Math 249B. Finiteness in codimension 1

We have reduced the task to constructing a generically étale alteration $\varphi : X' \to X$ of a normal projective variety X of dimension $d \geq 2$ over an algebraically closed field k. Although it is generally hopeless to construct φ to be étale or even flat over a specific point $x \in X$, we can at least exert some mild control over the locus of $x \in X$ with finite fibers: its complement has codimension ≥ 2. This is the content of the following result.

Proposition 0.1. *Any alteration $\varphi : X' \to X$ between projective varieties over $k = \overline{k}$ with normal X is finite over a dense open $U \subset X$ with complement of codimension ≥ 2.*

Proof. Let $R = \mathscr{O}_{X,x}$ for a point $x \in X$ with codimension 1; i.e., $\dim R = 1$. By normality, any such R is a discrete valuation ring. We will show that the proper map $X'_R = X \times_X \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ obtained by localization to R is a finite morphism. Once this is shown, by "spreading out" principles we obtain an open neighborhood $V \subset X$ around x such that $\varphi^{-1}(V) \to V$ is finite. The non-empty union U of all such V's is then an open subset of X for which $\varphi^{-1}(U) \to U$ is finite (as may be verified over each member V of an open cover of U), and the proper closed set $X - U$ contains no points in X with a 1-dimensional local ring on X, so $X - U$ has codimension ≥ 2 at all of its points.

Noting that $X'_R \to \mathrm{Spec}(R)$ has generic fiber $\eta' \to \eta$ that is finite, we are reduced to proving finiteness of any proper map $f : Y \to \mathrm{Spec}(R)$ between integral schemes such that R is an excellent discrete valuation ring (such as $\mathscr{O}_{X,x}$ above) and the generic fiber Y_η is η-finite. Finiteness for f is the same as quasi-finiteness since f is proper, so our task concerns only the closed fiber Y_0 of Y: we just need to check that it has only finitely many points. Let R' be the normalization of R in the finite extension $k(\eta')$ of the fraction field $k(\eta)$ of R; this is R-finite by excellence of R. