. *Let $(R, \mathfrak{m}, k)$ be a local noetherian ring and Z a non-empty proper flat R-scheme so that the special fiber Z_0 is geometrically reduced and geometrically connected over the residue field k. Then, the natural map $\phi\colon R \to H^0(Z, \mathscr{O}_Z)$ is an isomorphism.*

Proof. Surjectivity of $R \to H^0(Z, \mathscr{O}_Z)$ will use cohomology and base change, and injectivity will use flatness. Note that f is both open and closed and Z is connected (and non-empty), so f is surjective.

First let's show ϕ is injective. Pick $z_0 \in Z_0$. By flatness of f, we have a map $R \to \mathscr{O}_{Z,z_0}$ which is a flat local map. Hence it is faithfully flat, and thus injective. But the map factors as

$$R \xrightarrow{\phi} H^0(\mathscr{O}_Z) \to \mathscr{O}_{Z,z_0}$$

so $\ker\phi = 0$.

We'll now verify surjectivity. If Y is proper over a field F and geometrically reduced and geometrically connected over F, then $F \simeq \Gamma(Y, \mathscr{O}_Y)$ (as we see via scalar extension to the case F is algebraically closed). This implies $k \simeq H^0(Z_0, \mathscr{O}_{Z_0})$, so via the factorization

$$\begin{array}{ccc} H^0(Z, \mathscr{O}_Z)/\mathfrak{m} & \xrightarrow{\alpha} & H^0(Z_0, \mathscr{O}_{Z_0}) \\ & \nwarrow \quad \nearrow_{\simeq} & \\ & k & \end{array} \tag{5.9}$$

involving the base change morphism α we see that α is surjective. The theorem on cohomology and base change applies to $\mathscr{O}_Z$ since Z is R-flat and proper, so α is an isomorphism. This implies that the module-finite ring map

$$\phi\colon R \to H^0(Z, \mathscr{O}_Z)$$

is surjective modulo $\mathfrak{m}$ and so is surjective (by Nakayama's Lemma). □

6. THREE-POINT LEMMA

Remark 6.1 (Summary of the situation thus far)**.** Before continuing, we review the current situation. We have established all of the following properties except $(vi)(e)$ and $(iv)(f)$ below:

(i)-(iv) X is a projective variety over $k = \overline{k}$ of dimension $d \geq 2$ with $Z \subset X$ the support of a non-empty Cartier divisor.

(v) X is normal

(vi) there exists a surjective map $X \to Y$ where Y is a projective variety of dimension $d-1$ satisfying the following properties:
  - (a) All fibers of f are geometrically connected of pure dimension 1,
  - (b) the smooth locus $\mathrm{sm}(X/Y)$ for f is fiberwise dense over Y,
  - (c) f is smooth over a dense open $U \subset Y$,
  - (d) $Z \to Y$ is finite (hence surjective) and generically étale,
  - (e) $\mathrm{sm}(X/Y) \cap Z$ meets each geometric fiber X_y for $y \in Y$ a geometric point in at least 3 points per irreducible component of X_y.
  - (f) We have

$$Z = \cup_{i=1}^{r} \sigma_i(Y)$$

    for sections $\sigma_i\colon Y \to X$ so that $\{\sigma_i(\eta_Y)\}$ are pairwise distinct (which is preserved under further alterations of Y).

The motivation for the condition with 3 points in (vi)(e), which we will arrange below by suitable enlargement of Z, is that irreducible components isomorphic to $\mathbf{P}^1$ in stable curves need 3 distinguished points.

Lemma 6.2 (The three-point lemma, [deJ, Lemma 4.13])**.** *Suppose we have $f\colon X \to Y$ is a map of projective varieties over $k = \overline{k}$ with $d = \dim X \geq 2$ and assume that $(vi)(a)$ and $(vi)(b)$ from Remark 6.1 hold. Then there exists a Cartier divisor $D \subset X$ so that*

(1) $D \to Y$ is finite and generically étale and

(2) for all geometric points $y \in Y$, $\mathrm{sm}(X/Y) \cap D$ meets each irreducible component of X_y in at least 3 points.

Remark 6.3. Once we complete the proof of Lemma 6.2, we can replace Z with $(Z \cup D)_{\mathrm{red}}$, which is still the support of a Cartier divisor. We also gain condition $(vi)(e)$ of Remark 6.1.

Proof. We'll build D as $X \cap H$ for a generic hyperplane H in a suitable embedding $\iota\colon X \to \mathbf{P}^M$ for M large.

Pick a very ample $\mathscr{L}$ on X. Consider

$$\begin{aligned} \iota_{\mathscr{L}}\colon X &\hookrightarrow \mathbf{P} = \mathbf{P}\left(\Gamma(X, \mathscr{L})\right), \\ X &\mapsto [\mathrm{ev}_x\colon \Gamma(X, \mathscr{L}) \to \mathscr{L}(x) := \mathscr{L}_x/\mathfrak{m}_x\mathscr{L}_x]. \end{aligned}$$

This is a map to the space of hyperplanes in $\Gamma(X, \mathscr{L})$. Note that hyperplanes $H \subset \mathbf{P}$ correspond to lines in $\Gamma(X, \mathscr{L})$, which correspond to hyperplanes in $\Gamma(X, \mathscr{L})^\vee$, the space of which is $\mathbf{P}(\Gamma(X, \mathscr{L})^\vee) =: \mathbf{P}^\vee$.

We seek H such that $H \cap X_y$ is 0-dimensional for all geometric points $y \in Y$ (and we will also seek additional properties). That is, we wish to avoid the situation where $H \cap X_y$ is 1-dimensional for some y. To do so, we consider the incidence correspondence:

$$T := \left\{(H, y) \in \mathbf{P}^\vee \times Y : \dim\left(H \cap X_y\right) = 1\right\} \subset \mathbf{P}^\vee \times Y.$$

This is closed because it is the jumping locus for fiber-dimension of a morphism: we have a map

$$\begin{array}{c} \{(H, x) \in \mathbf{P}^\vee \times X : x \in H\} \\ \big\downarrow \\ \{(H, y) \in \mathbf{P}^\vee \times Y\} \end{array} \tag{6.1}$$

for which the fiber over a point (H_0, y_0) is equal to $\{H_0\} \times \left(H_0 \cap X_{y_0}\right)$. Thus, all fibers of this map have dimension 1 or 0, and so by definition T is the locus where the fiber dimension has at least 1. This proves that T is closed, by upper-semicontinuity of fiber dimension.

Consider the projections

$$\begin{array}{ccccc} & & T & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathbf{P}^\vee & & & & Y \end{array} \tag{6.2}$$

To find H making $H \cap X_y$ be 0-dimensional for all $y \in T$, we only need to check $\mathrm{pr}_1(T) \subset \mathbf{P}^\vee$ is not all of $\mathbf{P}^\vee$, since we know the image is closed by properness. We will study the fibers of $h := \mathrm{pr}_2$ to show $\dim T < \dim \mathbf{P}^\vee$, provided we replace $\mathscr{L}$ by a large tensor power.

For a geometric point $\overline{y}_0 \in Y$, label the irreducible components of $X_{\overline{y}_0}$ as $C_1, \ldots, C_r$, so

$$\begin{aligned} h^{-1}(\overline{y}_0) &= \left\{ H \in \mathbf{P}^\vee : H \text{ contains some } C_j \right\} \\ &= \cup_{1 \le j \le r} \left\{ H \supset \operatorname{Span} \iota_{\mathscr{L}}(C_j) \right\} \end{aligned}$$

Define

$$\Lambda_{C_j}^{\mathscr{L}} := \operatorname{Span}\left(\iota_{\mathscr{L}}\left(C_j\right)\right).$$

Now, observe that $h^{-1}(\overline{y}_0)$ is a union of linear subspaces of $\mathbf{P}^\vee$ of codimension $1 + \Lambda_{C_j}^{\mathscr{L}}$, so

$$\begin{aligned} \dim T &\le \dim Y + \sup_{\overline{y} \in Y(k)} \dim h^{-1}(\overline{y}) \\ &= \dim Y + \sup_{\overline{y} \in Y(k), C \subset X_{\overline{y}} \text{ irreducible components}} \left(\dim \mathbf{P}^\vee - \left(1 + \dim \Lambda_C^{\mathscr{L}} \right) \right) \\ &= \dim Y + \dim \mathbf{P}^\vee - \inf_{y \in Y(k), C \subset X_{\overline{y}}} \left(1 + \dim \Lambda_C^{\mathscr{L}} \right). \end{aligned}$$

We just need that all $\dim \Lambda_C^{\mathscr{L}}$ are uniformly bigger than $\dim Y$.

The linear span $\Lambda_C^{\mathscr{L}} \subset \mathbf{P}\left(\Lambda\left(X, \mathscr{L}\right)\right)$ is

$$\mathbf{P}\left(\Lambda\left(X, \mathscr{L}\right) / W_C^{\mathscr{L}}\right)$$

for

$$W_C^{\mathscr{L}} := \ker\left(\Lambda\left(X, \mathscr{L}\right) \to \Gamma\left(C, \mathscr{L}|_C\right)\right).$$

Hence,

$$\begin{aligned} 1 + \dim \Lambda_C^{\mathscr{L}} &= \dim \Gamma\left(X, \mathscr{L}\right) / W_C^{\mathscr{L}} \\ &= \dim\left(\operatorname{im}\ \left(\Gamma\left(X, \mathscr{L}\right) \to \Gamma\left(C, \mathscr{L}|_C\right)\right)\right). \end{aligned}$$

We want the dimension of these images to be uniformly large. Even better, the dimensions of these images are uniformly big for *all* irreducible closed curves $C \subset X$:

Lemma 6.4. *With the notation as above, for $n \ge 1$, we have*

$$\operatorname{im}\ \left(\Gamma(X, \mathscr{L}^{\otimes n}) \to \Gamma(C, \mathscr{L}^{\otimes n}|_C)\right)$$

has dimension at least $n + 1$.

Proof. Since $\mathscr{L}$ is very ample, we can find two hyperplane slices of C in $\mathbf{P}$ that are disjoint and non-empty.

This corresponds to two global sections

$$s, s' \in \Gamma(X, \mathscr{L}) - \{0\}$$

such that

$$s|_C, s'|_C \in \Gamma\left(C, \mathscr{L}|_C\right)$$

have disjoint non-empty zero loci. Therefore, they are linearly independent over k, so

$$f := \frac{s'|_C}{s|C} \in k(C) - k.$$

Observe that f is transcendental over k, as k is algebraically closed. Therefore, the monomials $s'^j|_C s^{n-j}|_C \in \Gamma(C, \mathscr{L}^{\otimes n}|_C)$ are linearly independent over k, since a nontrivial dependence relation would give an algebraic dependence on f over k. So, $s'^j s^{n-j}|_C$ form the desired $n+1$ independent sections in the image. □

Now, rename $\mathscr{L}^{\otimes n}$ as $\mathscr{L}$ for such big enough n, taken to be at least 3. Thus, for generic $H \in \mathbf{P}^\vee$, $D := H \cap X$ meets each $X_{\overline{y}}$ in a finite set.

Remark 6.5. For any irreducible component C of any fiber $X_{\overline{y}}$, we know D meets C in at least 3 points with multiplicity by Bezout's theorem, since $\deg \mathscr{L}|_C^{\otimes n} \geq n$ in the earlier situation. Thus, if $D \cap X_{\overline{y}_0} \subset \mathrm{sm}(X/Y)$ and is in the étale locus for D over Y then then we are in good shape.

Note that $D \to Y$ is finite for $D := H \cap X$ with H as above, as $D \to Y$ is proper and quasi-finite. Further, D has pure dimension $d-1$ due to being Cartier in X. We want to arrange that D is generically étale over Y for most $H \in \mathbf{P}^\vee$:

Lemma 6.6. *For generic $H \in \mathbf{P}^\vee$, the map $D := H \cap X \to Y$ is generically étale.*

Proof. Choose $y_1 \in Y(k)$ and consider $X_{y_1} \subset \mathbf{P}$. Let H_1 be a hyperplane so that $H \cap X \to Y$ is quasi-finite (and hence finite) and $D_{y_1} := H_1 \cap X_{y_1} \subset H_1 \cap \mathrm{sm}(X/Y)$ is étale and meets each irreducible component of X_{y_1} in at least three points. The condition that $H \cap X$ be Y-finite holds for generic H due to our earlier passage to a large tensor power of an initial choice of very ample $\mathscr{L}$ on X, and the other conditions (involving just X_{y_1} and $\mathrm{sm}(X/Y)_{y_1}$) happen for a generic choice of H_1 due to Bertini's theorem (since by design $\mathrm{sm}(X/Y)$ meets the fiber X_{y_1} in a dense open subset). Set $D_1 := H_1 \cap X$.

It remains only to show (as we will do using a variant of earlier calculations upgrading smoothness of a point in a fiber to smoothness of a morphism at that point, under some hypotheses on the base) that $D \to Y$ is étale at points that are étale in its fibers. Once we show this, by Y-finiteness of D we would obtain an open $U_1 \subset Y$ containing y_1 so that $(D_1)_{U_1} \to U_1$ is étale. In particular, for all geometric points $\overline{y} \in U$ we would have that $(D_1)_{\overline{y}}$ is étale and thus (for degree reasons) has at least 3 points per irreducible component of $X_{\overline{y}}$.

We will next exploit that

$$D_{y_1} \subset \mathrm{sm}(X/Y).$$

Let $h \in \mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the local equation for $H \cap X$ near x, so the structure map $D_1 \to Y$ induces

$$\mathscr{O}_{Y,y_1} \to \mathscr{O}_{X,x}/(h) = \mathscr{O}_{D_1,x}.$$

Proving étaleness at x amounts to the natural map of completions

$$\widehat{\mathscr{O}_{Y,y_1}} \to \widehat{\mathscr{O}_{D_1,x}}$$

being an isomorphism. By design $x \in \mathrm{sm}(X/Y)$, so

$$\widehat{\mathscr{O}_{D,x}} = \widehat{\mathscr{O}_{X,x}}/(h) = \widehat{\mathscr{O}_{Y,y_1}}[\![t]\!]/(h)$$

and hence it suffices to show $h \in k^\times \cdot t + \widehat{m}_{y_1}[\![t]\!]$ (as then we would get the desired isomorphism between completions for $D \to Y$ at x and y_1 via successive approximation calculations). This task for h only involves its image in

$$\begin{aligned}\widehat{\mathscr{O}_{X,x}}/\widehat{\mathfrak{m}_{y_1}}[\![t]\!] &= \widehat{\mathscr{O}_{X,x}}/\mathfrak{m}_{y_1} \cdot \widehat{\mathscr{O}_{X,x}} \\ &= \widehat{\mathscr{O}_{X_{y_1},x}} \\ &= k[\![t]\!].\end{aligned}$$

Thus, we just need that the image of h in here is tu for a unit u. But

$$k[\![t]\!]/(h) = \widehat{\mathscr{O}_{(D_1)_{y_1},x}}$$

and

$$\mathscr{O}_{(D_1)_{y_1},x} = k$$

since $(D_1)_{y_1}$ is k-étale. Therefore, $k[\![t]\!]/(h) = k$, so h has the desired form modulo $\widehat{\mathfrak{m}_{y_1}}$.

□

By Lemma 6.6 for our arbitrary initial choice of point $y_1 \in Y(k)$ we have found H_1 and an open neighborhood U_1 of y_1 so that $D_1 := H_1 \cap X$ is Y-finite and étale over U_1. But away from U_1 we have not controlled the fiber of $D_1 \to Y$ at $y \in (Y - U_1)(k)$, and so for such y the fiber $(D_1)_y$ might not have at least 3 points on each irreducible component C of X_y (even though the *scheme* $(D_1)_y \cap C$ has large degree; perhaps it is a very fat point).

If the open set U_1 from Lemma 6.6 is equal to Y then we are done. If not, choose $y_2 \in (Y - U_1)(k)$, and run the same procedure to find $D_2 := H_2 \cap X$ that is Y-finite and étale over some open neighborhood U_2 of y_2 in Y. We can also arrange (by genericity of the choice of H_2) that H_2 does not contain any irreducible component of D_1, so the effective Cartier divisor $D_1 + D_2$ (i.e., the closed subscheme of X corresponding to $\mathscr{I}_{D_1} \cdot \mathscr{I}_{D_2}$) agrees with $(D_1 \cup D_2)_{\mathrm{red}}$ on a dense open. (Note that each D_i is generically reduced due to generic étaleness over the reduced scheme Y.) Thus $D_1 + D_2 \to Y$ is still generically étale. If $U_1 \cup U_2 \neq Y$, we continue in the same way. we conclude using noetherian induction. □

By replacing Z with $(Z \cup D)_{\mathrm{red}}$, as we may certainly do, we have arranged that property (iv)(e) From Remark 6.1 holds.

7. Passage to a union of sections

We next arrange that condition Remark 6.1(iv)(f) holds, by finally applying a (mild) generically étale alteration to Y (and adjusting X and Z in a suitable manner to be described).

Remark 7.1. If $\psi \colon Y' \to Y$ is a generically étale alteration, then the data

$$X' := \left(X \times_Y Y'\right)_{\mathrm{red}}$$
$$Z' := \left(Z \times_Y Y'\right)_{\mathrm{red}}$$
$$Y'$$

satisfies all of our running conditions except possibly that X' may not be normal. The only aspect of this that requires some thought is to justify that X' is irreducible, but this is done in a handout (the point being that flatness of the fiberwise-dense $\mathrm{sm}(X/Y)$ allows us to detect irreducible components of $X \times_Y Y'$ by looking at the generic fiber over Y', which in turn is a base change of the generic fiber of $X \to Y$ that is smooth and geometrically connected (so geometrically irreducible!).

As a simple illustration, we can take ψ to be the finite normalization $\widetilde{Y} \to Y$ to arrange that Y is normal (at the possible cost of losing normality of X).

Lemma 7.2. *By passing to a further alteration of Y, we may arrange that* (iv)(f) *of Remark 6.1 holds.*

Proof. We want to arrange for a generically étale alteration $\psi : Y' \to Y$ that

$$Z' = \cup_{1 \le j \le r} \sigma_j(Y')$$

for distinct sections $\sigma_j \colon Y' \to X'$.

Let η be the generic point of Y, so Z_η is a nonempty finite étale η-scheme. We can choose $\eta' \to \eta$ a big finite Galois point such that $(Z_\eta) \times_\eta \eta'$ is a finite disjoint union of copies of η'.

Let $\psi\colon Y' \to Y$ be the normalization of Y in the finite Galois extension $K(\eta')$ of $K(\eta) = K(Y)$. Since $Y'_\eta = \eta'$ as schemes, we can use Remark 7.1 to arrange that

$$Z_\eta = \coprod_{i=1}^{N} \eta.$$

Let $Z_1, \ldots, Z_r$ be the irreducible components of the pure-dimensional Z with reduced structure. Thus, the finite maps $Z_i \to Y$ are surjective for dimension reasons, so the generic points of the Z_i's are precisely the points of Z_η. Therefore each finite map $Z_i \to Y$ is birational. But Y is normal, so each $Z_i \to Y$ is an isomorphism. Thus, the inverse maps $Y \simeq Z_i \to X$ define sections σ_i satisfying

$$Z = \bigcup_i \sigma_i(Y)$$

as subsets in X. □

Remark 7.3. Above, we crucially used (generically étale) alterations involving function field extensions in order to split Z into a union of sections. Later we will require such alterations for less explicit (and deeper) geometric reasons.

To go further, we need to digress and review some basic facts concerning relative stable marked curves and their associated moduli spaces/stacks (which admits a "smooth scheme chart" built via Hilbert schemes, so we also need to review some facts about Hilbert schemes).

8. Stable curves

8.1. Definitions and examples.

Definition 8.1. Fix $n, g \geq 0$ so that $2g - 2 + n > 0$. (That is, either $g \geq 2$ or $g = 1$ with $n \geq 1$, or $g = 0$ with $n \geq 3$.) An *n-pointed stable genus-g curve* over a scheme S is a proper flat finitely presented map $f\colon \mathscr{C} \to S$ equipped with sections $\sigma_1, \ldots, \sigma_n \in \mathscr{C}(S)$ so that:

(1) all geometric fibers $\mathscr{C}_{\overline{s}}$ are connected semistable and the (arithmetic) genus $h^1(\mathscr{C}_{\overline{s}}, \mathscr{O})$ is equal to g,
(2) the sections σ_i are pairwise disjoint and supported in the Zariski-open S-smooth locus $\mathscr{C}^{\mathrm{sm}}$.

(3) for any geometric point $\bar{s}$ of S, any irreducible component C of $\mathscr{C}_{\bar{s}}$ isomorphic to $\mathbf{P}^1$ contains at least three **special points**: some $\sigma_i(\bar{s})$ or a point where C meets another irreducible component of $\mathscr{C}_{\bar{s}}$.

Exercise 8.2. Show that C obtained from $\mathbf{P}^1$ via n "semistable self-gluing" are semistable irreducible curves of genus n over an algebraically closed field. We can take "semistable self-gluing" to mean semistable with normalization $\mathbf{P}^1$, and the content is to prove that C can be canonically reconstructed from its normalization $\widetilde{C} \simeq \mathbf{P}^1$ equipped with the data of the pairs $\left\{\widetilde{x_i}, \widetilde{x_i'}\right\}$ over each singularity of C (and that any such finite collection of pairs gives rise to a "self-gluing" $\mathbf{P}^1 \to C$ that is initial for maps out of $\mathbf{P}^1$ having the same restriction to each k-point in such a pair).

The idea is to consider the exact sequence

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \nu_* \mathscr{O}_{\widetilde{C}} \longrightarrow \mathscr{Q} \longrightarrow 0 \tag{8.1}$$

for ν the normalization map. Here, the coherent sheaf $\mathscr{Q}$ is a skyscraper supported at C^{sing} and the semistability implies

$$\mathscr{Q} \simeq \oplus_{x \in C^{\text{sing}}} \kappa(x)$$

(which we leave as an exercise, using the description of $\widehat{\mathscr{O}}_{C,x}$ and that normalization commutes with completion in these situations, as can be proved by elementary means without recourse to hard theorems involving excellence). Passing to the cohomology sequence, we get

$$0 \longrightarrow k \longrightarrow k \longrightarrow h^1(\mathscr{O}_C) \longrightarrow h^1(\mathscr{O}_{\widetilde{C}}) \longrightarrow 0, \tag{8.2}$$

so

$$h^1(\mathscr{O}_C) = \#C^{\text{sing}}.$$

Remark 8.3. In general, if C is an irreducible semistable proper curve over an algebraically closed field then the same calculation with the normalization sequence yields

$$h^1(\mathscr{O}_C) = h^1(\mathscr{O}_{\widetilde{C}}) + \#C^{\text{sing}}.$$

Example 8.4. Consider the union $\mathbf{P}^1 \cup C$ where $\mathbf{P}^1$ has two sections and meets a nodal C_1 at a point. One can use the normalization sequence to show this has (arithmetic) genus

$$g = 1 + g(\widetilde{C_1}).$$

Exercise 8.5. Let $k = \bar{k}$.

(1) Show that a connected proper semistable curve over k with genus 0 is precisely a connected finite tree of copies of $\mathbf{P}^1$. The point is that the homology you get from the exact sequences for the normalization comes from the homology of the dual graph (so arithmetic genus 0 amounts to the dual graph having no loops; i.e. it is a tree).
(2) The only stable curve over k with $g = 0, n = 3$, is $\mathbf{P}^1$ with the ordered triple of markings $(0, 1, \infty)$ up to *unique* isomorphism.
(3) A *stable* curve of genus 1 with $n = 1$ is a single elliptic curve with a marked point or a nodal cubic (i.e., $\mathbf{P}^1$ with one semistable self-gluing) having a marked point.

    If one allows $n > 1$, there are many more possibilities. We will just describe the underlying semistable curve, since the stable curves are obtained by adding marked points. Any such curve can be written as $C \cup D_1 \cup \cdots \cup D_n$, with C, D_i defined as follows: D_i are all semistable genus 0 curves as described above (a connected finite tree of copies of $\mathbf{P}^1$) so that each D_i meets C at points $p_1, \ldots, p_n$, and D_i does not meet D_j, and C is a smooth genus-1 curve or a so-called *Néron polygon*: a curve whose normalization is a collection of copies of $\mathbf{P}^1$ so that the dual graph is a polygon (a nodal $\mathbf{P}^1$, or a banana – two copies of $\mathbf{P}^1$ glued at two points, or a triangle – three copies of $\mathbf{P}^1$ each glued to the other two at a single point – and so on).
(4) For a stable curve with irreducible component C satisfying $h^1(\mathscr{O}_C) = 1$, there must be at least one special point on C.

Remark 8.6. The two key references on the topic of stable curves are the following: the case $n = 0$ with any $g \geq 2$ is [DM, §1], and the case of arbitrary n is addressed in [Knu, pp. 161-199]. Both involve much use of coherent duality over rings (not just over fields).

8.2. **Moduli for stable curves.** A key input for the construction of a "universal stable marked curve" (for a given g and n) Will require applying the following openness result to a universal object over a variant of a Hilbert scheme:

Lemma 8.7. *Let $f \colon X \to S$ be a proper flat finitely presented map of schemes. Then*

$$U := \{s \in S : X_{\bar{s}} \text{ is a connected semistable curve of arithmetic genus } g.\}$$

is open in S.

Proof. The openness results here are special cases of general openness theorems on the base for properties of geometric fibers of flat finitely presented proper scheme maps (though in the present case one may be able to give more direct arguments; we just wish to convey that there are general principles at work here, applicable way beyond the setting of relative curves).

Openness of the set of pure 1-dimensional fibers is [EGA, IV_3, 12.2.2(ii)]. Openness for the condition for geometrically reduced and geometrically connected fibers is [EGA, IV_3, 12.2.4(vi)]. The idea of these proofs is usually by first attacking problems locally on the source (proving openness results in X without any properness hypotheses for flat finitely presented maps, by bootstrapping from constructibility results via generization considerations), and then in the proper setting the bad locus in the base is the image of the complementary closed bad locus in X.

Once we have passed to the open locus cut out by the preceding conditions, we have the equality $h^1(\mathscr{O}) = 1 - \chi(\mathscr{O})$ on geometric fibers and can invoke the constancy Zariski-locally on the base for the fibral Euler characteristic $s \mapsto \chi(X_s, \mathscr{F}_s)$ for any proper finitely presented $f : X \to S$ and S-flat finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ (see [Mum, Ch. II, §5, Cor. 1(b)] for noetherian S; one reduces to this case by some nontrivial results in EGA concerning spreading-out for flatness and properness).

Finally, what about openness for the semistability condition? We just need to prove openness of the semistable locus on X (implying openness on S via properness considerations, due to the *closed* non-semistable in X having closed image in S), and this is shown using Artin approximation in [FK, Ch. III, Prop. 2.7] as a special case of a general étale-local description for ordinary double-points in geometric fibers (in any odd relative dimension). □

Example 8.8 (Why stable curves can't be classified by a universal object over a moduli scheme)**.** The obstruction to existence of a universal stable marked curve (for a given g and n) is, roughly speaking, that there exist non-constant families with pairwise isomorphic geometric fibers. (This is a more precise version of the informal obstruction that objects can admit non-trivial automorphisms.)

Consider $g = n = 1$ and a field k with $\operatorname{char} k \neq 2$. Let E be an elliptic curve over k and consider the map

$$f \colon (E \times \mathbf{G}_m)/\langle -1 \rangle \to \mathbf{G}_m$$
$$(x, t) \mapsto t^2.$$

where the action of -1 is given by

$$(-1)\,(x,t) = (-x,-t)\,.$$

(Let's briefly discuss the existence of this quotient by the action of $\mathbf{Z}/(2)$. That action is free on geometric points since negation on $\mathbf{G}_m$ over k is fixed-point free, and for any quasi-projective scheme V over a field equipped with an action by a finite group Γ that is free on geometric points over any algebraically closed field, one can cover V by Γ-stable affine opens and thereby form a quasi-projective V/Γ for which $V \to V/\Gamma$ is a finite étale Γ-torsor; see [Mum, ?].)

We take the marking σ to be the image of the zero section $t \mapsto (0, t)$. All k-fibers of f are isomorphic to E. If there were actually a universal 1-point stable genus-1 curve over k then we could recover f as the left side of a fiber square

$$\begin{array}{ccc} X & \longrightarrow & (\mathscr{E}^{\text{univ}}, \sigma^{\text{univ}}) \\ \downarrow & & \downarrow \\ S & \xrightarrow{q} & \overline{M}_{1,1} \end{array} \tag{8.3}$$

and hence the classifying map $q : S = \mathbf{G}_m \to \overline{M}_{1,1}$ would send the entirety of $S(\bar{k})$ through a single k-point (corresponding to $(E, 0)$). But then q would have to factor through that k-point, forcing $X \to S$ to be a constant stable marked curve. The following is an instructive exercise:

Exercise 8.9. Complete the above example by verifying that X is not S-isomorphic to $E \times S$.

8.3. **Equivalent definitions of stable curves.** We'd like to explain why we require three points to be special on each $\mathbf{P}^1$, and also the reason that we require $2g - 2 + n$ to be positive. Keep in mind that we are just surveying some key highlights in the basic theory of stable curves and their moduli so that we can use these ideas to progress further into deJong's proof; hence, we will not have time to develop this material in full and so will provide literature references for various details.

Lemma 8.10. *Let X be a connected semistable curve over an algebraically closed field k, let $g := h^1(\mathscr{O})$, and let $\sigma_1, \ldots, \sigma_n \in X^{\text{sm}}(k)$ be disjoint sections. The following conditions are equivalent.*

(1) $(X; \sigma_1, \ldots, \sigma_n)_{k[\varepsilon]}$ (viewed as a curve over Spec $k[\varepsilon]$ with $\varepsilon^2 = 0$) has no nontrivial $k[\varepsilon]$-automorphism reducing to the identity modulo ε.

(2) The locally finite type k-scheme (actually always finite type, due to working with curves)

$$\underline{\text{Aut}}_{(X;\sigma_1,\ldots,\sigma_n)/k}$$

is étale.

(2′) The group Aut $(X; \sigma_1, \ldots, \sigma_n)$ *is finite.*

(3) The following two conditions hold:

(i) every irreducible component isomorphic to $\mathbf{P}^1$ *has at least* 3 *special points,*
(ii) every irreducible component with arithmetic genus at least 1 *has at least* 1 *special point.*

Remark 8.11. Note that when $2g-2+n>0$, condition (ii) in (3) automatically holds (so one never sees (ii) mentioned in literature on stable pointed curves); this is *not* the reason that $2g-2+n$ is assumed to be positive. Also, the equivalence of (1) and (2) is due to each expressing the vanishing of the tangent space at the identity on the Aut-scheme (i.e., a group scheme locally of finite type over a field is étale if and only if its tangent space at the identity point vanishes).

The fact that the Aut-scheme in (2) is really finite type and not just locally finite type can be proved in several ways, but perhaps the most conceptually satisfying way is to exploit the fact that fixing a Hilbert polynomial carves out a finite-type clopen subscheme of a Hilbert scheme in the projective setting. We will come back to this point later in the proof of Lemma 9.6 (which does not required the present considerations, so there is no circularity involved).

Proof. An automorphism as in (1) is a map

$$\begin{aligned}\phi\colon \mathscr{O}_X[\varepsilon] &\to \mathscr{O}_X[\varepsilon]\\ f+g\varepsilon &\mapsto f+(g+D(f))\,\varepsilon\end{aligned}$$

for a k-linear derivation

$$D:\mathscr{O}_X\to\mathscr{O}_X.$$

So we can view D instead as an $\mathscr{O}_X$-linear map $D:\Omega^1_{X/k}\to\mathscr{O}_X$. Hence, on X^{sm} we can view D as a vector field $\vec{v}$, since $\Omega^1_{X/k}|_{X^{\mathrm{sm}}}=\Omega^1_{X^{\mathrm{sm}}/k}$ is a vector bundle with local frame

$$\left\{\frac{\partial}{\partial t}\right\}$$

for étale $t:U\to\mathbf{A}^1_k$ for Zariski-open $U\subset X^{\mathrm{sm}}$.

If $\widehat{\phi}$ denotes the $k[\varepsilon]$-automorphism of $X_{k[\varepsilon]}$ corresponding to ϕ then the condition that $\widehat{\phi}((\sigma_j)_{k[\varepsilon]})=(\sigma_j)_{k[\varepsilon]}$ as $k[\varepsilon]$-points is exactly the condition that $\vec{v}(\sigma_j)=0$. That is, the vector field on X^{sm} associated to D has zeros at these k-points of X^{sm}. Under this constraint, we want to show that necessarily $D=0$ when (3) holds (so (3) implies (1)).

The case $n=0, g\geq 2$ is treated [DM, §14] (where (3)(ii) is never mentioned since it holds automatically, and there is no appearance of marked points). The method amounts to some calculations in coherent duality on semistable curves, generalizing the classical fact that a smooth connected proper curve over k with genus at least 2 has no nonzero global vector fields. The same method generalizes without much difficulty to show (3) implies (1) in general (with the marked points, and incorporating (3)(ii)).

The other equivalences are easier and are omitted. □

Remark 8.12. Suppose we are given a projective scheme Y over an algebraically closed field and form its Aut-scheme, which is made out of a Hom-scheme that in turn is built from the Hilbert scheme of $Y \times Y$ (by viewing endomorphisms through their graphs). The Hilbert scheme is built as a countable union of finite type k-schemes, so the same holds for this Aut-scheme. Hence, the identity component of $\underline{\mathrm{Aut}}_{Y/k}$ is of finite type (though one can also show abstractly that any locally finite type k-group scheme which is connected is in fact finite type and irreducible) and more importantly the component group Γ_Y of $\underline{\mathrm{Aut}}_{Y/k}$ is countable. But might Γ_Y have some reasonable finiteness properties, such as being finitely generated or finitely presented?

Note that if E is a non-CM elliptic curve then the Aut-scheme of $E \times E$ (without respecting $(0,0)$!) has identity component $E \times E$ via translations and component group $\mathrm{GL}_2(\mathbf{Z})$ that is finitely presented (as are all arithmetic groups). In general what can be said about the structure of Γ_Y, say if Y is a variety (i.e., reduced and irreducible)? Even the case of smooth projective surfaces Y over $\mathbf{C}$ is very mysterious!

There is one natural idea that comes quickly to mind for attacking this but almost as quickly dies: the Néron–Severi group $\mathrm{NS}(Y) = \mathrm{Pic}_{Y/k}/\mathrm{Pic}^0_{Y/k}$ is a finitely generated abelian group (Theorem of the Base, due to Néron and Lang), and Γ_Y naturally acts on this, so it provides a natural representation

$$\Gamma_Y \to \mathrm{GL}(\mathrm{NS}(Y)/\mathrm{torsion}) = \mathrm{GL}_N(\mathbf{Z}).$$

If the image were an arithmetic subgroup, hence finitely presented, that might give us a handle on the problem. Alas, Borcherds gave examples of smooth projective surfaces over $\mathbf{C}$ for which this image is *not* arithmetic, so those hopes are dashed.

I once came upon this puzzle as a side issue during some work on finiteness questions over global function fields, and lacking any idea on how to proceed with it I decided to ask around. I asked a lot of people, including Mumford, Oort, de Jong, Gabber, and so on. Nobody had any idea. What a stumper! I posed this baffler on Math Overflow to raise awareness about such a natural question. The happy ending is that John Lesieutre saw that MO question and recently came up with smooth geometrically connected counterexamples with trivial geometric étale fundamental group over any field of characteristic 0 (a bit beyond dimension 2 for now, but he has some candidates in dimension 2 as well).

8.4. **Dualizing sheaves.** We'd next like to give a criterion for ampleness of a line bundle attached to a marked semistable curve. To do so, we'll first need a brief digression into coherent duality.

Consider

$$(X; \sigma_1, \ldots, \sigma_n)$$

As in the setup for Lemma 8.10. the k-scheme X is Gorenstein, as the Gorenstein property of local Noetherian rings can be checked on the completion and we know the completions at closed points of semistable curves over $k = \overline{k}$ are either $k[\![t]\!]$ or $k[\![u, v]\!]/(uv)$. Consequently, the relative dualizing complex for X/k is an invertible sheaf.

Remark 8.13. The Cohen-Macaulay property of a finite-type scheme over a field (or more generally, but we omit that here) is equivalent to the condition that its dualizing complex in the sense of Grothendieck's theory of coherent duality (which is characterized by features not requiring properness!) is simply a sheaf (thereby called the *dualizing sheaf*), denoted $\omega_{X/k}$. The Gorenstein property (stronger than the CM condition) is equivalent to the dualizing sheaf being invertible.

The formation of the dualizing sheaf $\omega_{X/k}$ commutes with étale pullback on X (this is a general property of the dualizing complex), so one can describe $\omega_{X/k}$ using étale models. There is a useful concrete description in the case of semistable curves via 1-forms with controlled poles and residues on the normalization, but since we are not getting into gritty details of calculations with dualizing sheaves here (though they are lurking in the references to [DM] below!) we will not discuss these explicit descriptions here.

Let

$$\widetilde{\omega}_{X/k} := \omega_{X/k}\left(\sum \sigma_i\right) = \omega_{X/k} \otimes \mathscr{I}_{\sigma_1}^{-1} \otimes \cdots \otimes \mathscr{I}_{\sigma_n}^{-1}, \tag{8.4}$$

where $\mathscr{I}_\sigma$ is the ideal sheaf of σ for $\sigma \in X^{\mathrm{sm}}(k)$.

For a line bundle $\mathscr{L}$ on X we have a notion of degree defined by the condition

$$\chi(\mathscr{L}^{\otimes m}) = \deg \mathscr{L} \cdot m + \chi(\mathscr{O}_X).$$

See [BLR, §9.1] for an elegant discussion of this notion, its relation to Cartier and Weil divisors and normalization, and its properties are rather arbitrary proper curves over fields (including additivity in $\mathscr{L}$). In particular, the degree of the inverse ideal sheaf of a rational point in the smooth locus is equal to 1, as in the familiar setting of smooth connected projective k-curves. In particular, if $\mathscr{L}$ is ample then for very large m there are many global sections of $\mathscr{L}^{\otimes m}$, so $\deg \mathscr{L} > 0$ in such cases. (The converse fails when X is reducible, as $\mathscr{L}$ might have degree-0 restriction to some irreducible components. Since ampleness can be checked on the irreducible components equipped with their reduced structure, ampleness on pure curves is equivalent to positivity of the degree of the restriction to each irreducible component equipped with its reduced structure.)

Lemma 8.14. *We have* $\deg \widetilde{\omega}_{X/k} = 2g - 2 + n$, *with* $\widetilde{\omega}_{X/k}$ *as defined in* (8.4).

Proof. Coherent duality implies

$$\deg \omega_{X/k} = 2g - 2$$

For $g := h^1(\mathscr{O}_X) = 1 - \chi(\mathscr{O}_X)$. Thus, $\deg \widetilde{\omega}_{X/k} = 2g - 2 + n$. □

Lemma 8.15. *Condition* (3) *in Lemma 8.10 is equivalent to* $\widetilde{\omega}_{X/k}$ *being ample,* (*which in turn implies* $2g - 2 + n > 0$).

Proof. For $n = 0, g \geq 2$ the implication " $\Rightarrow$" is [DM, Theorem 1.2], proved via computations with coherent duality; the converse in such cases is easier. This method adapts to any (g, n) with $g \geq 2$ or when $g = 0, 1$ with with $n \geq 4$. For $g \leq 1, n \leq 3$, see [Knu, Corollary 1.10]. □

Remark 8.16. For $f : X \to S$ proper of finite presentation, an invertible sheaf $\mathscr{L}$ on X is ample over all affine opens in S (or equivalently on each member of an open affine cover of S) if and only if $\mathscr{L}_s$ on X_s is ample over every $s \in S$. The amazing implication here (in view of the absence of any flatness hypotheses, thereby preventing any use of base change theorems for coherent cohomology) is "$\Leftarrow$", which is [EGA, IV_3, 9.6.4]. This condition on $\mathscr{L}$ is called *S-ampleness*.

9. Moduli of stable curves

To work in the relative setting, we need a generalization of $\omega_{X/k}$ from the earlier considerations with semistable curves over fields. Here is "the bitter pill." For finitely presented maps with Cohen-Macaulay fibers there is a finitely presented quasi-coherent dualizing sheaf $\omega_{X/S}$ compatible with any base change on S and with étale pullback on X and having some nice properties with respect to coherent duality (that we do not have time to flesh out here). Furthermore, $\omega_{X/S}$ is invertible if all fibers X_s are Gorenstein.

Consider a proper finitely presented map

$$\begin{array}{c} X \\ \big\downarrow{\scriptstyle f} \\ S \end{array} \tag{9.1}$$

whose geometric fibers are connected semistable curves, and suppose we are also given pairwise disjoint sections $\sigma_1, \ldots, \sigma_n \in \mathrm{sm}(X/S)$.

Proposition 9.1. *In the above situation,*

$$S^{\mathrm{st}} := \{s \in S : (X_s; \{\sigma_i(s)\}) \text{ is stable } \}$$

is Zariski-open in S.

Proof. Consider the invertible sheaf $\mathscr{L} = \omega_{X/S}(\sum \sigma_i)$ whose formation is compatible with base change. (Here we are twisting by the inverse ideal sheaves of the sections σ_i, leaving it as an exercise to check that the ideal sheaf of a section through the relative smooth locus of a relative flat finitely presented curve is always *invertible* and that the formation of this ideal sheaf commutes with any base change.) Our preceding discussion in the context of semistable curves over algebraically closed fields gives the crucial equality

$$S^{\mathrm{st}} = \{s \in S : \mathscr{L}_s \text{ is ample on } X_s\}.$$

But for any proper finitely presented map $Y \to S$ and any invertible $\mathscr{L}$ whatsoever on Y, it is a general fact that

$$\{s \in S : \mathscr{L}_s \text{ is ample on } Y_s\}$$

is Zariski-open: see [EGA, IV_3 9.9.6]. □

Generalizing classical arguments with cohomology and base change by which a smooth proper relative curve with geometrically connected fibers of genus $g \geq 2$ admits a closed immersion ("tri-canonical embedding") into a projective-space bundle of dimension $5g - 6$, the preceding results on the relative setting lead to:

Theorem 9.2. *Let $(X \to S, \{\sigma_i\})$ be an n-pointed stable genus-g curve over a scheme S. For any integer $m \geq 4$, the sheaf $f_*(\mathscr{L}_{X/S}^{\otimes m})$ is a vector bundle whose formation commutes with any base change on S, and the natural map $f^* f_*(\mathscr{L}_{X/S}^{\otimes m}) \to \mathscr{L}_{X/S}$ is surjective. Moreover, the resulting natural map*

$$X \to \mathbf{P}(f_*(\mathscr{L}_{X/S}^{\otimes m}))$$

into a projective-space bundle is a closed immersion, the rank of $f_*(\mathscr{L}_{X/S}^{\otimes m})$ is equal to

$$N(n, g, m) = m(2g - 2 + n) + 1 - g.$$

There is a universal such structure equipped with the data of an isomorphism

$$\mathbf{P}(f_*(\mathscr{L}_{X/S}^{\otimes 4})) \simeq \mathbf{P}^{N(n,g,4)-1}.$$

Proof. See the handout "Universal Stable Curve" on the course website. (The only reason we use the 4th tensor power rather than the 3rd is to avoid some headaches that arise in certain cases with irreducible geometric fibers when $g \leq 1$. by artful use of the stability condition for such cases, as explained in [Knu], one can push through the use of $m = 3$ in general. For our purposes, even using $m = 538$ would be sufficient.) $\square$

The proof of the preceding result provides a universal 4-canonically embedded n-pointed stable genus g curve,

$$(\mathscr{X}; \{\tau_i\}) \longrightarrow \mathscr{S} \tag{9.2}$$

with $\mathscr{S}$ a $\mathbf{Z}$-scheme that is a quasi-projective $\mathrm{PGL}_{N(g,n,4)-1}$-torsor over an open inside the n-fold fiber product $\mathscr{Z} \times_{\mathscr{H}_\Phi} \cdots \times_{\mathscr{H}_\Phi} \mathscr{Z}$, where $\mathscr{H}_\Phi$ is Hilbert scheme for a specific projective space and a specific degree-1 polynomial $\Phi = \Phi_{g,n}(t) \in \mathbf{Q}[t]$.

Any n-pointed stable genus-g curve $f : (X; \{\sigma_i\}) \to S$ over any scheme is a pullback of

$$(\mathscr{X}; \{\tau_i\}) \to \mathscr{S}$$

Zariski-locally on S upon trivializing $f_*(\omega_{X/S}(\sum \sigma_i)^{\otimes 4})$ Zariski-locally on S; there is a $\mathrm{PGL}(N(g,n,4))$'s worth of choices. In particular, using the natural action of $\mathrm{PGL}(N(n,g,4))$ on $\mathscr{S}$ arising from its universal property, the vague dream is that the quotient $\mathscr{S} / \mathrm{PGL}(N(n,g,4))$ (whatever it might mean) is a moduli space for such data of n-pointed stable genus-g curves $(X \to S, \{\sigma_i\})$ over variable base schemes.

Remark 9.3. The preceding method is often used to build moduli spaces: we have something too structureless to make a geometric construction, so we put extra structure on it to connect it to something we can work with (such as a trivialization of a vector bundle or a projective-space bundle, or a level-structure on an abelian variety, etc.). Then, we quotient the output about by a suitable group action to obtain the desired moduli space for the original more structureless data.

There are two potential problems with the preceding dream involving $\mathscr{S} / \mathrm{PGL}(N(n,g,4))$:

(1) The $\mathrm{PGL}(N(n,g,4))$-action on $\mathscr{S}$ is very far from free (say on geometric points with values in a fixed but arbitrary algebraically closed

field), since stable curves (and even smooth curves) can have non-trivial automorphisms. Namely, by using a "4-canonical embedding", if the data $(X_0, \{s_i\})$ over an algebraically closed field k admits non-trivial automorphisms then such automorphism transfer to automorphisms of the ambient projective space preserving the marked curve inside that projective space. This gives a nontrivial element of $\mathrm{PGL}(N(n, g, 4) - 1)(k)$ that fixes the point in $\mathscr{S}(k)$ corresponding to $(X_0, \{s_i\})$ equipped with the trivialization of its ambient projective space.

(2) Even on a dense open of $\mathscr{S}$ where the PGL-action is free (if such were to exist, though in this case does not) it is not at all clear how to make sense of such a quotient as a scheme. This gets involved with very difficult matters involving "GIT over $\mathbf{Z}$" (not to be confused with GIT-hub), but anyway is totally inadequate here since we cannot ignore the stable marked curves with nontrivial automorphisms.

Instead, we require a "geometry of functors" (or of fibered categories), which is Artin's approach to moduli problem: we abandon trying to make actual schemes and instead identify a class of functors (or fibered categories) near enough to schemes that we can develop for them much of the familiar machinery of algebraic geometry. Of course, making definitions is not enough: one also needs an "EGA" for these things (largely done in the case of algebraic spaces by Donald Knutson in his PhD thesis under Artin).

Definition 9.4. For a scheme S, we let $\overline{\mathscr{M}}_{g,n}(S)$ denote the category of n-pointed stable genus g curves $(X, \underline{\sigma})$ over S, using only isomorphisms over S as the morphisms (so this category is a groupoid).

For maps $S' \to S$ we have a base change functor

$$b_{S'/S} : \overline{\mathscr{M}}_{g,n}(S) \to \overline{\mathscr{M}}_{g,n}(S')$$

and for $S'' \to S' \to S$ we have an isomorphism of functors

$$b_{S''/S'} \circ b_{S'/S} \simeq b_{S''/S},$$

meaning an isomorphism of the two maps

$$\overline{\mathscr{M}}_{g,n}(S) \rightrightarrows \overline{\mathscr{M}}_{g,n}(S''),$$

and an evident associativity condition relative to $S''' \to S'' \to S' \to S$.

Remark 9.5. The assignment $S \rightsquigarrow \overline{\mathscr{M}}_{g,n}(S)$ is a sheaf of categories (aka stack) for the fpqc topology in the following sense:

Suppose we are given an fpqc map $S' \to S$ and a pair

$$(X', \theta)$$

for $X' \in \overline{\mathscr{M}}_{g,n}(S')$ and θ a **descent datum** relative to $S' \to S$. Let us now explain what we mean by a descent datum. By definition, θ is an isomorphism

$$\theta : p_1^*(X') \simeq p_2^*(X')$$

in the category $\overline{\mathscr{M}}_{g,n}(S' \times_S S')$, where p_1, p_2 are the base change functors along the two projections $S' \times_S S' \rightrightarrows S'$, satisfying the cocycle condition on the induced isomorphisms in

$$\overline{\mathscr{M}}_{g,n}\left(S' \times_S S' \times_S S'\right)$$

with respect to the three pullbacks

$$S' \times_S S' \times_S S' \to S' \times_S S'.$$

This is a commutativity condition on maps, which we won't write out here.

For any such (X', θ), the "stack" condition is that there exists a unique object $X \in \overline{\mathscr{M}}_{g,n}(S)$ *up to unique isomorphism* equipped with an isomorphism $X_{S'} \simeq X'$ in $\overline{\mathscr{M}}_{g,n}(S')$ carrying the canonical S'/S-descent datum on $X_{S'}$ (arising from X) over to θ.

In our context with objects $(X, \underline{\sigma})$ we have the associated line bundle $\widetilde{\omega}_{X/S} = \omega_{X/S}(\sum \sigma_i)$ that is S-ample on X and *canonically attached* to the given data over S in the sense of being functorial with respect to base change on S and canonically compatible with *all* isomorphisms in such $(X, \underline{\sigma})$. In particular, $\widetilde{\omega}_{X/S}$ is naturally compatible with *any* descent datum on $(X, \underline{\sigma})$, so by Grothendieck's effectivity criterion for fpqc descent in the presence of ample line bundles compatible with the descent datum it follows that for any fpqc $S' \to S$ the above unique effective descent property holds for any $(X', \underline{\sigma}') \in \overline{\mathscr{M}}_{g,n}(S')$ equipped with a descent datum relative to $S' \to S$.

We therefore say $\overline{\mathscr{M}}_{g,n}$ is a **stack** (in groupoids) for the fpqc topology on the category of schemes.

We have the following key lemma.

Lemma 9.6. *The diagonal map*

$$\begin{aligned} \Delta \colon \overline{\mathscr{M}}_{g,n} &\to \overline{\mathscr{M}}_{g,n} \times \overline{\mathscr{M}}_{g,n} \\ x &\mapsto (x, x) \end{aligned}$$

is relatively representable in quasi-compact and even finite type scheme maps.

Proof. For any S we have the fiber square

$$\begin{array}{ccc} \underline{\mathrm{Isom}}_S(x,y) & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle (x,y)} \\ \overline{\mathscr{M}}_{g,n} & \xrightarrow{\Delta} & \overline{\mathscr{M}}_{g,n} \times \overline{\mathscr{M}}_{g,n} \end{array} \tag{9.3}$$

for $x, y \in \overline{\mathscr{M}}_{g,n}(S)$, by definition of fiber products of fibered groupoids, where the upper-left is the Isom-functor (carrying any S-scheme T to the set $\mathrm{Isom}_T(x_T, y_T)$).

Grothendieck proved Isom-functors are represented by locally finitely presented S-schemes, constructing such Isom-schemes from Hom-schemes that in turn are built inside Hilbert schemes. The point is that if X and Y are the curves underlying x and y then one builds $\mathrm{Isom}_S(x,y)$ inside $\mathrm{Hilb}_{X\times Y/S}$ via the graphs of isomorphisms $\Gamma_f \to X \times Y$. Since Hilbert schemes are built (in the presence of a relatively ample line bundle) as a countable disjoint union of finitely presented schemes over the base (even quasi-projective, at least Zariski-locally on the base), the same holds for the Isom-schemes by design.

The crucial point is to show that in our setting with *curves*, the relevant Isom-schemes are even finite type over the base, not merely locally of finite type. This amounts to controlling Hilbert polynomials of graphs of isomorphisms. More specifically, for a field k and $(X, \sigma), (X', \sigma') \in \overline{\mathscr{M}}_{g,n}(k)$ we claim that the graph $\Gamma_f \subset X \times X'$ of any isomorphism

$$f : (X, \underline{\sigma}) \simeq (X', \underline{\sigma}')$$

has precisely one possibility for its Hilbert polynomial with respect to the ample line bundle

$$\mathscr{L} := p_1^*(\widetilde{\omega}_{X/k}) \otimes p_2^*(\widetilde{\omega}_{X'/k})$$

on $X \times X'$: it must be $\Phi(2t)$ (where $\Phi = \Phi_{g,n}$ is as in our earlier discussion).

This comes down to an elementary calculation: via the identification of Γ_f with X using the inclusion $(1, f) : X \to X \times X'$, we have

$$\begin{aligned} \mathscr{L}|_{\Gamma_f} &\simeq \widetilde{\omega}_{X/k} \otimes f^*\widetilde{\omega}_{X'/k} \\ &\simeq \widetilde{\omega}_{X/k}^{\otimes 2}. \end{aligned}$$

□

Proposition 9.7. *The map $\mathscr{S} \to \overline{\mathscr{M}}_{g,n}$ is a representable in smooth scheme covers; more precisely, for $S \to \overline{\mathscr{M}}_{g,n}$ corresponding to an n-pointed stable genus-g curve*

$f : (X, \underline{\sigma}) \to S$, the fiber product

$$\begin{array}{ccc} F & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathscr{S} & \longrightarrow & \overline{\mathscr{M}}_{g,n} \end{array} \tag{9.4}$$

is a scheme that is a $\mathrm{PGL}(N(g,n,4))$*-torsor over* S.

In particular, $\overline{\mathscr{M}}_{g,n}$ *is a quasi-separated Artin stack of finite type over* $\mathbf{Z}$.

Proof. By the universal property of the structure over $\mathscr{S}$ that underlies the definition of the bottom horizontal arrow (via "forgetting" the trivialization of a projective-space bundle), the fiber product is the Isom-functor over S for $\mathbf{P}^{N(n,g,4)-1}$ and $\mathbf{P}(f_*(\widetilde{\omega}_{X/S}^{\otimes 4}))$. This is a $\mathrm{PGL}(N(g,n,4))$-torsor over S.

By definition, a quasi-separated Artin stack of finite type over $\mathbf{Z}$ is precisely a stack in groupoids M for the fppf (let alone fpqc) topology on the category of schemes such that (i) $\Delta_M : M \to M \times M$ is represented in *quasi-compact* scheme maps, and (ii) there exists a "smooth cover by a scheme of finite type over $\mathbf{Z}$": a map $Y \to M$ from a $\mathbf{Z}$-scheme Y of finite type such that for any scheme S and map $S \to M$ corresponding to an object in $M(S)$ the fiber product $Y \times_M S$ is a scheme that is smooth surjective onto S. Hence, $\overline{\mathscr{M}}_{g,n}$ is an Artin stack of the asserted type. □

In [DM] the notion of Artin stack was not discussed and instead the smooth scheme cover of $\overline{\mathscr{M}}_{g,n}$ was (without too much explanation) "sliced" down to an étale cover by a scheme. This slicing ultimately relied on the fact that Aut-schemes at geometric points are étale, a property we noted in Lemma 8.10. The same technique works more generally:

Theorem 9.8. *If $\mathscr{X}$ is an Artin stack of finite presentation over a scheme T and its finite-type Aut-schemes at geometric points are étale then $\mathscr{X}$ admits an étale cover by a scheme; i.e., it is a Deligne–Mumford stack.*

We note that "finite presentation" includes quasi-separatedness (so $\overline{\mathscr{M}}_{g,n}$ is finitely presented over $\mathbf{Z}$, which is stronger than "finite type" precisely because of the quasi-compactness of the diagonal that we have seen involved an actual argument, not just trivialities). The Aut-schemes are locally finite type due to the Artin stack being locally of finite presentation over the base, and they are finite type (or equivalently, quasi-compact) due to the quasi-separatedness of the Artin stack. Indeed, if k is a field and $x \in \mathscr{X}(k)$ is an object then its Aut-functor is the base change of $\Delta_{\mathscr{X}_k/k}$ along $(x, x) : \mathrm{Spec}\,(k) \to \mathscr{X}_k \times \mathscr{X}_k$:

$$x \times_{\mathscr{X}} x = \mathrm{Spec}\,(k) \times_{\mathscr{X}_k \times_k \mathscr{X}_k} \mathscr{X}_k.$$

Proof. To prove Theorem 9.8, we first make an important observation about the fiber product

$$\mathscr{X} \times_{\Delta_{\mathscr{X}}, \mathscr{X} \times_T \mathscr{X}} S$$

for any T-scheme S and T-map $(x, y) : S \to \mathscr{X} \times \mathscr{X}$: this is not only a finitely presented algebraic space over S (as follows from the very definition of $\mathscr{X}$ being a finitely presented Artin stack over T), but has *vanishing* relative Ω^1, or in other words is *unramified* over S in the sense of Grothendieck.

This is a property that is sufficient to check on geometric fibers over S, so it comes down to the property that for any algebraically closed field k and objects $x, y \in \mathscr{X}(k)$ the finite type Isom-scheme $\underline{\mathrm{Isom}}(x, y)$ is étale. But if non-empty then (since $k = \overline{k}$) this is a copy of the Aut-scheme $\underline{\mathrm{Aut}}(x)$ that is étale by hypothesis. With the unramifiedness of $\Delta_{\mathscr{X}/T}$ established, the Deligne–Mumford property of $\mathscr{X}$ is exactly [LMB, Thm. 8.1]. □

The preceding work yields the following result of Deligne and Mumford that we shall use extensively:

Corollary 9.9. *The finitely presented Artin stack* $\overline{\mathscr{M}}_{g,n} \to \mathrm{Spec}\,(\mathbf{Z})$ *is actually a Deligne-Mumford stack.*

Here is an important refinement:

Theorem 9.10. *The finitely presented Deligne-Mumford stack* $\overline{\mathscr{M}}_{g,n}$ *over* $\mathbf{Z}$ *is smooth and proper.*

The proof involves an induction on n (the base cases being $n = 0, g \geq 2$ and $(n, g) = (1, 1), (3, 0)$, with the base cases for $g \geq 2$ involving serious work in deformation theory in [DM, §1]). We'll say a bit about the proof soon, especially a beautiful geometric idea of Knudsen.

Question 9.11. Where are we at?

Recall that we have $X \to Y$ with sections $\tau_1, \ldots, \tau_n$ satisfying conditions coming out of the 3-point lemma (and $Z = \cup \tau_i(Y)$). For a dense open $U \subset Y$ over which X is smooth and the τ_i's are pairwise disjoint, this corresponds to a map $h : U \to \mathscr{M}_{g,n} \subset \overline{\mathscr{M}}_{g,n}$ into the open substack classifying smooth marked curves. Inspired by the standard device of extending the domain of a rational map via blow-up of the source, we can dream of altering Y so that h extends to a map $\overline{h} : Y \to \overline{\mathscr{M}}_{g,n}$. This is a *geometric problem*.

Once that is done, the plan is to exploit the "3-point lemma" property of the τ_i's to show that the resulting pullback by $\overline{h}^*$ of the universal marked stable family, which agrees over U with $(X, \underline{\tau})|_U$, can dominate $(X, \overline{\tau})$ over the entirety of Y via a birational map (possibly after a further alteration of Y). Then we would be in very good shape: studying a semistable fibration with smooth geometrically connected generic fiber over Y of dimension $d-1$. Then we apply *induction* to alter Y to be smooth, and so are brought to such a fibration over a *smooth* base; that seems very tractable for even running a concrete resolution of singularities via blow-ups! (But don't forget: we need to keep track of $Z = \cup \tau_i(Y)$ throughout this process.).

Before returning to our regularly scheduled program of de Jong's proof, we say a few things concerning the proof of:

Theorem 9.12 (Deligne-Mumford, Knudsen)**.** *The finitely presented Deligne-Mumford stack $\overline{\mathscr{M}}_{g,n}$ over $\mathbf{Z}$ is smooth and proper over $\mathbf{Z}$.*

Remark 9.13. Smoothness is detected by deformation rings, which are completions of strictly henselian local rings at points of $\overline{\mathscr{M}}_{g,n}$. (These deformation rings might not exist at points for Artin stacks in general, though there are versal deformations.) Properness is detected by a valuative criterion.

Proof. We first discuss the case $n = 0, g \geq 2$. We then induct on $n \geq 0$ with fixed $g \geq 2$. At the end we treat the cases with $g \leq 1$.

The case $n = 0, g \geq 2$ is treated in [DM, §1]. The issue for smoothness is to understand the deformation theory of reducible semistable curves. This is used to compute the deformation rings to check $\mathbf{Z}$-smoothness. In fact, they even describe the deformation rings for $\mathscr{M}_{g,n} \subset \overline{\mathscr{M}}_{g,n}$, and the description of deformation rings yields $\overline{\mathscr{M}}_{g,n} - \mathscr{M}_{g,n}$ is a $\mathbf{Z}$-flat Cartier divisor, so $\mathscr{M}_{g,n} \subset \overline{\mathscr{M}}_{g,n}$ is fiberwise dense over Spec $\mathbf{Z}$ (which will be crucial for the valuative criterion).

Using the uniqueness of stable models, Deligne and Mumford showed the quasi-finite <u>Isom</u>-schemes for stable genus-g curves are proper (hence finite). This expresses properness for the quasi-compact diagonal of $\overline{\mathscr{M}}_{g,n}$, so this stack is separated. (The diagonal for an Artin stack is represented in algebraic spaces, and the *definition* of separatedness for stacks is that the diagonal is proper. Since the valuative criteria for separatedness and properness continue to hold for quasi-separated algebraic spaces, it follows that the usual formulation of the valuative criterion for separatedness holds for Artin stacks.)

There is a Chow's Lemma for separated DM stacks of finite type over a noetherian ring A, with the birational condition relaxed to a generically étale condition. This is recorded in the handout on avoiding Gabber's Theorem, and it gives that for $\mathscr{M}$ a DM stack separated and finite type over A, there exists a surjective proper generically étale map $Y \to \mathscr{M}$ with Y quasi-projective over A.

This yields a valuative criterion for properness of $\mathscr{M}$ in which we allow finite extensions on the discrete valuation ring (which can be taken to be complete with an algebraically closed residue field). The idea is that by Chow's lemma, $\mathscr{M}$ fails to be proper if and only if the quasi-projective Y is not proper, and the latter provides a point outside Y that is contained in the closure of Y in some projective space over A. We pick a map to Y from the spectrum of a discrete valuation ring in a such a way that the generic point maps into the dense open locus étale over $\mathscr{M}$ and the closed point lands outside Y.

Exercise 9.14. In the setting of a map of schemes, if we weaken the valuative criterion to only require that maps extend after a finite extension on the discrete valuation ring then in fact the map extended before extension of the discrete valuation ring. Hence, when the valuative criterion of properness for stacks is specialized to schemes it recovers exactly the criterion usually used for schemes.

To verify the valuative criterion of properness for $\overline{\mathscr{M}}_{g,n}$, it is enough to consider discrete valuation rings whose generic point maps into $\mathscr{M}_{g,n}$ The semistable reduction theorem for smooth geometrically connected curves of genus $g \geq 2$ over the fraction field of a discrete valuation ring R involves exactly such an extension on R. In [DM, §2] it is shown in cases with $R/\mathfrak{m}$ algebraically closed (as is sufficient for us) how to find a stable R-model after a suitable finite extension on $\operatorname{Frac}(R)$; this construction rests on manipulation of the minimal regular proper model, guided by the theory of abelian varieties (especially work of Raynaud relating the Néron model of the Jacobian to the minimal regular proper model).

Now, for a fixed g, we will induct on n. Knudsen's strategy is to show that the fibered category $\overline{\mathscr{M}}_{g,n+1}$ is isomorphic to the universal curve $\mathscr{Z}_{g,n}$ over $\overline{\mathscr{M}}_{g,n}$.

Warning 9.15. We have to describe the map $\overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$; this entails a bit of a subtlety because simply forgetting a point may lose stability.

As motivation, consider $Y \to S$ proper and finitely presented (even projective) with universal family

$$\begin{array}{ccc} \mathscr{Z} & \longrightarrow & Y \times \operatorname{Hilb}_{Y/S} \\ & \searrow \quad \swarrow & \\ & \operatorname{Hilb}_{Y/S} & \end{array} \tag{9.5}$$

What is the functor of points of $\mathscr{Z}$?

Lemma 9.16. *The functor of points of $\mathscr{Z}$ is*

$$\mathscr{Z}(T) = \{(Z, \sigma) : Z \in \operatorname{Hilb}_{Y/S}(T), \sigma \in Z(T)\}.$$

That is, $Z \subset Y_T$ is finitely presented and T-flat and σ is a section over T of the map $Z \to T$.

Proof. To give

$$\begin{array}{ccc} T & \longrightarrow & \mathscr{Z} \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{9.6}$$

amounts to giving a map

$$\begin{array}{ccc} T & & \mathscr{Z} \\ & h \searrow \quad \swarrow \phi & \\ & S & \end{array} \tag{9.7}$$

and a lift of h through ϕ The map h is precisely the T-flat $Z \subset Y_T$, with $Z = T \times_{h, \mathrm{Hilb}_{Y/S}, \phi} \mathscr{Z} \subset Y_T$. But then to give a section $\sigma : T \to Z$ over T is precisely to give a map

$$T \to \mathscr{Z}$$

lifting h through ϕ. □

Similarly to Lemma 9.16, $\mathscr{Z}_{g,n}$ is a fibered category of data

$$((X; \sigma_1, \ldots, \sigma_n), \sigma_{n+1})$$

Beware that σ_{n+1} may not be in $\mathrm{sm}(X/S)$ and may meet some σ_i for $i \leq n$. Thus, this data is not an $(n+1)$-pointed stable curve in general. To get around this issue, Knudsen defined contraction and stabilizations functors. That is, he defined maps of fibered categories

(1) $\mathscr{Z}_{g,n} \to \overline{\mathscr{M}}_{g,n+1}$ (which needs "stabilization") and
(2) $\overline{\mathscr{M}}_{g,n+1} \to \mathscr{Z}_{g,n}$ forgetting σ_{n+1} (which needs "contraction").

The idea of the second map is that if forgetting a marked point on a $\mathbf{P}^1$-component in a geometric fiber ruins stability then we should contract that fibral $\mathbf{P}^1$-component. To do this in the relative setting for $f : X \to S$ we have

$$X = \mathrm{Proj}\left(\bigoplus_{m \geq 0} f_*(\omega_{X/S}(\sum_{i=1}^{n+1} \sigma_i)^{\otimes m})\right)$$

by S-ampleness of $\omega_{X/S}(\sum_{i=0}^{n+1}\sigma_i)$, so we are led to try defining a "relative contraction" by

$$c(X) := \text{Proj}\left(\bigoplus_{m\geq 0} f_*(\omega_{X/S}(\sum_{i=1}^{n}\sigma_i)^{\otimes m})\right)$$

with sections

$$\overline{\sigma}_i : S \to X \to c(X).$$

One has to check this is is a semistable S-curve whose formation commutes with base change and has the desired effect over algebraically closed fields, and that forgetting $\overline{\sigma}_{n+1}$ yields a stable n-pointed genus-g curve.

Defining stabilization as a functor in the other direction involves blow-ups and more work (to ensure good behavior under base change, stability, and so on). These two operations turn out to be naturally inverse to each other, so we obtain an isomorphism

$$\begin{array}{ccc} \overline{\mathscr{M}}_{g,n+1} & \xrightarrow{\simeq} & \mathscr{Z}_{g,n} \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & \overline{\mathscr{M}}_{g,n} & \end{array} \tag{9.8}$$

with

$$\pi\left(X;\sigma_1,\ldots,\sigma_{n+1}\right) = \left(c(X);\overline{\sigma}_1,\ldots,\overline{\sigma}_n\right).$$

This gives that π is proper, so $\overline{\mathscr{M}}_{g,n}$ is $\mathbf{Z}$-proper for all $n \geq 0$ (with the fixed value of $g \geq 2$) by induction.

What about $\mathbf{Z}$-smoothness of $\overline{\mathscr{M}}_{g,n}$ when $g \geq 2$? Proceeding by induction on $n \geq 0$ with fixed $g \geq 2$, by the preceding isomorphism it is the same to show that $\mathscr{Z}_{g,n}$ is $\mathbf{Z}$-smooth when $\overline{\mathscr{M}}_{g,n}$ is $\mathbf{Z}$-smooth. The issue is local at the geometric points ξ of $\mathscr{Z}_{g,n}$ is singular in the fibers over $\overline{\mathscr{M}}_{g,n}$. Knudsen's deformation theory analysis and Artin approximation (to be discussed later) show that étale locally $f : \mathscr{Z}_{g,n} \to \overline{\mathscr{M}}_{g,n}$ near ξ looks like

$$\begin{array}{c} \text{Spec}\left(R[t,x,y]/(xy-t)\right) \\ \big\downarrow \\ \text{Spec}\, R[t] \end{array} \tag{9.9}$$

for suitable rings R, with ξ corresponding to $(x,y,t) = (0,0,0)$ over some $s \in \text{Spec}\, R$. The ring $R[t]$ is $\mathbf{Z}$-smooth near $t = 0$ over s by induction on n, since it shares an étale neighborhood in common with $(\overline{\mathscr{M}}_{g,n}, f(\xi))$, so

R is $\mathbf{Z}$ smooth near s because $\operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$ is a smooth cover. Note that although $\operatorname{Spec}(R[t,x,y]/(xy-t))$ is not smooth over $\operatorname{Spec} R[t]$, it is smooth over $\operatorname{Spec} R$! Hence, $R[t,x,y]/(xy-t)$ is smooth over $\mathbf{Z}$ near $(0,0,0)$ over s. The latter shares an étale neighborhood in common with $(\xi, \mathscr{Z}_{g,n})$, so $\mathbf{Z}$-smoothness of $\mathscr{Z}_{g,n}$ is proved. The cases with $g \geq 2$ (and any $n \geq 0$) are now settled.

The cases with $g = 1$ (so $n \geq 1$) proceed by induction in exactly the same way once the base case $n = 1$ is settled. The smoothness and properness of $\overline{\mathscr{M}}_{1,1}$ over $\mathbf{Z}$ was proved by Deligne and Rapoport in their work on generalized elliptic curves.

Finally, suppose $g = 0$ (so $n \geq 3$). In this case one can likewise carry out the same inductive arguments after the case $n = 3$ is settled. In that base case a little miracle happens: $\overline{\mathscr{M}}_{0,3} = \operatorname{Spec} \mathbf{Z}$! This says that the only object over *any* ring (or equivalently over any scheme) is the automorphism-free object

$$(\mathbf{P}^1, \{0, 1, \infty\}).$$

That this is the only object over an algebraically closed field was noted some time ago, and it then follows that the same holds over any field. Thus, any such object $(X; \sigma_1, \sigma_2, \sigma_3)$ over any base S has $X \to S$ smooth with its geometrically connected fibers of genus 0, so X is a Zariski $\mathbf{P}^1$-bundle by deformation-theoretic arguments with coherent base change. The 3 markings rigidify this uniquely up to unique isomorphism over any local ring, and then Zariski-locally on the base by spreading-out, and then globally due to the local rigidity. □

10. Application of moduli stacks

For our needs, it is the properness of $\overline{\mathscr{M}}_{g,n}$ over $\operatorname{Spec} \mathbf{Z}$ rather than its smoothness that matters. We discussed its smoothness both for general awareness and because knowledge of smoothness aspects of this stack was used in the proof of properness (to justify the sufficiency of checking the valuative criterion when the generic point lands in $\mathscr{M}_{g,n}$).

We shall use this proper stack to make a generically étale alteration on Y so that the restriction of $(X, \underline{\tau}) \to Y$ over some dense open $U \subset Y$ extends to a *stable* n-pointed genus-g family $(\mathscr{C}; \underline{\sigma})$ over Y. We'll then have two families over Y, the stable family and the "3-point lemma" family, and an isomorphism between them over U. This isomorphism will be extended to a proper birational map $\mathscr{C} \to X$ from the stable family onto the "3-point lemma" family after a suitable further modification of Y; it is in constructing this Y-map $\mathscr{C} \to X$ that the true importance of the 3-point lemma will be seen.

Recall that $Z = \cup_{i=1}^n \sigma_i(Y) \subset X$ for $f : X \to Y$ with sections σ_i collectively satisfying the "3-point" property; in particular, $n \geq 3$. By design, for some dense open $U \subset Y$ the map $X_U \to U$ is smooth with geometrically connected fibers of some genus $g \geq 0$.

We can shrink U if necessary so that $\{\sigma_i|_U\}$ are pairwise disjoint, since $\{\sigma_i(\eta_Y)\}$ are pairwise distinct. Since $2g - 2 + n \geq n - 2 > 0$, it makes sense to consider the universal stable marked curve

$$(\mathscr{Z}_{g,n}, \underline{\zeta}) \to \overline{\mathscr{M}}_{g,n}$$

with $\overline{\mathscr{M}}_{g,n}$ now denoting the proper stack over our field $k = \overline{k}$ obtained by base change from $\mathbf{Z}$. (This base change has the same moduli-theoretic meaning on the category of k-schemes.) Note that

$$(X_U \to U, \underline{\sigma}|_U)$$

arises via pullback

$$\begin{array}{ccc} (X_U, \underline{\sigma}|_U) & \longrightarrow & \left(\mathscr{Z}_{g,n}, \underline{\zeta}\right) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\phi} & \overline{\mathscr{M}}_{g,n} \end{array} \tag{10.1}$$

and ϕ factors through $\mathscr{M}_{g,n}$ since the fibers over U are smooth.

Here is our first goal:

Goal 10.1. Our dream (which will not quite be fulfilled) is to find a U-modification fiber square

$$\begin{array}{ccc} U' & \longrightarrow & Y' \\ \downarrow \simeq & & \downarrow \\ U & \longrightarrow & Y \end{array} \tag{10.2}$$

so that there exists a commuting triangle

$$\begin{array}{ccc} U' & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & \overline{\mathscr{M}}_{g,n} & \end{array} \tag{10.3}$$

or in other words

$$(X_{U'}, \underline{\sigma}|_{U'})$$

extends to a stable family $(\mathscr{C}, \underline{\tau})$ over Y', and then further normalize Y'. (The reason to express this task in terms of extending a map, rather than in terms of its raw moduli-theoretic meaning, is because extending a map is a more familiar type of geometric problem and is the viewpoint that will lead us to a solution.)

If $\overline{\mathscr{M}}_{g,n}$ were a scheme we could just take the graph $Y' = \overline{\Gamma}_\phi \subset Y \times \overline{\mathscr{M}}_{g,n}$ (which is a U-modification because its restriction over U is the graph $\Gamma_\phi \to U \times \overline{\mathscr{M}}_{g,n}$ that is a closed immersion since $\overline{\mathscr{M}}_{g,n}$ is separated); note that Y' is Y-proper because $\overline{\mathscr{M}}_{g,n}$ is proper.

Warning 10.2. There are at least two problems with forming this graph closure, due to $\overline{\mathscr{M}}_{g,n}$ being a stack and not a scheme:

(1) the diagonal $\Delta_{\overline{\mathscr{M}}_{g,n}}$ is really not even monic (let alone not a closed immersion), yet the graph of ψ is a base change of this diagonal, so this graph is generally not a substack of $U \times \overline{\mathscr{M}}_{g,n}$.
(2) Even if we ignore the first problem by simply forming the "schematic image" of the graph (which is unlikely to be at all useful), this is merely a stack and not a scheme.

These two issues are overcome in the 1-page handout that makes $Y' \to Y$ as a generically étale alteration (so not generally as a modification); the version of Chow's Lemma for DM stacks involves such an alteration rather than a birational map. The appearance of alterations here is no surprise, much as the valuative criterion for properness of $\overline{\mathscr{M}}_{g,n}$ requires making a finite separable extension on the fraction field of a discrete valuation ring in order to extend the map as in the valuative criterion.

The upshot is that after a suitable generically étale alteration on Y (and then further normalizing it) we can assume there exists a stable family

$$(\mathscr{C}, \underline{\tau}) \to Y$$

and an isomorphism

$$\psi : (\mathscr{C}, \underline{\tau})_U \simeq (X, \underline{\sigma})_U$$

over U.

Here is a further goal.

Goal 10.3. At the cost of a U-modification (or maybe a generically étale alteration) $Y' \to Y$ and then replacing X with $(X \times_Y Y')_{\text{red}}$ as usual (which preserves all of our running hypotheses upon arranging for Y' to also be normal), we want to extend ψ to a Y-map

$$\widetilde{\psi} : \mathscr{C} \to X$$

that is necessarily birational (as it is an isomorphism over U) and satisfies $\tau_i \mapsto \sigma_i$ because it does so over U.

Remark 10.4. Note that $\mathscr{C}$ is Y-proper with a Y-ample line bundle, so since Y is projective over k it follows that $\mathscr{C}$ is also projective over k. Thus, $\mathscr{C}$ is a projective variety (it is integral since it is semistable over Y with smooth geometrically connected generic fiber).

To make such a $\widetilde{\psi}$ after a suitable change of Y, we will use the stability of

$$(\mathscr{C}, \underline{\tau})$$

and the "3-point" property of

$$(X, \underline{\sigma}).$$

Often, when one wants to extend a map to a proper target after a modification of the source, a natural strategy is to pass to the closure of the graph in an appropriate fiber product. We will do the same here.

Goal 10.5. Consider the schematic closure

$$T := \overline{\Gamma}_\psi \subset \mathscr{C} \times_Y X.$$

Observe that $T_U = \Gamma_\psi \subset \mathscr{C}_U \times_U X_U$ so that $T_U \to U$ is smooth with geometrically connected fibers of dimension 1. The ideal situation is that the Y-morphism $\mathrm{pr}_1 : T \to \mathscr{C}$ is an isomorphism. We know pr_1 is at least proper birational, since it is an isomorphism over U.

We'll see later that if Y is normal (as we may arrange) then $\mathscr{C}$ is normal as well (using Serre's "R_1 + S_2" criterion). More generally, it will be shown that any semistable curve over a normal base with smooth generic fiber has normal total space. Granting this, it would suffice for $\mathrm{pr}_1 : T \to \mathscr{C}$ to be quasi-finite (and hence finite, by properness) since it would then be a finite birational map from a variety to a normal variety and thus an isomorphism. Such quasi-finiteness is a geometric problem for $T_{\overline{y}} \subset \mathscr{C}_{\overline{y}} \times X_{\overline{y}}$ for all geometric points $\overline{y} \in Y$.

To make progress on that geometric problem for $T_{\overline{y}}$, there are at least two things we must address beforehand:

(1) Is each $T_{\overline{y}}$ at least a curve (rather than of dimension 0, or filling up $\mathscr{C}_{\overline{y}} \times X_{\overline{y}}$)?
(2) If so, is each $T_{\overline{y}}$ connected?

The key issues this will come down to are to arrange that:

(1') T is Y-flat,
(2') $T \to Y$ is its own Stein factorization.

Remark 10.6. We do have a bit of geometric information - the sections

$$(\tau_i, \sigma_i) : Y \to \mathscr{C} \times_Y X$$

factor through $T = \overline{\Gamma}_\psi$ since we can check that over U. Therefore, the fiber

$$T_{\overline{y}} \subset \mathscr{C}_{\overline{y}} \times X_{\overline{y}}$$

contains the points

$$(\tau_i(\overline{y}), \sigma_i(\overline{y}))$$

for all i. This will provide some geometric control on the possibilities due to the stability and 3-point conditions on these markings, provided that $T_{\overline{y}}$ really is a connected curve.

Lemma 10.7. *In the above setup, the map $T \to Y$ is its own Stein factorization.*

(Recall that we normalized Y before forming T.)

Proof. This is a map of projective varieties over k and over U it is $\Gamma_\psi \to U$ or equivalently $X_U \to U$. Therefore, the map $\mathscr{O}_Y \to h_*\mathscr{O}_T$ is an equality over U. But, $h_*\mathscr{O}_T$ is a coherent sheaf of domains that coincides with $\mathscr{O}_U$ over U, so Hence, $h_*\mathscr{O}_T = \mathscr{O}_Y$ since Y is normal. □

The preceding lemma accomplishes $(2')$ in Goal 10.5, and hence (2) in Goal 10.5. To show (1) and $(1')$ in Goal 10.5, we first note that $T_U \to U$ is just the smooth map $X_U \to U$ whose geometrically connected fibers have dimension 1. The following lemma thereby shows that $(1')$ implies (1).

Lemma 10.8. *Let $Z \to Y$ be flat and finite type between irreducible noetherian schemes. For $d := \dim Z_{\eta_Y} \geq 0$, all geometric fibers $Z_{\overline{y}}$ have pure dimension d.*

Proof. This is standard - see a handout on the course website. □

Remark 10.9. One can drop irreducibility (and even the noetherian hypotheses) if instead $Z \to Y$ is *proper* flat and finitely presented, in which case $y \mapsto \dim Z_y$ is locally constant; see [EGA, IV_3, 12.2.1(ii)].

Example 10.10. Without properness, one cannot drop irreducibility from the lemma (even with separatedness). For example, consider $Y = \operatorname{Spec} R$ for R a discrete valuation ring with fraction field K and residue field κ. Glue $\mathbf{A}^1_R$ along inversion on $(\mathbf{G}_m)_K \subset \mathbf{A}^1_R$ to a K-scheme that is a union of $\mathbf{A}^1_K$ and a plane meeting at the origin (so the gluing has generic fiber that is a connected union of $\mathbf{P}^1_K$ and a plane). This gluing process takes place entirely over the generic fiber, so the glued scheme Z is R-flat, and it is separated. See Figure 2.

This gluing Z is not R-proper because we "glued in" a new irreducible component on the generic fiber so that this new component is closed in the entire space (as we may check Zariski-locally!) but is supported entirely over the generic point of Y. By design, the generic fiber of the gluing is reducible of dimension 2 but the (irreducible) special fiber is 1-dimensional.

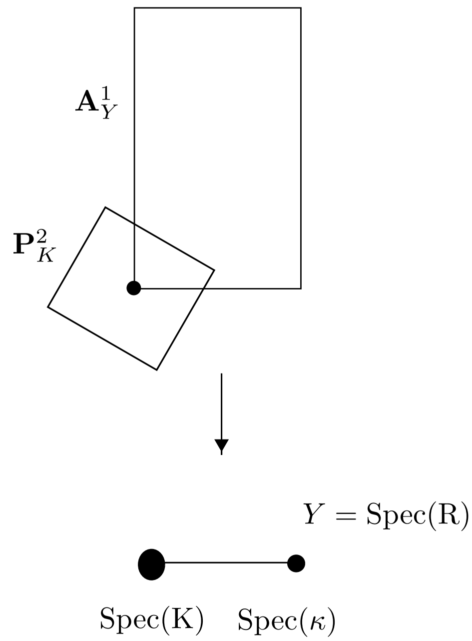


FIGURE 2. A picture of a flat (but non-proper) morphism to a dvr whose geometric fibers are not pure dimensional.

We now want to show $(1')$ in Goal 10.5. Consider $T_U = \Gamma_\psi \subset V_U$ for $V := \mathscr{C} \times_Y X \subset \mathbf{P}^N_Y$. We seek a U-modification $Y' \to Y$ so that the schematic closure

$$\overline{T_{U'}} \subset V_{Y'}$$

is Y'-flat. Once this is achieved and Y' is renamed as Y then the formation of such closure commutes with any further modification on Y since a Y-flat separated scheme is always the closure of its restriction over a dense open in Y. By this same principle, it suffices to find $Y' \to Y$ so that there exists a Y'-flat closed $\mathscr{T} \subset V_{Y'}$ that restricts to $T_{U'}$ over $U' \simeq U$ (as $\mathscr{V}$ is then *automatically* the schematic closure of $T_{U'}$, since a flat closed subscheme of a scheme over a reduced base is the schematic closure of its restriction over any dense open in the base).

There is a deep general result [RG, 5.2.2] by Raynaud and Gruson on "flattening by blow-up" which provides the desired flat closed $\mathscr{T} \subset V_{Y'}$ for a suitable Y', but in the projective setting we can use the following simpler trick with Hilbert schemes. Observe that the U-flat closed subscheme $T_U \subset V_U \subset \mathbf{P}^N_U$ has (by U-flatness) all fibers with a common Hilbert polynomial

$\Phi \in \mathbf{Q}[t]$. Therefore, $T_U \to U$ is a pullback

$$\begin{array}{ccc} T_U & \longrightarrow & \mathscr{Z} \subset V \times \mathrm{Hilb}^{\Phi}_{V/Y} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathrm{Hilb}^{\Phi}_{V/Y} \end{array} \tag{10.4}$$

of a universal flat family over a projective Hilbert scheme over Y. Then, take Y' to be the graph closure $\overline{\Gamma}_f \subset Y \times \mathrm{Hilb}^{\Phi}_{V/Y}$ and re-normalize this Y' (and rename that as Y) before forming T, which will now be Y-flat! In this way, both goals have been achieved, so $T_{\overline{y}}$ is a connected curve for all $\overline{y}$.

To recap where we are now, via the isomorphism

$$\begin{array}{ccc} \mathscr{C}_U & \xrightarrow{\psi} & X_U \\ & \searrow \quad \swarrow & \\ & U & \end{array} \tag{10.5}$$

we defined $T := \overline{\Gamma}_\psi \subset \mathscr{C} \times_Y X$ and arranged that $T \to Y$ is flat. It was deduced from such properties and the normality of Y that $T \to Y$ has geometrically connected fibers of dimension 1, and that we have a factorization

$$(\tau_i, \sigma_i) : Y \to T \subset \mathscr{C} \times_Y X.$$

Our aim is to show $p_1 : T \to \mathscr{C}$ is quasi-finite because this has been shown to imply $T \to \mathscr{C}$ is an isomorphism (so as to extend ψ to a proper birational Y-morphism) provided that also $\mathscr{C}$ is normal. Thus, let's now address why $\mathscr{C}$ is normal, and then we will take up the geometric task of proving p_1 is quasi-finite. More generally, we have the following normality result:

Lemma 10.11. *If $X \to S$ is a semi-stable curve over a normal noetherian S (no properness assumptions) with smooth generic fibers then X is normal.*

Example 10.12. Here is a model case that can be done by bare hands, to appreciate that some actual work is involved. Consider a normal noetherian domain A with fraction field K, and for a choice of $a \in A - \{0\}$ let $R = A[u,v]/(uv - a)$. By the fibral flatness criterion this is A-flat (as $uv - c$ is not a zero-divisor in $k[u,v]$ for any field k and $c \in k$), so it is a semistable curve; its generic fiber is a hyperbola.

We leave it as an instructive exercise to the reader to directly prove that R is a domain, and even integrally closed. Here is a hint: R is A-free with basis $1, u^i, v^j$ for $i, j \geq 1$ (since $uv = a \in A$), so we have an injection

$$\begin{aligned} R &\hookrightarrow R \otimes_A K \\ &= K[u,v]/(uv-a) \\ &= K[u,u^{-1}]. \end{aligned}$$

Thus, R is certainly a domain, and since $K[u, u^{-1}$ is already an integrally closed domain we only need to study elements of $R \otimes_A K$ integral over R.

The preceding example will arise in an important way when we study resolution of singularities for relative semistable curves.

Proof of Lemma 10.11. In the general case we shall use Serre's necessary and sufficient normality criterion "R1 + S2." Without loss of generality S is connected, hence irreducible with generic point denoted η. For $x \in X$, f is flat at x and so by the dimension formula for flat maps we have

$$\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{S,f(x)} + \dim \mathscr{O}_{X_{f(x)},x}.$$

Note that

$$\dim \mathscr{O}_{X_{f(x)},x} = \begin{cases} 1 & \text{if } x \text{ is closed in } X_{f(x)}, \\ 0 & \text{if } x \text{ is a generic point of} X_{f(x)}. \end{cases}$$

Recall that R_1 means $\mathscr{O}_{X,x}$ is regular when its dimension is at most 1, and given R_1 the condition S_2 is equivalent to saying that when $\mathscr{O}_{X,x}$ has dimension at least 2 then it has a regular sequence of length 2.

If $\dim \mathscr{O}_{S,f(x)} \geq 2$, then $\mathscr{O}_{S,f(x)}$ has a regular sequence of length 2 by normality of S (using Serre's "R_1 + S_2" criterion). By flatness of f, the pullback of these elements to $\mathscr{O}_{X,x}$ is a regular sequence. Thus, these cases are settled and so we may and do now assume that either $\mathscr{O}_{S,f(x)}$ is a field (i.e., $f(x) = \eta$) or is 1-dimensional. Recalling that X_η is assuming to be a smooth curve, all local rings are regular for x in the generic fiber.

To complete the proof, it suffices to deal with the case when $\mathscr{O}_{S,f(x)}$ is 1-dimensional, and hence a dvr (as S is normal). So now we may assume $S = \operatorname{Spec} R$ for R a dvr, and consider $x \in X_0$ a point in the special fiber. Since $\dim \mathscr{O}_{X,x} = 1 + \dim \mathscr{O}_{X_0,x}$, there are two cases:

(1) Suppose x is generic in X_0. In this case, $\mathscr{O}_{X,x}$ is 1-dimensional and so we want to show that it is a dvr. Let $\pi \in R$ be a uniformizer. Note that $\pi \in \mathscr{O}_{X,x}$ is a regular element of $\mathfrak{m}_x$ by R-flatness. Also, $\mathscr{O}_{X,x}/(\pi) = \mathscr{O}_{X_0,x}$ is the local ring at a generic point of X_0, which is a field. Therefore, π generates $\mathfrak{m}_x$. Since $\mathscr{O}_{X,x}$ is a 1-dimensional local noetherian ring in which the maximal ideal is generated by an element that is not a zero-divisor, $\mathscr{O}_{X,x}$ is a dvr by [Se, Ch. I, §2, Prop. 2].

(2) Say x is closed in X_0, so $\mathscr{O}_{X,x}$ has dimension 2. We seek a 2-term regular sequence. Since $\pi \in \mathfrak{m}_x$ is a not a zero divisor by R-flatness, it suffices to find a regular element in the maximal ideal of the quotient

$$\mathscr{O}_{X,x}/(\pi) = \mathscr{O}_{X_0,x},$$

which is a 1-dimensional local noetherian ring. That is, we are asking whether $\mathscr{O}_{X_0,x}$ satisfies S_1. By Serre's reducedness criterion, a local noetherian ring is reduced if and only if it satisfies "$R_0 + S_1$". Thus, we are done because X_0 is reduced (by definition of semistability for curves).

□

We next aim to study the fibers of $T \to \mathscr{C}$. Recall $T \subset \mathscr{C} \times_Y X$. We have

$$\begin{array}{ccccc} & & T & & \\ & \swarrow^{p_1} & \downarrow^{h} & \searrow^{p_2} & \\ \mathscr{C} & & & & X \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array} \tag{10.6}$$

with h flat and Y is normal, and all geometric fibers

$$T_{\overline{y}} \subset \mathscr{C}_{\overline{y}} \times X_{\overline{y}}$$

are connected curves. Our aim is to show that each map

$$(p_1)_{\overline{y}} : T_{\overline{y}} \to \mathscr{C}_{\overline{y}}$$

has finite fibers (as we saw this would imply that $p_1 : T \to \mathscr{C}$ is an isomorphism, as desired).

Hence, we want to show that $(p_1)_{\overline{y}}$ doesn't crush any irreducible component of $T_{\overline{y}}$ to a point.

10.1. Setup for Lemma 10.13.

Consider the irreducible components

$$\begin{array}{ccccc} & & T_{\overline{y}} = T_1 \cup \cdots \cup T_t \subset \mathscr{C}_{\overline{y}} \times X_{\overline{y}} & & \\ & \swarrow^{(p_1)_{\overline{y}}} & & \searrow^{(p_2)_{\overline{y}}} & \\ \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_s = \mathscr{C}_{\overline{y}} & & & & X_1 \cup \cdots \cup X_r = X_{\overline{y}}. \end{array}$$

Note that p_1, p_2 are proper birational, hence surjective, so $(p_1)_{\overline{y}}, (p_2)_{\overline{y}}$ are surjective. Thus, for all X_i there exists some j for which T_j surjects onto X_i and for all $\mathscr{C}_\alpha$ where exists a T_γ that surjects onto $\mathscr{C}_\alpha$. We need to know (among other things) that such j, γ are unique:

Lemma 10.13. *In the preceding setup, the following results hold.*

(i) For each $1 \leq i \leq r$ there exists a unique $1 \leq j(i) \leq t$ so that $T_{j(i)} \twoheadrightarrow X_i$. Also, there exists an open $V \subset X$ meeting $X_{\overline{y}}$ densely such that $p_2^{-1}(V) \simeq V$.

(*) Moreover,

$$T_{j(i)} \to \mathscr{C}_{\overline{y}} \tag{10.7}$$

is not constant.

(ii) For each $1 \leq \alpha \leq s$ there exists a unique $1 \leq \gamma(\alpha) \leq t$ so that $T_{\gamma(\alpha)} \twoheadrightarrow \mathscr{C}_\alpha$. Also, there exists an open $W \subset \mathscr{C}$ meeting $\mathscr{C}_\alpha$ densely such that $p_1^{-1}(W) \to W$ is an isomorphism.

Remark 10.14. The proof of the condition (10.7) will crucially use the properties from the 3-point Lemma.

Before we prove the preceding lemma, we record its main consequence for our needs:

Corollary 10.15. *The map $(p_1)_{\overline{y}}$ is quasi-finite.*

Proof. If $(p_1)_{\overline{y}}$ is not quasi-finite, it crushes some T_j to a point in $\mathscr{C}_{\overline{y}}$. Therefore, by (10.7), T_j cannot map onto any X_i, so it must be crushed to a point by $(p_2)_{\overline{y}}$. But $T_j \subset \mathscr{C}_{\overline{y}} \times \mathscr{C}_{\overline{y}}$, and since the map to each component is a point, T_j must itself be a point. However, T_j is a curve! Contradiction. □

Using that p_1 is quasi-finite, we can even say it is an isomorphism:

Corollary 10.16. *The map $p_1 : T \to \mathscr{C}$ is an isomorphism.*

Proof. Since $(p_1)_{\overline{y}}$ is quasi-finite for all geometric points $\overline{y}$ (Corollary 10.15), p_1 is quasi-finite. Note also that $\mathscr{C}$ is normal by Lemma 10.11. Therefore, p_1 is a proper birational quasi-finite map to a normal variety $\mathscr{C}$, hence an isomorphism. □

10.2. **Proof of Lemma 10.13(i) and (ii).** We now prove (i) without (*), and the same exact method will give (ii) since the argument will not use the sections $\sigma_i(\overline{y})$ or $\tau_j(\overline{y})$ that break the symmetry between X and $\mathscr{C}$ (in other words, the argument will treat $X_{\overline{y}}$ with methods that apply equally well to $\mathscr{C}_{\overline{y}}$).

Suppose for some $j' \neq j$ that $T_j, T_{j'}$ both map onto X_i under $(p_2)_{\overline{y}}$. Since $T_j \cap T_{j'}$ is finite (and possibly empty) this would imply the fibers of

$$p_2^{-1}(X_i) \to X_i$$

would have size at least 2 away from a finite subset of X_i. Therefore, to get a contradiction it is enough to find an open $V \subset X$ meeting $X_{\overline{y}}$ densely (in particular, meeting X_i) such that

$$p_2^{-1}(V) \to V$$

is an isomorphism

Recall that $\mathrm{sm}(X/Y)$ meets all $X_{\overline{y}}$ in a dense open. Also, $(p_2)_{\overline{y}} : T_{\overline{y}} \to X_{\overline{y}}$ has only finitely many positive dimensional fibers. Hence, the subset

$$\Omega = \left\{ x \in X : p_2^{-1}(x) \text{ is 0-dimensional} \right\}$$

of X meets $X_{\overline{y}}$ away from a finite set of closed points. By upper semicontinuity of fiber dimension, Ω is open in X.

We conclude that

$$V := \Omega \cap \mathrm{sm}(X/Y) \subset X$$

is an open subset of X meeting $X_{\overline{y}}$ densely and

$$p_2^{-1}(V) \to V$$

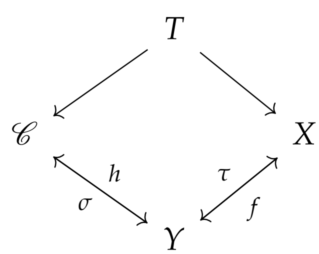
is a finite birational map (as it is proper quasi-finite, and $p_2 : T \to X$ is birational between varieties because $T = \overline{\Gamma}_\psi$ with $\psi : \mathscr{C}_U \simeq X_U$).

Thus, it remains to show that this finite birational map between varieties is an isomorphism. For that purpose it is enough to show V is normal. But $V \to Y$ is smooth (due to how V was built) and Y is normal, so V is also normal! (Here we essentially used that the smooth locus of $X \to Y$ meets *every* fiber in a dense open subset.)

It remains to prove Lemma 10.13(*).

10.3. **Setup and proof of Lemma 10.13(*).** Recall our setup: We have

(10.8)



with T flat over Y, both Y and $\mathscr{C}$ normal, and T defined to be the closure in $\mathscr{C} \times_Y X$ of the graph of a U-isomorphism

$$\begin{array}{ccc} \mathscr{C}_U & \xrightarrow{\psi} & X_U \\ & \searrow \quad \swarrow & \\ & U & \end{array} \tag{10.9}$$

that carries $\tau_i|_U$ to $\sigma_i|_U$ for all i. We have seen all $T_{\overline{y}}$ are connected curves inside $\mathscr{C}_{\overline{y}} \times X_{\overline{y}}$ and the section

$$(\tau_i, \sigma_i) : Y \to \mathscr{C} \times_Y X$$

factors through T (as may be checked on $U \subset Y$). In particular, $T_{\overline{y}}$ contains $(\tau_i(\overline{y}), \sigma_i(\overline{y}))$ for all i.

10.3.1. *Proof of Lemma 10.13(*).* Using the notations of Lemma 10.13, we want to show that for each i, the unique $T_{j(i)}$ mapping onto X_i is not crushed to a point in $\mathscr{C}_{\overline{y}}$. Assume to the contrary for some i that

$$(p_1)_{\overline{y}}(T_{j(i)}) = \{c\}$$

for some $c \in \mathscr{C}(\overline{y})$. We seek a contradiction.

The surjective map $T_{j(i)} \twoheadrightarrow X_i$ hits all markings $\sigma_j(\overline{y}) \in X_i$, and there are at least 3 of these; let's label them as

$$x_\alpha := \sigma_\alpha(\overline{y}), \;\; x_\beta := \sigma_\beta(\overline{y}), \;\; x_\gamma := \sigma_\gamma(\overline{y})$$

in $X_i \cap \mathrm{sm}(X/Y)_{\overline{y}}$. In particular, $x_\alpha, x_\beta, x_\gamma$ do not lie in any components of X other than X_i.

We will use crucially that

$$t_\alpha := (\tau_\alpha(\overline{y}), \sigma_\alpha(\overline{y})) \in T_{\overline{y}}$$

and likewise for β, γ. We'll argue separately depending on whether or not

$$c \in \left\{\tau_\alpha(\overline{y}), \tau_\beta(\overline{y}), \tau_\gamma(\overline{y})\right\} \subset (\mathscr{C})^{\mathrm{sm}}_{\overline{y}}.$$

That is, we shall treat two separate cases:

Case 1. $c \notin \left\{\tau_\alpha(\overline{y}), \tau_\beta(\overline{y}), \tau_\gamma(\overline{y})\right\}$. In this case, we'll find three irreducible components of $\mathscr{C}_{\overline{y}}$ through c, which is a contradiction since $\mathscr{C}_{\overline{y}}$ is semistable.

Case 2. $c \notin \left\{\tau_\alpha(\overline{y}), \tau_\beta(\overline{y}), \tau_\gamma(\overline{y})\right\}$. In this case $c \in (\mathscr{C})^{\mathrm{sm}}_{\overline{y}}$ and we'll find two irreducible components of $\mathscr{C}_{\overline{y}}$ through c, again a contradiction.

Note that $t_\alpha \mapsto \sigma_\alpha(\overline{y}) = x_\alpha$, and likewise for β, γ. We label Cases 1 and 2 as Proposition 10.17 and Proposition 10.19 below.

Proposition 10.17. *If*

$$c \notin \left\{\tau_\alpha\left(\overline{y}\right), \tau_\beta\left(\overline{y}\right), \tau_\gamma(\overline{y})\right\}.$$

then there are three irreducible components of $\mathscr{C}_{\overline{y}}$ through c.

Proof. For this, we need the following lemma.

Lemma 10.18. *The non-empty fibers*

$$p_2^{-1}(x_\alpha), p_2^{-1}(x_\beta), p_2^{-1}(x_\gamma)$$

are each connected chains of irreducible components of $T_{\overline{y}}$. Moreover, their p_1-images

$$p_1(p_2^{-1}(x_\alpha)), p_1(p_2^{-1}(x_\beta)), p_1(p_2^{-1}(x_\gamma))$$

are also connected chains of irreducible components of $\mathscr{C}_{\overline{y}}$.

Proof. Since $p_2^{-1}(x) \subset \mathscr{C}_{\overline{y}} \times \{x\}$ for $x \in X(\overline{y})$, clearly $p_2^{-1}(x) \to p_1(p_2^{-1}(x))$ is an isomorphism. Hence, we can focus on the claim concerning p_2-fibers (though in fact it will proceed via the p_1-images of such fibers).

First, we show that the visibly pairwise disjoint fibers

$$p_2^{-1}(x_\alpha), p_2^{-1}(x_\beta), p_2^{-1}(x_\gamma)$$

are each either a point or a connected chain of irreducible components of $T_{\overline{y}}$. Since $T_{j(i)} \to X_i$ is surjective, certainly each of these p_2-fibers meets $T_{j(i)}$. We claim that each is connected, for which it suffices to show that $p_2 : T \to X$ is its own Stein factorization over the open subset $\mathrm{sm}(X/Y) \subset X$ whose $\overline{y}$-fiber contains $x_\alpha, x_\beta, x_\gamma$. Since p_2 is a proper birational map and $\mathrm{sm}(X/Y)$ inherits normality from Y, the claim concerning Stein factorization follows. We conclude that each of

$$p_2^{-1}(x_\alpha), p_2^{-1}(x_\beta), p_2^{-1}(x_\gamma)$$

is either a point or a connected chain of irreducible components of the connected curve $T_{\overline{y}}$.

To complete the proof, we want to show none of

$$p_2^{-1}(x_\alpha), p_2^{-1}(x_\beta), p_2^{-1}(x_\gamma)$$

are points. Let's just proceed to show $p_2^{-1}(x_\alpha)$ is not a point, as the situation is symmetric with respect to α, β, γ.

It suffices to show $p_1(p_2^{-1}(x_\alpha))$ is not a point. By hypothesis $(p_1)_{\overline{y}}(T_{j(i)}) = \{c\}$ and

$$T_{j(i)} \twoheadrightarrow X_i \ni x_\alpha$$

so $c \in p_1(p_2^{-1}(x_\alpha))$. Since $x_\alpha = \sigma_\alpha(\overline{y})$, we have

$$\tau_\alpha(\overline{y}) = p_1\left(\tau_\alpha(\overline{y}), \sigma_\alpha(\overline{y})\right) \in p_1(p_2^{-1}(x_\alpha)).$$

This point is distinct from c by hypothesis, so $p_2^{-1}(x_\alpha)$ is indeed not a point. □

Consider $p_1(p_2^{-1}(x_\alpha))$. By Lemma 10.18, we know this is a connected chain of irreducible components of $\mathscr{C}_{\overline{y}}$. In particular this has an irreducible component through c in $\mathscr{C}_{\overline{y}}$. To get three distinct irreducible components of $\mathscr{C}_{\overline{y}}$ through c, (using x_β, x_γ also), we just need to ensure

$$p_1(p_2^{-1}(x_\alpha)), p_1(p_2^{-1}(x_\beta)), p_1(p_2^{-1}(x_\gamma))$$

have no irreducible components in common. Since

$$p_2^{-1}(x_\alpha), p_2^{-1}(x_\beta), p_2^{-1}(x_\gamma)$$

are pairwise disjoint, and each p_2-fiber has p_1-image that is a union of irreducible components of $\mathscr{C}_{\overline{y}}$ obtained as p_1-images of irreducible components of $T_{\overline{y}}$ that it contains, we only need to show that for $j \neq j'$ if $p_1(T_j)$ and $p_1(T_{j'})$ happen to be irreducible components of $\mathscr{C}_{\overline{y}}$, then these components are distinct. But this is exactly Lemma 10.13(ii). □

Proposition 10.19. *If*

$$c \in \left\{\tau_\alpha\left(\overline{y}\right), \tau_\beta\left(\overline{y}\right), \tau_\gamma(\overline{y})\right\}.$$

then there are two irreducible components of $\mathscr{C}_{\overline{y}}$ *through* c*.*

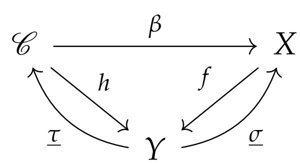
Proof. The proof of this is completely analogous to that of Proposition 10.17, using that if $c = \tau_\alpha(\overline{y})$, then

$$c \notin \left\{\tau_\beta(\overline{y}), \tau_\gamma(\overline{y})\right\}$$

(and similarly with α replaced by β or γ). □

10.4. **Reducing to the case when $X \to Y$ is a semistable curve.** By Corollary 10.16 the map $p_1 : T \to \mathscr{C}$ is an isomorphism, so composing its inverse with $p_2 : T \to X$ yields a Y-map

(10.10)



such that β_U is an isomorphism (for a dense open $U \subset Y$), $\beta \circ \tau_i = \sigma_i$ for all i, and

$$Z = \cup_i \sigma_i(Y)$$

is the support of a Cartier divisor in X.

Since β is proper birational (a "modification"), we can replace (X, Z) with

$$\left(\mathscr{C}, \beta^{-1}(Z)\right)$$

except for the issue is that $\beta^{-1}(Z)$ may not be Y-finite. But the fibration over Y is just an auxiliary device in the service of our real goal, which is to find a generically étale alteration of X that is smooth and in which the preimage of Z is the support of a strict normal crossings divisor.

The properties that $Z \to Y$ is finite and generically étale will never be used anymore (the only role for those conditions were to create the sections σ_i satisfying the properties in the 3-point lemma that has finally served its role above to create β via $p_1 : T \to X$ being an isomorphism). Thus, we now *drop* those running hypotheses on $Z \subset Y$. Since moreover

$$\beta^{-1}(Z) \subset \mathscr{Z} := \left(\cup_j \tau_j(Y)\right) \cup h^{-1}(D)$$

for $D := (Y - U)_{\text{red}}$, for the purpose of our real goal of finding a suitable generically étale alteration for (X, Z) it is enough to work with

$$(\mathscr{C}, \mathscr{Z}).$$

By induction on dimension for our main goal in this course, there exists a generically étale alteration

$$F : Y' \to Y$$

so that

(1) Y' is smooth,
(2) $F^{-1}(D)_{\text{red}}$ is a strict normal crossings divisor in Y'.

Since $\mathscr{C}_{Y'} \to \mathscr{C}$ is a generically étale alteration, instead of working with $(\mathscr{C}, \mathscr{Z})$ it suffices to work with

$$\left(\mathscr{C}_{Y'}, \mathscr{Z}' := \left(\cup_j \tau'_j(Y')\right) \cup h'^{-1}(D')\right)$$

where

$$h' : \mathscr{C}_{Y'} \to Y'$$

is the structure map, $\tau'_i = (\tau_i)_{Y'}$, and $\mathscr{C}_{Y'}$ is a semistable Y'-curve and $D' \subset Y'$ is a sncd. The stability of $\mathscr{C}_{Y'} \to Y'$ is not required in what follows, only that it is semistable with smooth generic fiber and that the sections τ'_i satisfy

$$\tau'_j(Y') \subset \text{sm}(\mathscr{C}_{Y'}/Y').$$

We are now in a geometrically very favorable situation! The next order of business is to understand a global intrinsic process for resolution of singularities for semistable curves with smooth generic fiber over a regular base scheme, beginning with the key case of a discrete valuation ring as the base. Eventually we need to incorporate "Z" into this process, but first we should understand such resolution without the distraction of "Z".

Remark 10.20. If we were only in the case of projective varieties, we would not need to keep track of Z throughout the argument. But, if we started with a non-proper X at the start of the main goal of this course then it is very important is useful to have Z in order to arrange that the final smooth generically étale alteration of X is the complement of a sncd in a smooth projective variety. Note that the way we applied the inductive hypothesis above was for the pair (Y, D) where we know nothing at all about D. For this reason, it is essential for the induction that the proper closed set Z is allows to be rather arbitrary.

Up to the issue of making Z an sncd, we have essentially reduced our task to resolving singularities on a semistable curve with smooth generic fiber over a regular base. As a warm-up, we'll start with the case that the base is a discrete valuation ring.

11. Semistable curves over a discrete valuation ring

Let's start with a toy (or local) version.

Example 11.1. Take R to be a discrete valuation ring, π a uniformizer, and $n \geq 1$. Consider

$$C_n := \operatorname{Spec} R[x, y] / (xy - \pi^n).$$

We'll see later using Artin approximation that any point on a semistable curve with smooth generic fiber over R has an étale neighborhood that is étale over C_n for some n.

The generic fiber of C_n is clearly smooth and its special fiber is smooth away from one point, call it ξ. Then,

$$\widehat{\mathscr{O}_{C_n,\xi}} = \widehat{R}[\![x, y]\!]/(xy - \pi^n)$$

with $xy - \pi^n \in \mathfrak{m}^2$ if and only if $n \geq 2$. Therefore, this is not regular if and only if $n \geq 2$.

We will later work out some blow-up calculations which will show that $\mathrm{Bl}_\xi(C_n)$ is regular if $n = 2$ and that this blow-up is covered by copies of C_{n-2} if $n \geq 3$ (hence also regular if $n = 3$, but not so if $n \geq 4$).

It follows that we have a coordinate-free intrinsic resolution process in this case: blow up all finitely many non-regular points, and repeat. This eventually ends.

We want a version of this in general, but we will have to link a given abstract semistable curve $X \to \operatorname{Spec} R$ to such C_n's in a way that permits transferring over information about blow-up calculations. There will be some issues to address due to non-rational non-regular points.

We'll also later have to generalize to a higher-dimensional base than R as above. For $X \to Y$ a semistable curve fibration over a connected regular Y with dimension more than 1, we'll have a blow-up process to push $X^{\text{sing}} := X - \operatorname{reg}(X)$ into codimension at least 3. Some calculations are required to affirm that the blow-ups we make of semistable curves are still semistable.

11.1. **Describing the étale local structure of semistable curves.** Let Y be any scheme and $f : X \to Y$ a semi-stable curve, with no properness assumptions; i.e., f is flat and locally finitely presented and for $\overline{y} \in Y$ a geometric point, $X_{\overline{y}}$ is a reduced curve such that if $\overline{x} \in X_{\overline{y}}$ is not smooth then

$$\widehat{\mathscr{O}_{X_{\overline{y}},\overline{x}}} \simeq \kappa(\overline{y})[\![u,v]\!]/(uv).$$

Our goal is to describe the étale-local structure of f. The key to linking this completion to our toy model via a common étale neighborhood will be Artin approximation, which is discussed in [deJ, §2.21-2.23,§3.1-3.5] and (with full proof) in [BLR, §3.6].

Remark 11.2. We will only need Artin's version below, which concerns for A essentially finite type over a field or an excellent Dedekind domain. But we will state the ultimate general version below (due to further deep work of Popescu). Note that excellent Dedekind domains include $\mathbf{Z}$ and any complete dvr. The final part of [deJ] requires Artin approximation over a complete dvr; for us the version for A essentially of finite type over a field will be sufficient (though is no easier to prove than Artin's result in his own generality).

Definition 11.3. Let A be a local ring. Then the **henselization** of A, denoted A^h is by definition

$$A^h := \varinjlim_{A \to A'} A'$$

where the limit is taken over all local-étale maps with trivial residue field. (The collection of such local-étale maps is directed and rigid: there is at most one local A-algebra map between any two such A', and any two are "dominated" by a common third one. Thus, such a direct limit makes sense.)

Fact 11.4. If A is noetherian then A^h is noetherian with the same completion of A; see [EGA, 0_{III}, 10.3.1.3]. Further, it is excellent when A is.

Theorem 11.5 (Artin-Popescu approximation). *Let $(A, \mathfrak{m})$ be an excellent local noetherian ring. Let $B = A[x_1, \dots, x_n]/(f_1, \dots, f_m)$ be a finite type A-algebra. The map*

$$\mathrm{Hom}_A(B, A^h) \to \mathrm{Hom}_A(B, \widehat{A}) = \varprojlim_n \mathrm{Hom}_A(B, A/\mathfrak{m}_A^{n+1})$$

has dense image (in the $\mathfrak{m}_A$-adic topology on the target).

Proof. See [BLR, §3.6]. □

Exercise 11.6. Using the ramification theory of dvr's, one can show without Artin approximation that if A is a dvr then $A^h \subset \widehat{A}$ is the algebraic closure of A inside $\widehat{A}$. Artin approximation yields the same with A any normal excellent local domain.

Remark 11.7. Let's deduce a very useful enhancement of the preceding result. For C essentially finite type over A, meaning (for our purposes) a local ring at a prime on a finite-type A-algebra, we have

$$\mathrm{Hom}_A(B, \widehat{C}) = \mathrm{Hom}_C(C \otimes_A B, \widehat{C})$$

Now, applying Theorem 11.5, we obtain a map

$$\mathrm{Hom}_A(B, C^h) = \mathrm{Hom}_C(C \otimes_A B, C^h) \to \mathrm{Hom}_C(C \otimes_A B, \widehat{C}) = \mathrm{Hom}_A(B, \widehat{C})$$

which has dense image and coincides with the evident natural map.

Corollary 11.8. *For such $(A, \mathfrak{m})$ as in Theorem 11.5 and $n \geq 2$, and C_1, C_2 two local essentially finite type A-algebras. Given an isomorphism*

$$f : \widehat{C_2} \simeq \widehat{C_1}$$

over $\widehat{A}$ (equivalently over A), there exists a residually trivial local-étale extension $C_1 \to C_1'$ and a local A-algebra map

$$\phi : C_2 \to C_1'$$

so that

(1) the induced map

$$\widehat{\phi} : \widehat{C_2} \to \widehat{C_1'} = \widehat{C_1}$$

is an isomorphism (so ϕ is local-étale and residually trivial),

(2) $\widehat{\phi} \equiv f \bmod \mathfrak{m}^n_{\widehat{C_1}}$.

Before proving the corollary, we consider an example.

Example 11.9. Suppose $A = k$ is a field and X_1, X_2 are two finite type schemes over k and $x_i \in X_i(k)$ for $i = 1, 2$. Given a k-algebra isomorphism

$$\widehat{\mathscr{O}_{X_1,x_1}} \simeq \widehat{\mathscr{O}_{X_2,x_2}}$$

there exists a common residually trivial étale neighborhood

$$\begin{array}{ccc} & (X_1', x_1') & \\ \swarrow & & \searrow \\ (X_1, x_1) & & (X_2, x_2) \end{array} \tag{11.1}$$

by setting $C_i = \mathscr{O}_{X_i,x_i}$ and spreading out C_1' to make (X_1', x_1').

Proof of Corollary 11.8. We have $C_2 = B_\mathfrak{p}$ for B of finite type over A. Then, letting $\mathrm{Hom}_{A,\mathrm{loc}}$ denote the set of local A-algebra maps, we have

$$\begin{aligned} f &\in \mathrm{Hom}_{A,\mathrm{loc}}\left(\widehat{C_2}, \widehat{C_1}\right) \\ &= \mathrm{Hom}_{A,\mathrm{loc}}(C_2, \widehat{C_1}) \\ &= \left\{ h \in \mathrm{Hom}_A(B, \widehat{C_1}) : h^{-1}(\mathfrak{m}_{\widehat{C_1}}) = \mathfrak{p} \right\} \\ &\approx \lim_{C_1 \to C_1'} \left\{ h \in \mathrm{Hom}_A(B, C_1') : h^{-1}(\mathfrak{m}_{\widehat{C_1}}) = \mathfrak{p} \right\} \\ &= \lim_{C_1 \to C_1'} \mathrm{Hom}_{A,\mathrm{loc}}(C_2, C_1') \end{aligned}$$

where the "$\approx$" step uses approximations modulo $\mathfrak{m}^n_{\widehat{C_1}}$ and so in particular maintains the condition on the h-preimage.

Given a local A-algebra map $\phi : C_2 \to C_1'$ so that

$$\widehat{\phi} \equiv f \bmod \mathfrak{m}^n,$$

we would like to show $\widehat{\phi}$ is an isomorphism. The composite map

$$\widehat{C_1} \xleftarrow{f} \widehat{C_2} \xrightarrow{\widehat{\phi}} \widehat{C_1'} = \widehat{C_1}$$

obtained by inverting the isomorphism f is the identity modulo $\mathfrak{m}^2$, so the composite map is surjective due to successive approximation. But if R is a noetherian ring then any surjective map $R \to R$ as rings is an isomorphism, so the composite map is an isomorphism and hence so is $\widehat{\phi}$. □

Example 11.10. Let k be a field and X a curve over k. Let $x \in X(k)$ with

$$\widehat{\mathscr{O}_{X,x}} \simeq k[\![u, v]\!]/(uv).$$

Then, for $C = \{uv = 0\} \subset \mathbf{A}^2_k$, then (X, x) and $(C, (0,0))$ have a common residually trivial étale neighborhood.

For example, X could even be irreducible, say $y^2 = x^3 + x^2$, the nodal cubic.

Example 11.11. We could ask for a variant of the previous example. Let k be a field and X a curve over k. Let $x \in X$ is a closed point with

$$\widehat{\mathscr{O}_{X_{\overline{k}}, \overline{x}}} \simeq \overline{k}[\![u, v]\!]/(uv).$$

Does the same conclusion of Example 11.10 hold, but without the residually trivial assumption? That is, can we find a common étale neighborhood? The issue is that we need to worry about inseparable extensions. If so, then $k(x)/k$ is separable (assuming that n is even if the characteristic is 2).

The reason we care about non-algebraically closed fields because we'll be applying this to the residue field at a generic point of a singular locus, and in varieties over any fields of positive characteristic (even algebraically closed) the residue fields at generic points of positive-dimensional closed subschemes are *never* perfect. So, we'll absolutely need to handle the above issue for general fields k.

11.2. **Quadratic forms in a geometric setting.**

Definition 11.12. Say $X \to S$ is a flat finitely presented map, $s \in S$ is a point. A closed point $x \in X_s$ is called an *ordinary double point* if for compatible "algebraic" geometric points $\overline{s}$ over s and $\overline{x}$ over x and $\overline{s}$ we have an $k(\overline{s})$-algebra isomorphism

$$\widehat{\mathscr{O}_{X_{\overline{s}, \overline{x}}}} \simeq k(\overline{s})[\![t_1, \ldots, t_n]\!]/q$$

for a nonzero quadratic form q that is non-degenerate in the sense that the projective quadric $(q = 0) \subset \mathbf{P}^{n-1}$ is smooth.

It is easy to check that in the preceding definition the choice of $\overline{x}$ doesn't impact the existence or not of such an isomorphism.

Remark 11.13. Given a quadratic form q on a nonzero finite-dimensional vector space V over a field F, consider the associated symmetric bilinear form

$$B_q(v, w) := q\,(v + w) - q(v) - q(w)$$

on V. Note that B_q is non-degenerate implies the projective quadric $(q = 0) \subset \mathbf{P}(V^*)$ is smooth, and the converse holds except precisely when n is odd and $\operatorname{char} \overline{s} = 2$; see HW2 Exercise 4 in the course "Algebraic Groups I". For example, $xy + z^2$ is a counterexample the converse in characteristic 2 (for this case, the z-axis is a line that is B_q-perpendicular to everything).

Example 11.14. Let A be a local ring and $Q = \sum a_y t_i t_j$ residually non-degenerate over A. Let $a \in \mathfrak{m}_A$. Consider

$$\begin{array}{c} X := \operatorname{Spec}\ (A[t_1, \ldots, t_n]/(Q-a)) \\ \big\downarrow \\ S := \operatorname{Spec} A. \end{array} \tag{11.2}$$

Let s be the closed point of S and $x = \vec{0} \in X_s$. If the quadratic form Q is residually non-degenerate then some coefficient of Q is a unit, so

$$Q - a \in A[t_1, \ldots, t_n]$$

is nowhere a zero-divisor on the fibers of the S-flat $\mathbf{A}^n_S \to S$ and hence X is S flat by the local flatness criterion after reducing to noetherian A.

Lemma 11.15. *In the above setup, for all $b \in S$, Q_b is non-degenerate over $k(b)$.*

Proof. The structure map $h : V(Q) \subset \mathbf{P}^{n-1}_A \to S$ is finitely presented, proper, and (by the reasoning in the preceding example, with $a = 0$) flat. Hence, verifying smoothness of h at a point $\xi \in V(Q)$ is equivalent to showing that the fiber $V(Q)_{h(\xi)}$ is smooth at ξ. We know that the locus $\Omega \subset V(Q)$ of points at which h is smooth is Zariski-open and it contains the entire special fiber by our assumption that Q_s is non-degenerate (with $s \in S$ the closed point). Therefore $\Omega = V(Q)$ by S-properness of $V(Q)$. □

Lemma 11.16. *Let F be a field, q a non-degenerate quadratic form on F^n, and $c \in F$ (e.g., $F = k(b)$ for $b \in S$, $c = a(b)$, $q = Q_b$). Then*

$$\operatorname{Spec}(F[t_1, \ldots, t_n]/(q-c))$$

is smooth except when

$$\begin{cases} c = 0 & \textit{at } \vec{0}, \\ c \neq 0 & \textit{at a unique generic point for } n \textit{ odd when } \operatorname{char} F = 2. \end{cases}$$

Proof. One can assume F is algebraically closed. The case $n = 1$ is easy, so we assume $n \geq 2$. We can choose coordinates so that

$$q = q_n := \begin{cases} t_1 t_2 + \cdots + t_{n-1} t_n & \text{if } n \text{ is even} \\ t_0^2 + q_{n-1} & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

Now one can apply the Jacobian criterion. □

Corollary 11.17. *When n is even and $X \to S$ as in Example 11.14 then $X \to S$ is smooth, except precisely along the* 0 *section over* $\operatorname{Spec} A/a \subset S$.

Proof. This follows immediately from Lemma 11.16. □

11.3. **The main structure theorem for ordinary double points.**

Theorem 11.18. *Say $X \to S$ is a flat finitely presented map. Let $x \in X_s$ be an ordinary double point with X_s of dimension at least 1 near x. Define $n = 1 + \dim_x(X_s)$, and assume n is even if* $\operatorname{char} s = 2$. *Then,*

(1) *Choose any residually non-degenerate quadratic form*

$$Q \in \mathscr{O}_{S,s}^{\mathrm{sh}}[t_1, \ldots, t_n],$$

where A^{sh} denotes the strict henselization of a local ring A. There exists $a \in \mathfrak{m}_{\mathscr{O}_{S,s}^{\mathrm{sh}}}$ so that there is an $\mathscr{O}_{S,s}^{\mathrm{sh}}$-algebra isomorphism

$$\mathscr{O}_{X,x}^{\mathrm{sh}} \simeq \mathscr{O}_{S,s}^{\mathrm{sh}}\{t_1, \ldots, t_n\}/(Q-a) = \left(\mathscr{O}_{S,s}^{\mathrm{sh}}[t_1, \ldots, t_n]/(Q-a)\right)^{\mathrm{sh}}_{(\mathfrak{m}_s, t_1, \ldots, t_n)}.$$

By spreading out, this says there is a commutative diagram of pointed maps

(11.3)

$$\begin{array}{ccccc}
 & & (X', x') & \xrightarrow{f_1} & \text{a spread out version of Example 11.14 over } \mathscr{O}_{S',s'} \\
 & \swarrow^{f_2} & & & \swarrow \\
(X, x) & & & & \\
\downarrow & & & & \\
(S, s) & \xleftarrow[f_3]{} & (S', s') & &
\end{array}$$

where the labeled maps f_1, f_2, f_3 are étale.

(2) *Further $a\mathscr{O}_{S,s}^{\mathrm{sh}}$ is intrinsic to the above setup. In particular, if some such a is not a zero divisor (so it generates an invertible ideal) then all such possible a are a unit multiple of that one.*

The proof is a hard but spectacular application of Artin approximation. For (1) we may assume S is noetherian, and even finite type over $\mathbf{Z}$, via limit considerations. The argument in this case via Artin approximation is given in [FK, §2, Ch III]. (Note that in [FK], they define "non-degenerate" to require n even in characteristic 2.) That proof characterizes

$$a\widehat{\mathscr{O}_{S,s}^{\mathrm{sh}}}$$

as an annihilator in $\widehat{\mathscr{O}_{S,s}^{\mathrm{sh}}}$ of a specific completed stalk of a module of relative differentials for X over S; such a uniqueness for that ideal in the completion implies uniqueness in general (by faithful flatness of completion for local noetherian rings).

Remark 11.19. In fact, the proof gives a little more: if $k(x) = k(s)$ then one can replace strict henselizations with henselizations, except one cannot control the (non-degenerate) reduction of Q over $k(s)$. This issue is already seen in the case of elliptic curves with non-split multiplicative reduction. The actual choice of Q lifting such an uncontrolled reduction does not matter because over $\mathscr{O}^{\mathrm{h}}_{S,s}$ any two quadratic forms in n variables with isomorphic non-degenerate reductions are in fact isomorphic, as is proved in a handout on quadratic forms.

Having talked about ordinary double points in more generality, we'll now come back to discussing the relative dimension 1 case, which for us is the main example of ordinary double points that we care about.

Let $X \to \operatorname{Spec} A$ be a flat semistable curve over a reduced local noetherian ring A. We *do not* make any properness or geometric connectivity hypotheses. Let's also assume the generic fibers are smooth. Let $x \in X_0$ be a non-smooth point in the special fiber X_0.

Proposition 11.20. *Under the above assumptions, there exists $a \in \mathfrak{m}_A$ unique up to $A^\times$-multiple so that (X, x) and*

$$(\operatorname{Spec} A[u,v]/(uv-a), \overline{0} := (u, v, \mathfrak{m}_A))$$

have a common étale neighborhood. Further, a is not a zero divisor in A.

Remark 11.21. The key thing is that we can find a inside A, and not just in a local-étale extension (as is the case in Theorem 11.18!).

Proof. First, we verify uniqueness of a, and then address its exists. By the $n = 2$ case of Theorem 11.18, there exists a' in the maximal ideal of a local-étale extension A' of A that does the job (using $A'[u,v]/(uv-a')$). Note that A' is reduced, being an étale A-algebra. We want to show that this a' is not a zero-divisor in A', which will imply a' is unique up to $A'^\times$-multiple by part (2) of Theorem 11.18. It would then follow by faithful flatness of A' over A that any possible $a \in \mathfrak{m}_A$ is unique up to $A^\times$-multiple and is not a zero divisor in A since necessarily $a \in a'A'^\times$ by the uniqueness over A'. (We will then have to find an $A'^\times$-multiple of a' that comes from A, necessarily from $\mathfrak{m}_A$ since $\mathfrak{m}_{A'} = A' \otimes_A \mathfrak{m}_A$ due to A' being local-étale over A).

To show a' is not a zero-divisor (and hence the uniqueness of a' up to unit multiple in A'), it amounts to showing that a' is nonzero each irreducible component of the reduced noetherian scheme $\operatorname{Spec} A'$. Hence, passing to such components with their reduced (and even integral) structure allows us to assume that $A = A'$ is a domain.

Now, since A is a domain, we just seek a contradiction if $a = 0$. Let K be the fraction field of A. We will use the vanishing of a to create a non-smooth point on X_K, contradicting our hypotheses on the generic fibers of X. With $a = 0$, we have pointed étale maps

$$\begin{array}{ccc} & (Y, y) & \\ \swarrow & & \overset{h}{\searrow} \\ (X, x) & & \left(\mathrm{Spec}\ A[u,v]/(uv), \overline{0}\right). \end{array} \tag{11.4}$$

Thus, to show X_K has a non-smooth point, it is enough to show that the non-smooth point $(0,0)$ in the K-fiber of Spec $A[u,v]/(uv)$ is in the image of h. We know the image $h(Y) \subset \mathrm{Spec}\ A[u,v]/(uv)$ is open by étaleness of h, and it meets the 0-section (which is isomorphic to Spec A) in at least the closed point $\overline{0} = h(y)$. But Spec A is local, so the open $h(Y)$ must contain the entirety of that 0-section (as a local scheme has no proper open subset containing the closed point). Thus, $h(Y)$ contains the generic point of the 0-section, which is the point $(0,0)$ in the K-fiber.

To summarize, we have shown that if such an a as desired exists, it must not be a zero divisor and is unique up to multiplication by $A^\times$. To construct a, consider the *invertible* ideal $I' := a'A' \subset \mathfrak{m}_{A'}$ (invertible since a' isn't a zero-divisor). Any generator of I' must be a unit multiple of I', so we seek a generator coming from A (necessarily then coming from $\mathfrak{m}_A$ because I' is a proper ideal of A'). It is enough to show $I' = I \otimes_A A'$ for an ideal $I \subset A$. Indeed, if we could show this then I would be an invertible A-module (since invertibility descends through faithfully flat ring extensions, such as $A \to A'$), and thus $I = aA$ for some $a \in A$ because A is local. This would show $aA' = I'$, so (as explained above) we'd be done.

To make the construction, inspired by the proof of Theorem 11.18, as given in [FK, §2, Ch. III] we shall try

$$I := \mathrm{ann}_A\left(\Omega^2_{X_A/A,x}\right)$$

where $\Omega^n_{B/A} := \wedge^n_B(\Omega^1_{B/A})$ for any ring map $A \to B$. By design of (Y, y) we have a pointed étale map $(Y, y) \to (X_{A'}, x')$ for some x' over x, so

$$\begin{aligned} I \otimes_A A' &= \mathrm{ann}_{A'}((\Omega^2_{X/A,x})_{A'}) && \text{by flatness of } A \to A' \\ &= \mathrm{ann}_{A'}(\Omega^2_{X_{A'}/A',x'}) && \text{for locality reasons} \\ &= \mathrm{ann}_{A'}\left(\Omega^2_{Y/A',y}\right) && Y \to X_{A'} \text{ étale} \end{aligned}$$

A pointed étale map $h : (Y, y) \to (\mathrm{Spec}\ A'[u,v]/(uv - a'), \overline{0})$ exists by design of Y, so likewise $\mathrm{ann}_{A'}(\Omega^2_{Y/A',y}) = \mathrm{ann}_{A'}(\Omega^2_{(A'[u,v]/(uv-a'))/A',\overline{0}})$. But this final annihilator is something we can compute! Indeed, direct calculation of $\Omega^1_{(A'[u,v]/(uv-a'))/A'}$ and passing to its second exterior power gives

$$\Omega^2_{(A'[u,v]/(uv-a'))/A'} = (A'[u,v]/(uv-a',u,v))du \wedge dv,$$

and as a module over $A'[u,v]/(uv-a')$ this collapses to $A'/(a')$ that is already local (so localizing at $\overline{0}$ has become redundant) and visibly has A'-annihilator equal to $(a') = I'$! $\square$

As an application, let $X \to \operatorname{Spec} R$ be an "open" (meaning no properness or geometric connectivity assumptions) semistable curve over a dvr R with uniformizer π, fraction field K, and residue field k. We assume X_K is smooth and $x_0 \in X_k$ is a non-smooth point. Recall from the general structure of ordinary double point singularities that *automatically* $k(x_0)/k$ is separable. (This is very important for later considerations!)

Since all nonzero elements of $\mathfrak{m}_R$ are $R^\times$-multiples of a unit π^n with $n > 0$, there exists a unique $n_{x_0} \geq 1$ such that (X, x) has an étale neighborhood in common with

$$\left(\operatorname{Spec} R[u,v]/(uv - \pi^{n_{x_0}}), \overline{0}\right).$$

Passing to completions of local rings, we see that $x_0 \in \operatorname{reg}(X)$ if and only if $n_{x_0} = 1$ (so one may call n_{x_0} a "measure of irregularity"). Our primary interest is therefore in the case $n_{x_0} \geq 2$.

Proposition 11.22. *Assume in the above setup that $n_{x_0} \geq 2$. Let*

$$X' := \operatorname{Bl}_{X_0}(X).$$

The R-scheme X' is a semistable curve over R and:

1. *if n_{X_0} is either 2 or 3 then X' is regular over x_0*
2. *if $n_{X_0} \geq 4$ then X' has a unique non-regular point x_0' over x_0 and*

$$n_{x_0'} = n_{x_0} - 2$$

and $k(x_0') = k(x_0)$.

The proof will consist of three steps:

1. A review of blow-up charts
2. Transfer our problem to the model case (keeping careful track of residue fields and points over x_0).
3. Doing blow-up calculations in the model case (which is also [Liu, §8.3, Ex. 3.53]).

We shall carry out (1), refer to a short handout for the (technically important but neither difficult nor interesting) details for (2), and then work out (3) based on (1).

Let A be a ring and $I = (f_1, \ldots, f_n)$ be a finitely generated ideal. Define $Z := \mathrm{Spec}\ A/I \to \mathrm{Spec}\ A := X$. The *blow-up* of X along Z, denoted

$$\widetilde{X} := \mathrm{Bl}_I(A) = \mathrm{Bl}_Z(X),$$

is final among X-schemes $Y \to X$ with the property that the quasi-coherent ideal sheaf

$$I\mathscr{O}_Y \subset \mathscr{O}_Y$$

is invertible.

Remark 11.23. We are really using the ideal sheaf $I\mathscr{O}_Y$ and not $I \otimes_A \mathscr{O}_Y$. That is, $I\mathscr{O}_Y$ is the image of $I \otimes_A \mathscr{O}_Y \to \mathscr{O}_Y$ (which may not be injective if Y is not flat over X).

One standard definition/construction is

$$\widetilde{X} = \mathrm{Proj}\ (\oplus_{m \geq 0} I^m) \subset \mathrm{Proj}\ A[T_1, \ldots, T_n] = \mathbf{P}_A^{n-1}$$

in which the graded algebra $\oplus_{m \geq 0} I^m$ is generated over A by $f_1, \ldots, f_n$ in degree 1. This looks uncomputable away from very special situations, so we want another description which yields more useful descriptions of

$$\widetilde{X} \cap D_+(T_i)$$

for $1 \leq i \leq n$.

The idea is to "cover" the moduli problem for the blow-up with several refinements (corresponding to "open subfunctors"), using that if $\mathscr{L}$ is invertible on a scheme W and generated by global sections $s_1, \ldots, s_n \in \Gamma(W, \mathscr{L})$, then

$$W_i := \{w \in W : s_i \text{ is a basis of } \mathscr{L} \text{ near } w\}$$

is an open cover of W.

For any Y as above, on the open locus Y_i where f_i is a local basis it is necessarily a global basis and so we can write $f_j = h_{ij} f_i$ for a unique $h_{ij} \in \mathscr{O}(Y_i)$ if $j \neq i$. Then f_i generates $\mathscr{L}$ and and in fact freely generates it over Y_i. Thus, if we consider the A-algebra

$$A_i := \left(A\left[T_{ij}|_{j \neq i}\right] / (f_j - T_{ji})\right) / (f_i^\infty\text{-torsion}) \subset A[1/f_i]$$

generated by the f_j/f_i modulo "f_i^∞-torsion" (i.e., the directed union of the f_i^n-torsion for all n) then the relation immediately yield that $Y_i = \mathrm{Spec}\ A_i$ is final among A-schemes Y on which f_i is a basis for $I\mathscr{O}_Y$.

For $i' \neq i$, we have an identification of open subschemes

$$Y_i \supset \{T_{i'i} \neq 0\} \simeq \{T_{ii'} \neq 0\} \subset Y_{i'}$$

via the relation $T_{ii'} = 1/T_{ii'}$ (and related transition formulas for T_{ji} and $T_{ji'}$ for $j \neq i, i'$) over the open sets of each on which $f_i, f_{i'}$ are both a basis for

$I\mathscr{O}_Y$. One can easily check this construction satisfying the triple overlap condition to glue them to make $Y = \mathrm{Bl}_I(A)$.

The gluing itself respects each closed immersion Spec $A_i \to \mathbf{A}^{n-1}_A$ so globalize to a closed subscheme

$$Y \to \mathbf{P}^{n-1}_A$$

making the commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}\, A_i & \longrightarrow & \mathbf{A}^{n-1}_A \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbf{P}^{n-1}_A \end{array} \tag{11.5}$$

be *Cartesian* (with $\mathbf{A}^{n-1}_A$ as $D_+(T_i)$).

Remark 11.24. If $\{f_1, \ldots, f_n\}$ is a regular sequence in A then the f_i^∞-torsion above vanishes [FL, Ch. IV, Thm. 2.2, Cor. 2.5]. That handles things such as blow-up of a regular scheme along a regular closed subscheme, but we will need go to beyond that setting.

Remark 11.25. The universal property of blow-up is *not* on all A-schemes! It only applies to A-schemes Y with $I\mathscr{O}_Y$ invertible. An analogue to keep in mind is normalization: normalization is final only with respect to dominant maps from normal schemes. Consequently, the behavior under base change can be delicate (since applying a base change may take us outside the rather limited category of test objects for the universal property).

Consider $A \to A'$ and define $I' = IA$ and $X' = \mathrm{Spec}\, A'$. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{X'} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array} \tag{11.6}$$

since $I\mathscr{O}_{X'} = \widetilde{I'}$ pulls back to an invertible ideal sheaf on $\widetilde{X'}$, but the natural map

$$A' \otimes_A I\mathscr{O}_{\widetilde{X}} \twoheadrightarrow I\mathscr{O}_{\widetilde{X}_{A'}}$$

may not be injective and in particular the target ideal sheaf might not be invertible! Hence, the diagram above may not be Cartesian. This problem does not arise if $A \to A'$ is flat: the construction (especially the formation of f_i^∞-torsion) is compatible with such base change in an evident manner, making the natural map $\widetilde{X'} \to \widetilde{X}_{A'}$ an isomorphism. This is compatible with the Proj description of blow-ups, using that naturally

$$A' \otimes_A I^m = (IA')^m$$

when A' is A-flat.

This globalizes, allowing us to make the blow-up of any scheme X along any finitely presented closed subscheme Z. By the universal property, the formation of blow-up is Zariski-local on X (over affine opens in X, it is the construction provided above). The natural map $q : \widetilde{X} \to X$ thereby restricts to an isomorphism over $U := X - Z$ because blow-up of the ideal sheaf (1) on U is tautologically an isomorphism; in other words, $\mathrm{Bl}_{\emptyset}(U) = U$.

On the other hand,

$$\begin{aligned} q^{-1}(Z) &= \mathrm{Proj}\ (\oplus_{m \geq 0} I^m) \otimes_A A/I \\ &= \mathrm{Proj}\ ((\oplus_{m \geq 0} I^m) \otimes_A A/I) \\ &= \mathrm{Proj}\ \left(\oplus_{m \geq 0} I^m / I^{m+1}\right). \end{aligned}$$

This is called the **exceptional divisor** in $\widetilde{X}$ (it is Cartier by design of blow-ups).

Example 11.26. Suppose X is finite type over a field k and $x \in X$ is a closed point, so the residue field $k' := \kappa(x)$ is k-finite. Let $\mathfrak{m} \subset \mathscr{O}_{X,x}$ be the corresponding maximal ideal. Let's blow up X along $\{x\}$. Its exceptional divisor fits into a fiber square

$$\begin{array}{ccc} \mathrm{Proj}\ \left(\oplus_{m \geq 0} \mathfrak{m}^m / \mathfrak{m}^{m+1}\right) & \longrightarrow & \mathrm{Bl}_x(X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k' & \longrightarrow & X. \end{array} \tag{11.7}$$

If x' is a regular point (not necessarily a k-smooth point!) we have as k'-algebras

$$\begin{aligned} k'[t_1, \ldots, t_d] &\simeq \oplus_{m \geq 0} \mathfrak{m}^m / \mathfrak{m}^{m+1} \\ \{t_i\} &\mapsto \text{ a } k'\text{-basis of } \mathfrak{m}/\mathfrak{m}^2, \end{aligned}$$

so $\mathrm{Bl}_x(X)$ has fiber $\mathbf{P}^{d-1}_{\kappa(x)}$ when $x \in \mathrm{reg}(X)$.

11.4. **A detailed blow-up computation.** This completes our review of generalities on blow-ups, so we now turn to the task of proving Proposition 11.22. A handout carries out the reduction to proving it in the following "model case": for a discrete valuation ring R with residue field k and uniformizer π, we consider

$$A := R[u,v] / (uv - \pi^n)$$

with $n \geq 2$. Thus, $X := \operatorname{Spec} A \to \operatorname{Spec} R$ is a flat semistable curve such that X is normal and also is R-smooth away from the point $\xi = \overline{0} \in X_k$ at which X is not regular because $n \geq 2$. We want to compute $\mathrm{Bl}_\xi(X) = \mathrm{Bl}_I(A)$ for $I = (u, v, \pi)$. and check that it satisfies the assertions in Proposition 11.22.

Note that $\mathrm{Bl}_I(A)$ is covered by

$$D_+(u), D_+(v), D_+(\pi),$$

the three affine opens where I is free on the elements u, v, and π respectively. We want to compute these three affine schemes, their open overlaps, and study their k-fibers over X_k (especially over ξ, to determine regularity at such points on the total space of the blow-up).

11.4.1. *$D_+(u)$ computation.* Let's start by computing $D_+(u)$. Introduce new variables v' and π' that are the universal multipliers against u:

$$v := v'u$$
$$\pi' := \pi' u$$

The coordinate ring of $D_+(u)$ is

$$(A[v', \pi'] / (v - v'u, \pi - \pi'u)) / (u^\infty\text{-torsion}).$$

We have

$$A[v', \pi'] / (v - v'u, \pi - \pi'u) = R[u, v, v', \pi'] / (uv - \pi^n, v - v'u, \pi - \pi'u)$$

and since $\pi^n = uv$ we also have

$$\pi'^n u^n = u^2 v'$$

and hence

$$u^2 \left(\pi'^n u^{n-2} - v' \right) = 0$$

is a relation in this quotient. Thus, by killing u^∞-torsion we acquire the relation $v' = \pi'^n u^{n-2}$, so the variable v' can be eliminated everywhere via that substitution.

Since $v = v'u = \pi'^n u^{n-1}$, we have

$$D_+(u) = \operatorname{Spec}\left((R[u, \pi'] / (\pi - \pi'u)) / (u^\infty\text{-torsion}) \right)$$

but $\operatorname{Spec} R[u, \pi'] / (\pi - \pi'u)$ is already a (regular) domain, so there is no more u^∞-torsion to kill. Hence,

$$D_+(u) = \operatorname{Spec} R[u, \pi'] / (\pi - \pi'u),$$

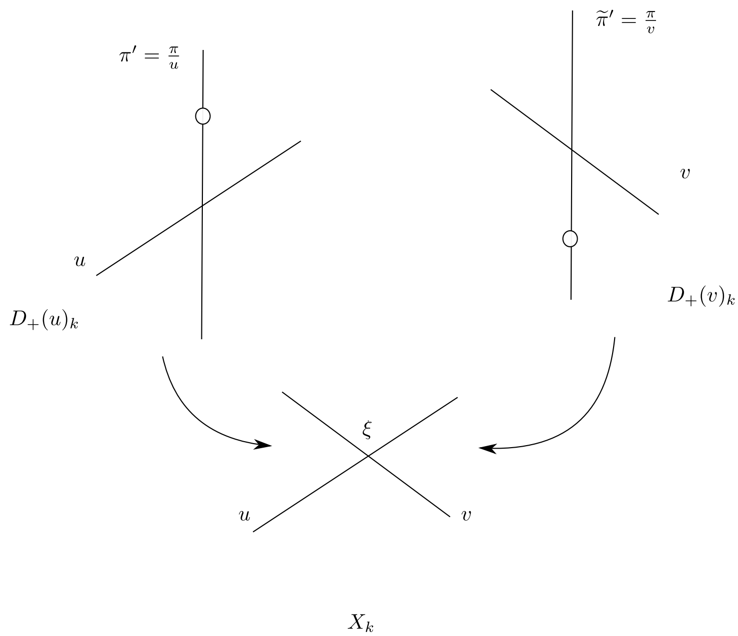


FIGURE 3. A depiction over X_k of the special fibers of the charts $D_+(u)$ and $D_+(v)$ in the blow-up.

so $D_+(u)_k = \text{Spec}\,(k[u,\pi']/(u\pi')) \to X_k = \text{Spec}\,(k[u,v]/(uv))$ identifies the u-axes on each but crushes the π'-axis into the origin $\xi \in X_k$.

11.4.2. *$D_+(v)$ computation.* The affine open $D_+(v)$ is essentially the same, up to some change of letters:

$$D_+(v) = \text{Spec}\,R[v,\widetilde{\pi}']/(\pi - v\widetilde{\pi}')$$

with a similar description of its k-fiber as Spec $k[v,\widetilde{\pi}']/(v\widetilde{\pi}')$ whose v-axis maps isomorphically onto that of X_k and whose $\widetilde{\pi}'$-axis is crushed into the origin $\xi \in X_k$. See Figure 3.

Note that both $D_+(u)$ and $D_+(v)$ are regular, as can be seen by inspection, but are not R-smooth: their special fibers are crossing axes.

11.4.3. *Overlap of $D_+(u)$ and $D_+(v)$.* Let's compute the overlap of $D_+(u)$ and $D_+(v)$ in the special fibers. Recall

$$D_+(u) = \text{Spec } R[u, \pi']/(u\pi' - \pi), \;\; D_+(v) = \text{Spec } R[v, \pi']/(v\widetilde{\pi}' - \pi)$$

with $v = v'u$ and $v' = \pi'^n u^{n-2}$ in the first chart and with $u = u'v$ and $u' = \widetilde{\pi}'^n v^{n-2}$. We have $D_+(u)_k \cap D_+(v)_k$ is the locus where $v' = \pi'^n u^{n-2}$ is invertible on $D_+(u)_k$ and $u' = \widetilde{\pi}'^n v^{n-2}$ is invertible mod $D_+(v)_k$. When $n \geq 3$ (so $n-2 > 0$) it follows that these special fibers do not meet at all, whereas if $n = 2$ they overlap exactly away from the complement of the origin on the lines $u = 0$ and $v = 0$, in each via the relation $\pi'\widetilde{\pi}' = \pi^2/(uv) = 1$ on the total space. Hence, the glued k-fiber contains a copy of $\mathbf{P}^1_k$ covered by the π'-axis and $\widetilde{\pi}'$-axis of each.

11.4.4. *Computing $D_+(\pi)$.* On this affine open chart we have $u = u''\pi, v = v''\pi$, so $\pi^n = uv = u''v''\pi^2$. Since $n \geq 2$, we have $\pi^2\left(\pi^{n-2} - u''v''\right) = 0$. Since π^∞-torsion vanishes on $D_+(\pi)$, in the coordinate ring of $D_+(\pi)$ we obtain the relation

$$u''v'' - \pi^{n-2}.$$

After killing this relation, we conversely obtain the old relation $\pi^n = uv$ from the relation $u''v'' = \pi^{n-2}$ and hence

$$D_+(\pi) = \text{Spec}\left(R[u'', v'']/(u''v'' - \pi^{n-2})\right).$$

(This is easily seen to be R-flat, so it really is the correct affine open part of the blow-up.) When $n = 2$ this gives $u''v'' = 1$ on $D_+(\pi)$, so

$$D_+(\pi)_k \subset D_+(u)_k \cup D_+(v)_k.$$

Hence, when $n = 2$, the third chart is redundant on the special fiber and the exceptional divisor is $\mathbf{P}^1_k$ with all points regular in the total space (as $D_+(u)$ and $D_+(v)$ are regular). Consider the more interesting case $n \geq 3$, so $D_+(\pi)_k$ on the special fiber misses the two points ∞ on the lines $u''_k = 0$ and $v''_k = 0$. The total special fiber contains affine lines with coordinates $u_k, u''_k, v''_k, v_k, \pi'_k, \widetilde{\pi}'_k$. Among these, the u_k-axis meets the u''_k-axis with k-rational crossing point inside the regular $D_+(u)$, the v_k-axis meets the v''_k-axis with k-rational crossing point contained in the regular $D_+(v)$, and the u''_k-axis meets the v''_k-axis with k-rational crossing point contained in $D_+(\pi)$ whose unique non-smooth point in the special fiber is a k-point with measure of irregularity $n-2$. See Figure 4 for a picture of the case $n = 2$ and Figure 5 for a picture of the case $n \geq 3$.

For $n \geq 3$, we conclude that the exceptional divisor is the union of two copies of $\mathbf{P}^1_k$ crossing at a single k-point whose measure of irregularity is

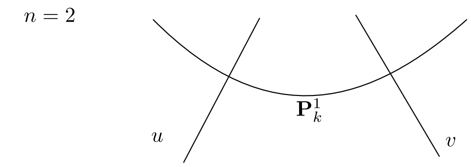


FIGURE 4. A depiction of the total special fiber when $n = 2$.

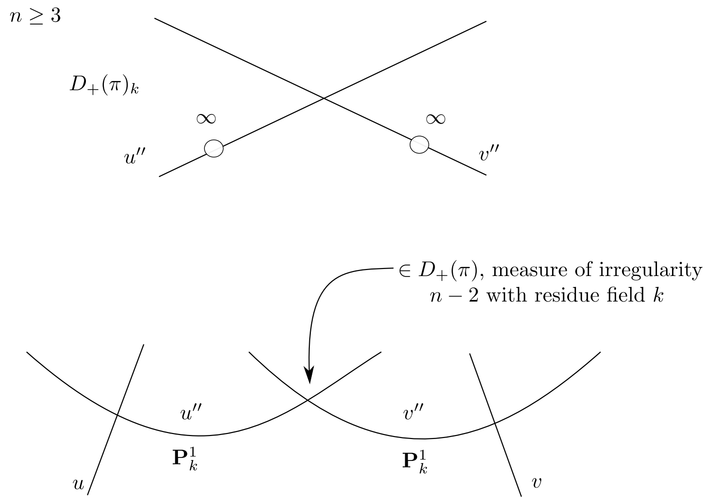


FIGURE 5. A depiction of $D_+(\pi)_k$ when $n \geq 3$, followed by a picture of the total special fiber (gluing of $D_+(u)_k$, $D_+(v)_k$, and $D_+(\pi)_k$) when $n \geq 3$.

$n-2$. We conclude that the blow-up is regular when $n \geq 3$ and that if $n \geq 4$ it has a unique non-regular point that is moreover k-rational with measure of irregularity $n-2$.

12. RESOLVING SEMISTABLE CURVES OVER A REGULAR BASE

For semistable curves over a general base, we shall now put a canonical scheme structure on the non-smooth locus, given by the annihilator of Ω^2.

Over a dvr this will recover the reduced structure on isolated non-smooth points in the special fiber.

Definition 12.1. An *open semistable curve over S* is a flat finitely presented surjection $f : X \to S$ whose fibers X_s are semistable curves.

Define $\mathrm{sm}(X/S)$ to be the Zariski-open S-smooth locus on X (which is fiberwise dense), and $\mathrm{sing}(X/S)$ to be $X - \mathrm{sm}(X/S)$. Note that even for regular S, $\mathrm{sing}(X/S)$ may meet $\mathrm{reg}\, X$.

Remark 12.2. When someone says "consider a singular point" on a scheme that isn't (locally) of finite type over a perfect field, you should always without exception ask: do you mean "not regular" or "not smooth"?

Lemma 12.3. *The annihilator ideal sheaf*

$$\mathrm{ann}_{\mathscr{O}_X}(\Omega^2_{X/S})$$

(quasi-coherent since $\Omega^2_{X/S}$ is finitely presented) is finitely generated with underlying zero-scheme $\mathrm{sing}(X/S)$. Moreover:

1. *The formation of $\mathscr{O}_X / \mathrm{ann}_{\mathscr{O}_X}(\Omega^2_{X/S})$ commutes with any base change on S.*
2. *Via this scheme structure, the finitely presented map $\mathrm{sing}(X/S) \to S$ is quasi-finite with étale fibers (equivalently, "unramified" in the sense of Grothendieck).*

Remark 12.4. The unramifiedness condition is equivalent to the vanishing of $\Omega^1_{\mathrm{sing}(X/S)/S}$. The reason for this is that the vanishing of this finitely presented quasi-coherent sheaf is suffices to check on fibers over S (due to Nakayama's Lemma), and for X of finite type over a field k we have $\Omega^1_{\mathrm{sing}(X/k)/k} = 0$ if and only if X is k-étale (as may be checked over $\bar{k}$).

Proof. To show

$$\mathrm{ann}_{\mathscr{O}_X}(\Omega^2_{X/S})$$

is finitely generated and satisfies (1), the problem is étale-local on X and S. Thus, it is enough to consider the model case.

That is, it suffices to consider

$$X = \mathrm{Spec}\, B := \mathrm{Spec}\, \left(A[u,v]/\left(uv - a\right)\right)$$

with $S = \operatorname{Spec} A$ and $a \in A$. Then,

$$\begin{aligned}\Omega^2_{B/A} &= \wedge^2_B \left(\Omega^1_{B/A}\right) \\ &= \wedge^2_B \left(\frac{B\,du \oplus B\,dv}{(u\,dv + v\,du)}\right) \\ &= (B/(u,v))\; du \wedge dv,\end{aligned}$$

so $\operatorname{ann}_B(\Omega^2_{B/A}) = (u,v) \subset B$ (visibly finitely generated). Thus,

$$B/\operatorname{ann}_{\mathscr{O}_X}(\Omega^2_{X/S}) = A/(a)$$

as A-algebras. It is clear that the formation of this quotient of B is compatible with any scalar extension on A, as desired.

To verify the support of $\mathscr{O}_X/\operatorname{ann}_{\mathscr{O}_X}(\Omega^2_{X/S})$ inside X is as claimed, and to verify (2), it now suffices to check on fibers. That is, we can assume $S = \operatorname{Spec} k$ for a field k. Further, we can assume $k = \overline{k}$, via flat base change.

On X^{sm} the sheaf Ω^1 is invertible, so $\Omega^2 = 0$. Therefore

$$(\operatorname{ann}_{\mathscr{O}_X} \Omega^2_{X/S})|_{X^{\mathrm{sm}}} = \mathscr{O}_{X^{\mathrm{sm}}},$$

so it only remains to verify what is happening at the non-smooth points. That is, we need to show $(\operatorname{ann}_{\mathscr{O}_X} \Omega^2_{X/S})_x = \mathfrak{m}_x$ for $x \in \operatorname{sing}(X/k)$. Since k is algebraically closed, so $k(x) = k$, we can work étale-locally to reduce to the model case

$$(X,x) = ((uv \subset \mathbf{A}^2_k), (0,0))$$

that is easily handled by a direct calculation. □

We return to the original situation. Recall we have a proper semistable curve

$$\begin{array}{ccc} & & X \\ & & \big\downarrow f \\ D & \longrightarrow & Y \end{array} \tag{12.1}$$

where D is an sncd in a smooth projective variety Y of dimension $d-1 \geq 1$ and X is a proper semi-stable curve over Y that is smooth over $U := Y - D$ with geometrically connected fibers.

Modulo the desired sncd condition for $Z = f^{-1}(D) \cup (\cup \sigma_i(Y))$ for some sections $\sigma_i \in X(Y)$, we need to resolve singularities of X as a k-scheme. It is this latter aspect that will be our primary focus; handling the task of making Z an sncd will come at the end, so we set Z aside for now.

If $d = 2$ then Y is a smooth curve and so $\mathrm{sing}(X/Y)$ maps quasi-finitely to some finite set of points $y_1, \ldots, y_n \subset Y(k)$. For $R_i := \mathscr{O}_{Y,y_i}$ a dvr, when we localize at y_i we obtain the situation

$$X_{R_i} \downarrow \mathrm{Spec}\, R_i \tag{12.2}$$

that is exactly our earlier focus (semistable curves with smooth generic fiber over discrete valuation rings).

Thus, by repeatedly blowing up points in X that are not k-smooth (hence not Y-smooth, so there are only finitely many of these), the process preserves semistability over Y and improves the measure of irregularity without ever increasing the number of non-regular points. Therefore, this process eventually terminates at a curve semistable over Y with measure of irregularity 1 at each of the finitely many points not smooth over Y, so it resolves singularities on the k-surface X.

Now suppose instead $d \geq 3$. In this case the proper map

$$\mathrm{sing}(X/Y) \to D$$

is quasi-finite and hence finite to the sncd D in the k-smooth Y, and D has pure dimension $d - 2$. Thus, each irreducible component T of $\mathrm{sing}(X/Y)$ has codimension at least 2 in X. Further, by dimension reasons, the codimension of such a T in X equals 2 if and only if T surjects onto some irreducible component $D_i \subset D$ (which we give the reduced structure, so D_i is k-smooth since D is an sncd in the k-smooth Y).

Our main focus is when such T of codimension 2 is in the non-regular locus of X (i.e., $\mathrm{sing}(X/k)$ has codimension 2 rather than ≥ 3). Given such a T, we want to "improve" its non-regularity via a blow-up, and repeat the process until we manage to reach a situation in which $\mathrm{sing}(X/k)$ has codimension ≥ 3 in X.

Consider an irreducible component T of $\mathrm{sing}(X/k)$ with codimension 2 in X (recall that X is normal, being a semistable curve with smooth generic fiber over a connected normal base, so $\mathrm{sing}(X/k)$ has codimension ≥ 2. Thus, T is also an irreducible component of $\mathrm{sing}(X/Y)$ with codimension 2 in X. Give T the reduced structure. We have a finite (since proper quasi-finite) surjection $T \twoheadrightarrow D_i$ for some i, so the resulting map between generic points $\eta_T \to \eta_{D_i}$ is étale because $R = \mathscr{O}_{Y,\eta_{D_i}}$ is a dvr for which η_T is a non-smooth point in the special fiber of X_R. The aims in this situation are:

(1) to show $T \to D_i$ is étale,

(2) to define a measure of irregularity along the entirety of T inside X, enabling us to carry out local calculation along T (not only near η_T).

If $\eta_T \notin \mathrm{Reg}(X)$ then we want to blow up X along T, but we need to verify that $\mathrm{Bl}_T(X)$ it remains semistable over Y (and that in a suitable sense this blow-up improves the non-regularity). Once we push $\mathrm{sing}(X/k)$ out of codimension 2 into codimension 3, the possibilities for the structure for $\mathrm{sing}(X/k)$ (such as the dimensions of its irreducible components and their overlaps) will turn out to be rather nice.

As in [deJ, 3.1], we now make the following extended setup for our current situation. We shall first proceed to resolve singularities in codimension 2 (in Theorem 12.5) after possibly passing to a completed local ring on the base, so we'll describe a setup that encompasses this.

Let S be a connected regular excellent scheme and $f : X \to S$ a proper semistable curve with $D \subset S$ sncd such that f is smooth over $S - D$ with geometrically connected fibers. (Recall that by Stein factorization, the geometric connectivity of all fibers follows from smoothness and geometric connectivity of the generic fiber since the base S is normal.) We know that X is normal so that $X^{\mathrm{sing}} := X - \mathrm{Reg}(X)$ has codimension at least 2. Further, write $D = \cup_i D_i$ with D_i the reduced irreducible components of D (all regular, since D is an sncd in S). Each $D_i \subset S$ is Cartier and for every subset $J \subset I$ of the index set I, the intersection

$$D_J := \bigcap_{j \in J} D_j \subset S$$

is regular with pure codimension $\#J$. Thus, for all points $d \in D_j$, generators of the invertible ideals $\mathscr{I}_{D_j,d} \subset \mathscr{O}_{S,s}$ constitute part of a regular sequence of parameters for $\mathscr{O}_{S,d}$.

Note that since S is regular,

$$\begin{aligned} \mathrm{sing}(X) &\subset \mathrm{sing}(X/S) \\ &:= X - \mathrm{sm}(X/S) \\ &\subset f^{-1}(D). \end{aligned}$$

For irreducible components T of $\mathrm{sing}(X)$, we want to get to the situation where $\mathrm{codim}_X(T) \geq 3$ for all T. Note that $\mathrm{sing}(X/S) \subset X$ has codimension at least 2 everywhere since X_{η_S} is smooth and all X_s are generically smooth. For irreducible components $T \subset \mathrm{sing}(X)$ of codimension 2 in X, we want to show $\mathrm{Bl}_T(X)$ "improves things."

Theorem 12.5. *There exists a modification $\phi : X_1 \to X$ (to be constructed as a composite of several blow-up maps) that is an isomorphism over*

$$\operatorname{reg} X \supset \operatorname{sm}(X/S) \supset f^{-1}(S - D),$$

such that

(1) $X_1 \to S$ *is a semistable curve*
(2) $\operatorname{codim}_X(\operatorname{sing}(X_1)) \geq 3$.

Here is the method of proof: Without loss of generality, there exists an irreducible component $T \subset X^{\text{sing}}$ with codimension 2 (or else there is nothing to do), and for $X' := \operatorname{Bl}_T(X) \to X$ we'll show

(i) $X' \to S$ is a semistable curve
(ii) in an appropriate sense, $\operatorname{sing}(X')$ are "less severe" than $\operatorname{sing}(X)$ by using a "measure of irregularity along an entire irreducible component of $\operatorname{sing}(X)$ with codimension 2.

Once we make (ii) precise and we prove it, we can iterate the process to reach the desired X_1 in finitely many steps. A source of some further complications beyond our calculations over dvr's is that we are blowing-up along the closed subscheme T that is no longer a point.

To prove actually prove Theorem 12.5, we will need to establish several ingredients. First, we need to describe (X, T) étale-locally without losing contact with irreducibility of T. This rests on the following subsidiary result:

Proposition 12.6. *The codimension-*2 *component T maps finitely onto some D_{i_T}, and $T \to D_{i_T}$ is étale. (Recall that $\eta_T \mapsto \eta_{D_{i_T}}$ is étale, with the generic point η_T of T being in codimension* 2*.)*

The integral T maps onto an irreducible component D_{i_T} of D. Indeed, the inclusion $T \subset \operatorname{sing}(X)$ implies that for all $t \in T$, $t \in X_{f(t)}$ is not smooth due to the regularity of S. Since the fibers X_s are generically smooth, $T \to D$ must be quasi-finite (so finite by properness) and then onto some D_{i_T} (via a finite surjection) for dimension reasons.

In particular, the generic point η_T of T maps to the generic point η_{i_T} of D_{i_T}. Since η_{i_T} has codimension-1 in S, the point η_T is a non-smooth point in the special fiber of $X_R \to \operatorname{Spec} R$ for the dvr $R = \mathscr{O}_{S,\eta_{i_T}}$. Thus, $\eta_T \mapsto \eta_{i_T}$ is étale, so $T \to D_{i_T}$ is étale when restricted to dense opens on the source and target. We want to show $T \to D_{i_T}$ is *everywhere* étale. We have

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{sing}(X/S) \\ \downarrow & & \downarrow \\ D_{i_T} & \longrightarrow & S \end{array} \tag{12.3}$$

with $\mathrm{sing}(X/S) \to S$ having étale fiber schemes. Thus, the map $T \to D_{i_T}$ to a regular (hence normal) scheme is a finite surjection between integral noetherian schemes with étale fibers. Now apply the following lemma:

Lemma 12.7. *Say $B \to C$ is a finite injection of noetherian domains with B normal and $\Omega^1_{C/B} = 0$(equivalent to* $\mathrm{Spec} C \to \mathrm{Spec} B$ *having étale fibers, by Nakayama's Lemma).*

Then, C is B-étale (so B-flat).

Proof. The key point is that a map of finite type between noetherian schemes with vanishing Ω^1 is Zariski-locally a closed immersion into a scheme étale over the base. Thus, Zariski-locally Spec C is a closed subscheme of an étale B-scheme E. By normality, the connected components of E are irreducible. By dimension reasons, the part of the dominant B-finite Spec C meeting any component of E must exhaust that component, so Spec B is C-étale. For details, see [FK, Ch. I, Lemma 1.5]. □

This completes the proof of Proposition 12.6. Next, we describe some completions. Choose $x \in \mathrm{sing}(X/S)$ over $s = f(x) \in D$ (e.g., any point in T). Let $D_{i_1}, \ldots, D_{i_m}$ be the irreducible components of D through s. We want to describe $\widehat{\mathscr{O}_x}$ as an $\widehat{\mathscr{O}}_s$-algebra using the generators t_j of $\mathscr{I}_{D_{i_j}} \subset \mathscr{O}_{S,s}$.

Warning 12.8. Keep in mind that $\kappa(x)$ is finite separable over $\kappa(s)$ (and usually this extension is non-trivial). We'll need to make a maneuver to pass to the case $\kappa(x) = \kappa(s)$.

We now reduce the task of describing such a completion to the case $\kappa(x) = \kappa(s)$. Let R be the unique local finite étale $\widehat{\mathscr{O}}_s$-algebra with residue field $\kappa(x)/\kappa(s)$. Explicitly,we pick a primitive element

$$\kappa(x) = \kappa(s)[U]/(f)$$

with f monic, so for a monic lift $F \in \widehat{\mathscr{O}}_s[U]$ the quotient ring

$$R = \widehat{\mathscr{O}}_s[U]/(F)$$

is finite flat over $\widehat{\mathscr{O}}_s$ with $R/\mathfrak{m}_0 R = \kappa(x)$ a field, so R is local and visibly finite étale over $\widehat{\mathscr{O}}_x$.

By the henselian property for $\widehat{\mathscr{O}}_x$, there is a unique map of $\widehat{\mathscr{O}}_s$-algebras

$$\begin{array}{ccc} \widehat{\mathscr{O}}_s & \longrightarrow & R \\ & \searrow \quad \swarrow_{t} & \\ & \widehat{\mathscr{O}}_x & \end{array} \tag{12.4}$$

lifting the equality of residue fields. We have a fiber diagram

$$\begin{array}{ccccc} Y' & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} \widehat{\mathscr{O}}_s & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow \\ D'_s & \longrightarrow & \widehat{D}_s & \longrightarrow & D \end{array} \tag{12.5}$$

Note that

$$\begin{aligned} Y' &\ni \operatorname{Spec} \kappa_R \times_S x \\ &= \operatorname{Spec} \left(\kappa(x) \otimes_{\kappa(s)} \kappa(x)\right) \end{aligned}$$

We can write $\operatorname{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(x))$ as a union of $\Delta(x)$ with some other scheme, and we define $y' := \Delta(x)$.

Exercise 12.9. Verify that as R-algebras,

$$\widehat{\mathscr{O}}_{Y',y'} \simeq \widehat{\mathscr{O}}_{X,x}.$$

The upshot is that we can pass to (Y', y') over $\operatorname{Spec} R$ to focus on the case $\kappa(x) = \kappa(s)$. In order to continue our proof of Proposition 12.6, we require the following lemma:

Lemma 12.10. *Suppose $S = \operatorname{Spec} A$ for a regular complete local noetherian ring A and $s \in S$ the closed point. Let $x \in X_s$ be non-smooth with $\kappa(x) = \kappa(s)$. Then there is an isomorphism of A-algebras*

$$\widehat{\mathscr{O}_x} \simeq A[\![u,v]\!]/(Q-a)$$

for a residually non-degenerate quadratic form Q and $a \in \mathfrak{m}_A - \{0\}$, and $a = \prod t_j^{n_j}$ for local generators t_j of the ideal sheaves

$$\mathscr{I}_{D_j,s} \subset A$$

of the D_j passing through s and $n_j \geq 0$. Also, $x \notin \operatorname{Reg}(X)$ if and only if $\sum_j n_j \geq 2$.

Remark 12.11. We can always change a by an $A^\times$-multiple by absorbing a unit into the quadratic form.

To prove Lemma 12.10, note that the refined structure theorem for ordinary double points (Theorem 11.18 and Remark 11.19) gives

$$\widehat{\mathscr{O}}_x \simeq A[\![u,v]\!]/(Q-a)$$

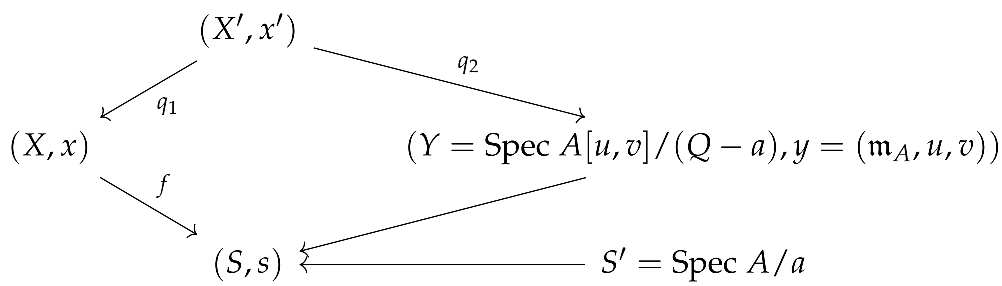
for some Q and $a \in \mathfrak{m}_A - \{0\}$ unique up to $A^\times$-multiple.

We shall show Spec $A/a \subset D$, which would imply $a \in A^\times \cdot \prod t_j^{n_j}$ for some $n_j \geq 0$ since A is regular local, hence a UFD. The main input in the proof will be Artin approximation and the smoothness over $S - D$:

Lemma 12.12. *We have* $|\mathrm{Spec}(A/a)| \subset D$*.*

Proof. It is enough to show that f is not smooth over any points in $S' :=$ Spec $A/a \subset S$. That is, we claim $X_{s'}$ is not smooth for all $s' \in S'$.

Using Artin approximation, there exists a common residually trivial étale neighborhood
(12.6)



with q_1, q_2 both étale. It is enough to show that X' has nonsmooth fiber over all $s' \in S'$, since then the q_1-images of such points do the job for $f : X \to S$. Therefore, it is enough to show that the open image $q_2(X') \subset Y$ contains the entire 0-section $\sigma : S' \to Y$ since the origin $u = v = 0$ in Y over points in the base where $a = 0$ are non-smooth points in their fibers over S' (as q_2 is étale).

The overlap $q_2(X') \cap \sigma(S')$ is open in the local scheme $\sigma(S')$ and contains the closed point $y = q_2(x')$. The only open subscheme of a local scheme that contains the closed point is the entire space, so $q_2(X')$ must contain all of $\sigma(S')$ as desired. ☐

This completes the proof of Lemma 12.10.

Corollary 12.13. *Prior to completing the base S, if $x \in \mathrm{sing}(X/S)$ and $s = f(x)$ then we obtain intrinsic integers $n_i(x) \geq 0$ attached to all $D_i \ni s$.*

Proof. For any point, we have a description of the completed local ring. We get the element $a = \prod_{i=1}^{m} t_i^{n_i}$, unique up to unit, via Lemma 12.10. This provides intrinsic multiplicities $n_i(x) \geq 0$ for each $D_i \ni s$. $\square$

Now it is time to analyze $n_{i_T}(x)$ in Lemma 12.10 for varying $x \in T$ when T is an irreducible component of $\mathrm{sing}(X/S)$ of codimension 2 in X. (For example, we could take T to be an irreducible component of X^{sing} of codimension 2 in X.)

The étaleness of T over the regular scheme D_{i_T} implies that T is regular, so for $x \in T$ the local ring $\widehat{\mathscr{O}_{T,x}}$ is regular, hence a domain.

The next lemma Lemma 12.14 defines a measure of irregularity $n_T \geq 1$.

Lemma 12.14. *For all $x \in T$, the exponent $n_{i_T}(x)$ is independent of x.*

Proof. Let $s = f(x)$ and define $\widehat{\mathscr{O}}'_s$ to be the a local finite étale $\widehat{\mathscr{O}}_s$-algebra with residue field $\kappa(x)$. So,

$$\widehat{\mathscr{O}}_x \simeq \widehat{\mathscr{O}}'_s[\![u,v]\!]/(Q_x(u,v) - \prod_i t_i^{n_i(x)})$$

as $\widehat{\mathscr{O}}'_s$-algebras, where t_i are local generators of $\mathscr{I}_{D_i} \subset \mathscr{O}_S$ near s for those $D_i \ni s$ and Q_x is a residually non-degenerate quadratic form (so $\mathrm{disc}(Q_x) \in (\widehat{\mathscr{O}}'_s)^\times$, a property retained under any Zariski-localization on the base). By using Artin approximation, we may use any Q_x with a specific reduction over $\kappa(x)$. Hence, for a suitable étale neighborhood

$$(\mathrm{Spec}\, A, \xi) \to (S, s)$$

we have

$$\begin{array}{ccc} & (X', x') & \\ \swarrow^{q_1} & & \searrow^{q_2} \\ (X,x) & & \mathrm{Spec}\, A[u,v]/(Q - \prod_i t_i^{n_i}, (\xi, (0,0))) \\ \searrow & & \swarrow \\ & (S,s) & \end{array} \tag{12.7}$$

with q_1 and q_2 étale, $x' \mapsto x \in T \subset X$, and $T \twoheadrightarrow D_{i_T}$. Let's localize S at η_{i_T}.

The open $q_1(X')$ meets T at x, so it contains all generizations of x in T (such as η_T). Therefore, there is some $y' \in X'$ with $q_1(y') = \eta_T$. Look at the image $q_2(y')$. After localizing at η_{i_T}, all t_j for $j \neq i_T$ become units. Therefore, the localization of the model case at $q_2(y')$ is a Zariski-localization of

$$R[u,v]/\left(Q - t_{i_T}^{n_{i_T}}\right)$$

with R a dvr with uniformizer t_{i_T} (explicitly, R is the localization of A at $q_2(y'))$ and $\mathrm{disc}(Q) \in R^\times$. Hence, the intrinsic measure of irregularity at η_T that has nothing to do with the specific choice of $x \in T$ is equal to $n_{i_T}(x)$. $\square$

12.1. **Proving Theorem 12.5.** Our goal is now to complete the proof of Theorem 12.5. Again, recall the setup: we have

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & & \downarrow f \\ D_{i_T} & \longrightarrow & Y \end{array} \tag{12.8}$$

with D_{i_T} an irreducible component of the sncd $D \subset Y$ for smooth projective Y over $k = \overline{k}$, f semistable proper with geometrically connected fibers and smooth over $Y - D$, and

$$T \subset X^{\mathrm{sing}} \subset \mathrm{sing}(X/Y)$$

an irreducible component of X^{sing} with codimension 2 in X (so T is also an irreducible component of $\mathrm{sing}(X/Y)$, the situation considered above).

Since $\eta_T \notin \mathrm{Reg}(X)$ we have $n_T \geq 2$ (recall $n_T = n_{i_T}(x)$ for all $x \in T$, by Lemma 12.14). We have

$$\widehat{\mathscr{O}_{X,x}} \simeq (\widehat{\mathscr{O}}_{f(x)})'[\![u,v]\!]/(Q - \prod_j t_j^{n_j}(x))$$

where $(\widehat{\mathscr{O}}_{f(x)})'$ is the unique local finite étale $\widehat{\mathscr{O}_{f(x)}}$ algebra with residue field $\kappa(x)$ and

(1) all D_j's are irreducible components of D through $f(x)$, with

$$(t_j) = \mathscr{I}_{D_j, f(x)} \subset \mathscr{O}_{Y, f(x)}$$

(2) $n_j(x) \geq 0$ (so $n_{i_T}(x) = n_T \geq 2$).

Our aim is to show:

Goal 12.15. The blow-up $\widetilde{X} := \mathrm{Bl}_T(X)$ is a proper semistable Y-curve with geometrically connected fibers, smooth over $Y - D$, with $\widetilde{X}^{\mathrm{sing}}$ "simpler" than X^{sing} with respect to the irreducible components of codimension 2: either the number of codimension-2 components has gone down, or there is a unique one over T, call it T', and $n_{T'} = n_T - 2$.

Remark 12.16. Certainly $\widetilde{X} \to Y$ is proper and over $Y - D$ is smooth with geometrically connected fibers of dimension 1 since $T \subset f^{-1}(D)$, so

$$\widetilde{X}|_{Y-D} \simeq X|_{Y-D}.$$

Therefore, to verify the desired semistability properties for $\widetilde{X} \to Y$ it is enough to check that $\widetilde{X} \to Y$ is open semistable, which is an étale-local problem on $\widetilde{X}$ and Y (as we have seen in earlier discussions via Stein factorization).

Provided the blow-up is Y-semistable, which we have noted is an étale-local problem, the following argument establishes Theorem 12.5. Granting this Y-semistability, it follows from our work over with blowing up semistable curves with smooth generic fiber over discrete valuation rings that the codimension-2 irreducible components $\widetilde{T}$ of $\widetilde{X}^{\text{sing}}$ correspond bijectively via the projection to codimension-2 irreducible components T_1 of X^{sing} with the exception that we omit T if n_T is either 2 or 3.

The point is that we can detect any such $\widetilde{T}$ looking over the generic points of D. Further, only over $\eta_{D_{i_T}}$ has anything changed, and there we understand how it has changed by localizing the base Y at that generic point, to arrive at the situation over a dvr. Then, over that point $\eta_{D_{i_T}}$ there is only one non-regular point in the blow-up over the given non-regular point on X, and its measure of irregularity drops by 2 by Proposition 11.22. If $n_T > 3$ then there is a unique $\widetilde{T}_1$ over T and it satisfies $n_{\widetilde{T}_1} = n_T - 2$, whereas if $n_T = 2, 3$ then there is no such codimension-2 component mapping onto T. Hence, there are at most as many codimension-2 irreducible components of $\text{sing}(\widetilde{X})$, and the total irregularity (the sum over all n_{i_T} for T irreducible components) drops in the blow-up.

12.1.1. *Justifying semistability of the blowup.* It remains to show the blow-up $\widetilde{X}$ is open-semistable over Y. This will be done in Proposition 12.17, in combination with a handout that works out calculations in a "model case". Our task is étale local on X around each $x \in T$. Recall the formation of $\widetilde{X} = \text{Bl}_T(X)$ commutes with flat base change on X, such as passage to an étale neighborhood of (X, x).

As a first step, it is harmless to pass to an étale neighborhood

$$\left(Y', y'\right) \to (Y, f(x)).$$

Indeed,

$$T \times_Y Y' \to T$$

is étale, so the connected components of $T \times_Y Y'$ are its irreducible components because regularity of T (see Proposition 12.6) implies $T \times_Y Y'$ is regular. Checking local properties of the blow-up along a disjoint union of closed subschemes amounts to checking the blow-up along each component separately. (Beware that we do *not* claim that blow-up along a reducible closed subscheme whose irreducible components meet can be achieved by successive blow-up of such components individually.)

We can thereby increase the residue field at $f(x)$ to make Q residually split and so may arrange that $Q = uv$. Thus, now by Artin approximation we have a common étale neighborhood

$$\begin{array}{ccccc} & & (W,w) & \xrightarrow{\ q_2\ } & \\ & & \downarrow{\scriptstyle q_1} & & \\ T & \longrightarrow & (X,x) & & \left(\operatorname{Spec} B = R[u,v]/(uv - \prod_j t_j^{n_j}), \xi\right) \\ \downarrow & & \downarrow & & \downarrow \\ D_{i_T} & \longrightarrow & (Y,y) & \longleftarrow & (\operatorname{Spec} R, y) \end{array} \tag{12.9}$$

with q_1, q_2 étale, $y = f(x)$, and $\xi = (y,0,0)$. We define $\widetilde{T} := q_1^{-1}(T)$ (which might be disconnected).

In the model case of Spec B, observe that

$$Y := V(t_1, u, v) = \operatorname{Spec} R/(t_1) \subset \operatorname{Spec} B$$

is a codimension-2 irreducible component of Spec B^{sing} passing through ξ. (The irregularity is clear since we can check it at the generic point, using that $n_1 = n_T \geq 2$ to ensure $\eta_T \notin \operatorname{reg} X$.)

Proposition 12.17. *In order to show* $\operatorname{Bl}_T(X)$ *over* x *is semistable, it suffices to show that* $\operatorname{Bl}_Y(\operatorname{Spec}(B))$ *over* ξ.

Proof. This proof proceeds along standard lines.

(1) First, we reduce to the case $\widetilde{T}$ is irreducible As we have initially constructed $\widetilde{T}$, it is possible that it may be reducible. But because $\widetilde{T} \to T$ is étale with $w \mapsto x$, and T is regular, it follows that $\widetilde{T}$ is regular with a unique connected component through w and that in turn must be irreducible by regularity. Hence, as far as studying what is happening in the blow up around x, we may as well shrink around w so that only the component passing through w appears. In other words, we may shrink W around w for the Zariski topology to make $\widetilde{T}$ irreducible.

(2) It is enough to show that $\operatorname{Bl}_{\widetilde{T}}(W)$ is Y-semistable at points over w. Indeed, by flat base change, $\operatorname{Bl}_{\widetilde{T}}(W) = \operatorname{Bl}_T(X) \times_X W$ with $w \mapsto x$, and $W \to X$ is étale.

(3) We can also shrink W further if necessary so that $q_2^{-1}(\xi) = w$. Then we have:

Lemma 12.18. *$Y = V(t_1, u, v) \subset \operatorname{Spec}(B)$ is the unique codimension-2 irreducible component of* $\operatorname{sing}(\operatorname{Spec}(B))$ *through ξ.*

Proof. Since $q_2 : (W, w) \to (\operatorname{Spec} B, \xi)$ is étale,

$$q_2^{-1}(\operatorname{sing}(\operatorname{Spec} B)) = W^{\text{sing}}.$$

Hence, the uniqueness of $\widetilde{T}$ implies the uniqueness of Y through ξ because we arranged that $q_2^{-1}(\xi) = \{w\}$. □

The preceding result implies $\widetilde{T} = q_2^{-1}(Y)$ since $q_2^{-1}(Y)$ is the union of the components of the non-regular locus through $q_2^{-1}(\xi) = \{w\}$ with codimension 2 in W, and we arranged that this is $\widetilde{T}$.

(4) We conclude that

$$\operatorname{Bl}_{\widetilde{T}}(W) = \operatorname{Bl}_Y(\operatorname{Spec} B) \times_{\operatorname{Spec} B} W,$$

so to show $\operatorname{Bl}_{\widetilde{T}}(W)$ is semistable near points over w it suffices to prove that

$$\operatorname{Bl}_{(u,v,t_1)}(R[u,v]/(uv - \prod_j t_j^{n_j})$$

is R-semistable for a regular local ring R (e.g., $R = \mathscr{O}_{Y,y}$ in our case) and $\{t_j\} \subset \mathfrak{m}_R$ part of a regular system of parameters with $n_1 \geq 2$. This is a hands-on calculation generalizing what we did over dvr's, and is done in a handout.

□

Now, by Theorem 12.5,

$$\operatorname{codim}\left(X^{\text{sing}}, X\right) \geq 3.$$

Lemma 12.19. *For all $x \in X^{\text{sing}}$, we have $n_i(x) \leq 1$ for all i.*

Proof. Suppose some $n_{i_0}(x) \geq 2$, so $\widehat{\mathscr{O}_x}$ has a local finite étale extension that is the completion of an étale neighborhood of

$$(W := \operatorname{Spec}(R[u,v]/(uv - \prod_i t_i^{n_i(x)})), w = (u, v, \mathfrak{m}_R))$$

with R local finite étale over $\widehat{\mathscr{O}_{f(x)}}$. Note that W is normal since it is an open-semistable curve over R with smooth generic fiber, so the non-regular locus of W (which is closed by excellence) is supported entirely in codimension ≥ 2. The integral closed subset

$$E := \left\{u = v = t_{i_0} = 0\right\} \simeq \operatorname{Spec}(R/(t_{i_0}))$$

has codimension 2 in W, and W is not regular at the generic point of E since the completion of E at that point is easily seen to be the visibly non-regular local noetherian ring $R'[\![u,v]\!]/(uv - t_{i_0}^{n_{i_0}(x)})$ where R' is the completion of the regular local ring $(R_{\mathfrak{p}})^\wedge$ for $\mathfrak{p} = (u, v, t_{i_0}) \subset R$ (note that $t_j \in R_{\mathfrak{p}}^\times$ for $j \neq i_0$). Thus, E is an irreducible component of $\mathscr{O}_{W,w}^{\text{sing}}$; i.e., the non-regular locus of W near w has an irreducible component of codimension 2.

The faithfully flat map from any excellent local noetherian ring to its max-adic completion is a *regular* map (i.e., flat with each fiber algebra regular and remaining so after any finite extension of the ground field of the fiber algebra). Hence, the non-regular locus of $\widehat{\mathscr{O}}_{X,x}$ is the full preimage of the non-regular locus of that of $\mathscr{O}_{X,x}$, and likewise for any local finite étale extension of $\widehat{\mathscr{O}}_{X,x}$. The same considerations apply to $\mathscr{O}_{W,w}$ and its completion, so we conclude that the non-regular locus of $\mathscr{O}_{X,x}$ has codimension 2 (even if some of its irreducible components become reducible after completion) since the dimension of a local noetherian ring is the same as that of its completion. We have contradicted that

$$\operatorname{codim}\left(X^{\text{sing}}, X\right) \geq 3.$$

□

12.2. **Setup for handling when** $\operatorname{codim}(X^{\text{sing}}, X) \geq 3$**.** We now have the following setup

$$\begin{array}{ccccccc} x & \in & X^{\text{sing}} & \subset & \operatorname{sing}(X/Y) & \longrightarrow & X \\ \downarrow & & & & \downarrow & & \downarrow f \\ y := f(x) & \in & \cup_{i=1}^n D_i & = & D & \longrightarrow & Y \end{array} \tag{12.10}$$

where $D \subset Y$ is an sncd, Y is connected excellent regular, and $f : X \to Y$ is an open semistable curve smooth over $Y - D$. We want to describe the local structure of X^{sing} near x, so we may shrink Y to ensure $y \in D_i$ for all i.

Let $A = \widehat{\mathscr{O}_{Y,y}}$ and A' the a local finite étale A-algebra with residue field $\kappa(x)$. Further,

$$\mathscr{I}_{D_i,y} = t_i \mathscr{O}_{Y,y},$$

with $\{t_i\}$ part of a regular system of parameters of $\mathscr{O}_{Y,y}$. As we have used multiple times already, $\widehat{\mathscr{O}_{X,x}}$ is uniquely an A'-algebra over its A-algebra structure compatibly with its residue field identification with that of A', and as A'-algebras

$$\widehat{\mathscr{O}_{X,x}} \simeq A'[\![u,v]\!]/(Q - \prod_{i=1}^{\mu} t_i)$$

for some residually non-degenerate quadratic form $Q \in A'[u,v]$ and $1 \leq \mu \leq n$. In fact, $2 \leq \mu \leq n$ since $\widehat{\mathscr{O}_{X,x}}$ is not regular (whereas $A'[\![u,v]\!]$ is regular) due to the hypothesis $x \in X^{\text{sing}}$.

Proposition 12.20. *(1) The closed set X^{sing} is covered by the closed sets (given the reduced structure)*

$$E_{ij} := X^{\text{sing}} \cap f^{-1}(D_i \cap D_j)$$

for $i < j$, and each E_{ij} is regular (hence its irreducible components are its connected components) with codimension 3 in X. Moreover, the finite map

$$E_{ij} \to D_i \cap D_j$$

is étale. In particular, all irreducible components E of X^{sing} with reduced structure are regular.

(2) If E, E' are distinct irreducible components of X^{sing} (taken with the reduced structure and $E \cap E' \neq \emptyset$ then $E \cap E'$ is regular with codimension 4 or 5 in X for each of its irreducible (or equivalently connected) components.

Proof. The idea is to use excellence to pass to the completion, then pass to an étale model where we can perform an explicit computation.

Note that passing to completion commutes the formation of nilradicals since these schemes are excellent, so the completion at a point for a reduced structure on a closed subscheme is the reduced structure on the completion of the closed subscheme at that point. We will implicitly use this repeatedly in the following.

For the first part, we pick $x \in X^{\text{sing}}$ and note that since $\text{Spec}\, \widehat{\mathscr{O}_x} \to \text{Spec}\, \mathscr{O}_x$ is regular, the non-regular locus of $\text{Spec}\, \widehat{\mathscr{O}_x}$ is the full preimage of the non-regular locus of $\text{Spec}\, \mathscr{O}_x$. Further, we can pass to a local finite étale extension of the local finite étale algebra A' over $A := \widehat{\mathscr{O}_{X,y}}$ with residue field $\kappa(x)$) so that are usual description of completions of étale neighborhoods of (X, x) has Q become uv residually, so Q is isometric to uv (as we are working over a complete base ring). This reduces us to studying the model case $\text{Spec}\, B$ with

$$B := R[u,v]/(uv - \prod_{i=1}^{\mu} t_i)$$

with R a complete regular local ring, $\{t_i\}$ a subset of a regular system of parameters in R (with t_iR the ideal for $D_i \subset \operatorname{Spec} R$), and $x = (u, v, \mathfrak{m}_R)$.

For the second part, we shall analyze $E_{ij} \cap E_{i'j'}$ in $\operatorname{Spec} \widehat{B_x}$ for $(i,j) \neq (i',j')$. We will show the intersection is codimension 4 if i or j belong to $\{i',j'\}$ and is codimension 5 if $i,j \notin \{i',j'\}$.

By excellence, the subset

$$\operatorname{Spec} B^{\text{sing}} \subset \operatorname{sing}(B/R) = \{u = v = 0, \prod_{i=1}^{\mu} t_i = 0\}$$

is closed in $\operatorname{Spec} B$, and the given description of the non-smooth locus for $\operatorname{Spec} B$ over R is correct because it is exactly the zero-section over $\cup_{i=1}^{\mu} D_i$. For points w of $\operatorname{Spec} B^{\text{sing}}$, we claim that at least two t_i's vanish at $s = f(w) \in \operatorname{Spec} R$. At least one of them must vanish (as $\prod_{i=1}^{\mu} t_i$ vanishes at w), and suppose some $t_{i_0}(s)$ vanishes but $t_j(s) \neq 0$ for all $j \neq i_0$. We seek a contradiction.

By design

$$\{u = v = t_{i_0} = 0\}$$

is non-regular at the point w away from $f^{-1}(D_j)$ for all $j \neq i_0$. The quotient $B/(u, v, t_{i_0}) = R/(t_{i_0})$ is regular and so remains regular after inverting all t_j for $j \neq i_0$. After inverting the t_j's for $j \neq i_0$, killing t_{i_0} is the same as killing $\prod_{i=1}^{\mu} t_i$. This forces regularity at w, a contradiction.

The converse also holds. That is, we claim that

$$\operatorname{Spec} R/(t_i, t_j) = \{u = v = t_i = t_j = 0\} \subset \operatorname{Spec} B^{\text{sing}}$$

for any $i < j$. This quotient of R is visibly local regular, so its spectrum is irreducible, and the singular locus in $\operatorname{Spec} B$ is closed (by excellence), so it is enough to check the completion of the total space of $\operatorname{Spec} B$ at the generic point (u, v, t_i, t_j) of the closed subset $\{u = v = t_i = t_j = 0\}$ of $\operatorname{Spec} B$ is non-regular. This completion is

$$\widehat{B_{(u,v,t_i,t_j)}} = \widehat{R_{(t_i,t_j)}}[\![u,v]\!]/(uv - t_it_j),$$

which is not regular since $uv - t_it_j$ is in the square of the maximal ideal.

We have described the non-regular locus of $\operatorname{Spec} B$ as the union of the regular irreducible closed subsets $Z_{ij} := \{u = v = t_i = t_j = 0\}$ for $i < j$, and so it is clear that $\operatorname{Spec} B^{\text{sing}} \cap f^{-1}(D_i \cap D_j) = Z_{ij}$ for any $i < j$. From this explicit description, we obtain the desired codimension assertions. Moreover,

$$(Z_{ij})_x^{\wedge} = \operatorname{Spec}\,(R/(t_i, t_j)) \to (D_i \cap D_j)_x^{\wedge}$$

is an isomorphism by inspection, so in the global setting (before passing to completions and increasing residue fields by finite separable extensions via suitable local-étale extensions) we see that the finite map $E_{ij} \to D_i \cap D_j$ is étale. $\square$

12.3. **Returning to the examination of Z.** In the original setup over $k = \overline{k}$, to achieve $\operatorname{codim}\left(X^{\text{sing}}, X\right) \geq 3$ we applied a composite of blowups

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi} & X \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array} \tag{12.11}$$

over the complement of the k-smooth locus X^{sm} of X (even over the complement of the Y-smooth locus of X) at each step. How does this interact with Z? Recall that

$$Z = f^{-1}(D) \cup (\cup_i \tau_i(Y))$$

for pairwise disjoint $\tau_i : Y \to \operatorname{sm}(X/Y)$, so each τ_i lands inside an open subscheme of X away from the blow-up locus. Therefore, the τ_i give rise to sections $\tau_i' : Y \to \operatorname{sm}(X_1/Y)$ such that

$$\phi^{-1}(Z) = f_1^{-1}(D) \cup \left(\cup_i \tau_i'(Y)\right) =: Z_1.$$

Thus, even accounting for Z, we can assume $\operatorname{codim}\left(X^{\text{sing}}, X\right) \geq 3$. In particular, for $d = 2$ we have reached the case of k-smooth X but we still need to do more work even there (to arrange that Z is an sncd in X, possibly after some further modification).

To wrap up the case $d = 2$ (which involves content via the sncd property we wish to achieve for Z), and more generally to get rid of the consideration of the pair (Y, D) that has largely done everything we need from it, we shall prove the following:

Lemma 12.21. *The proper reduced closed subset*

$$Z \cap X^{\text{sm}} \subset X^{\text{sm}}$$

is an ncd.

Proof. The property of being ncd is étale-local on X^{sm} near each $z \in Z(k)$. We shall do some mild case-analysis. First consider the situation away from $f^{-1}(D)$; i.e., over $Y - D$. Over this Y-smooth open subset of X, Z is a disjoint union of sections to a smooth curve over Y, and such a section is always sncd. That is, near any $z \in Z \cap f^{-1}(Y - D)$ in the section $\tau_i(Y)$ for a unique

i there is a Zariski-open neighborhood U in $f^{-1}(Y - D)$ admitting an étale map h as in the diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & \mathbf{A}^1_V \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & V & \end{array} \tag{12.12}$$

for open $V := f(U) \subset Y$ such that τ_i restricts to a section $\tau_i|_V : V \to U$ whose composition with h is the zero-section 0_V of the affine line over V. Since h is étale, it follows that $h^{-1}(0_V)$ is étale over the regular $0_V = V$ and hence $\tau_i(V)$ is a connected component of $h^{-1}(0_V)$. Thus, the global coordinate on $\mathbf{A}^1_V$ pulls back to a local function on U near z that cuts out Z near z.

Next, we consider the case $z \in f^{-1}(D)$, which is to say $f(z) \in D$. This breaks into two sub-cases, depending on whether or not $z \in \mathrm{sm}(X/Y)$. Suppose z does belong to the Y-smooth locus in X, so if z belongs to some $\tau_i(Y)$ (with i then unique) the data (X, Z, z) is étale over

$$(\mathbf{A}^1_Y, 0_Y \cup \mathbf{A}^1_D = f^{-1}(D), 0_{f(z)})$$

with $D \subset Y$ an sncd, so the ncd property is clear. If such z doesn't belong to any $\tau_i(Y)$, so Z near z coincides with $f^{-1}(D)$ then the proof of the ncd property is essentially the same but easier (as we don't need to pay attention to the zero-section of the affine line over Y).

There remains the subcase $z \in X^{\mathrm{sm}}$ with z not smooth in the fiber $X_{f(z)}$. Hence, $z \notin \tau_i(Y)$ for all i (as the τ_i are supported in $\mathrm{sm}(X/Y)$), so Z near z coincides with

$$f^{-1}(D) \cap X^{\mathrm{sm}}$$

The task is thereby reduced to showing that the closed subset

$$f^{-1}(D) \cap X^{\mathrm{sm}} \subset X^{\mathrm{sm}}$$

(given the reduced structure) is ncd at each of its k-points z that is non-smooth in $X_{f(z)}$.

As we saw in the étale-local model case, X near its regular k-point $z \in f^{-1}(D)$ has an étale neighborhood in common with

$$(W = \mathrm{Spec}\, A[u, v]/(uv - t_1), w)$$

for A a regular local ring, $D = \mathrm{Spec}\,(A/(\prod_{j=1}^{r} t_j))$ for $\{t_1, \ldots, t_r\}$ part of a regular system of parameters of A, and $w = (u, v, \mathfrak{m}_A)$.) Clearly the preimage

$$f^{-1}(D) = V(\prod_{j=1}^{r} t_j) \subset W$$

coincides with

$$\mathrm{Spec}\,(A/(\prod_{j=1}^{r} t_j))[u,v]/(uv - t_1) \subset W.$$

In the local ring at w, this closed subscheme is also cut out by $uv \prod_{j=2}^{r} t_j$ (the product over j means 1 if $r = 1$), and so to verify the ncd (even sncd) property for $f^{-1}(D)$ near w it suffices to show the collection of $r + 1$ elements $\{u, v, t_2, \ldots, t_r\}$ in the maximal ideal of the regular local ring

$$(A[u,v]/(uv - t_1))_w$$

is part of a regular system of parameters. The quotient by $(u, v, t_2, \ldots, t_r)$ is $A/(t_1, t_2, \ldots, t_r)$ that is regular (!) with dimension $r + 1$ less than that of the regular local ring $(A[u,v]/(uv - t_1))_w$, so we are done. □

12.4. **Completing the proof of the main result.** To complete the proof for $d = 2$, we have already reduced to the case of smooth X, and it remains to turn the ncd $Z \subset X$ into an sncd. The final handout shows that for smooth X of any dimension and $Z \subset X$ an ncd, blow up along a suitable closed subscheme of Z preserves smoothness of the ambient scheme and makes the preimage of Z in the blow-up"closer" to being sncd; more specifically, the handout provides an algorithm that reaches a k-smooth blow-up of X in finitely many steps for which the preimage of Z is an sncd. Beware that these blow-ups of X typically destroy the semi-stability property over Y, but that doesn't matter anymore when $d = 2$ (and similarly won't matter in what follows when $d > 2$). This completes the case $d = 2$.

To complete the proof when $d \geq 3$, we have the following new setup, suppressing any mention Y and D. We want to make a statement only involving X and Z because the subsequent blow-ups will usually ruin semistability over Y.

We are in the following new axiomatic situation. Let X be a projective variety over an algebraically closed field k, with $d = \dim X \geq 3$. Let $Z \subset X$ be a proper reduced closed subset so that

(i) The closed subset $Z \cap X^{\mathrm{sm}} \subset X^{\mathrm{sm}}$ is ncd.

(ii) The reduced closed subset $X^{\mathrm{sing}} \subset X$ has pure codimension 3 (if it is nonempty), with all irreducible components E_α smooth and the nonempty scheme-theoretic intersections $E_\alpha \cap E_\beta$ for $\alpha \neq \beta$ also smooth (but possibly not transverse) with each irreducible component of such $E_\alpha \cap E_\beta$ of codimension 4 or 5 in X (see Proposition 12.20).

(iii) For all $z \in (Z \cap X^{\mathrm{sing}})(k)$, we have

$$\widehat{\mathscr{O}_{X,z}} = k[u,v,t_1,\dots,t_{d-1}]/(uv - \prod_{j=1}^{s} t_j)$$

for some $2 \leq s \leq d-1$ (we have $s \neq 1$ since z is not a regular point on X) and

$$\widehat{\mathscr{O}_{Z,z}} = \widehat{\mathscr{O}_{X,x}}/(\prod_{j=1}^{r} t_j)$$

for some $s \leq r \leq d-1$. (The reason we have $r \geq s$ is that Z near such z in our situation of interest coincides with $f^{-1}(D)$ and some $n_i(z)$ might equal 0 rather than 1.)

Our remaining goal is to construct blow-ups

$$\widetilde{X} \xrightarrow{\phi} X$$

such that for

$$\widetilde{Z} := \phi^{-1}(Z)_{\mathrm{red}}$$

the pair $(\widetilde{X}, \widetilde{Z})$ satisfies (i)-(iii) with $\widetilde{X}$ "more nearly" k-smooth in a sense we will make precise below.

In finitely many steps, we'd then reach the case of (i)-(iii) with X smooth everywhere, so we could use the final handout (as we did at the end of the treatment of the case $d = 2$) to make the preimage of Z an sncd in some smooth modification of X. In particular, if X is already smooth everywhere then (as in the case $d = 2$) we are done by the final handout. Thus, we may and do now assume $X^{\mathrm{sing}} \neq \emptyset$.

The preceding axioms have two immediate consequences:

(1) For $x \in X^{\mathrm{sing}}(k)$, by our usual arguments with Artin approximation we know that (X, x) has a common étale neighborhood with

$$(\mathrm{Spec}\, k[u,v,t_1,\dots,t_{d-1}]/(uv - t_1,\dots,t_s), \vec{0})$$

(with $2 \leq s \leq d-1$).

(2) The reduced closed subset $Z \subset X$ is Cartier (i.e., $\mathscr{I}_Z$ is invertible); this can be checked in the completed local ring of X at all points of $Z(k)$ (treating separately those that lie in X^{sm} and in X^{sing}).

Now a miracle happens (as explained in the final handout); for each irreducible component E of X^{sing}, the pair

$$(\text{Bl}_E(X), Z\text{-preimage})$$

satisfies (i)-(iii), and with gain that the irreducible components of $\text{Bl}_E(X)^{\text{sing}}$ are precisely the strict transforms of the irreducible components E' of X^{sing} distinct from E. Therefore, the total number of irreducible components of the singular locus drops under this process, so after blowing up finitely many times we arrive at the situation of (i)-(iii) with X now k-smooth everywhere, a situation from which we have already discussed how to conclude (via some computations for which we punted to a handout).

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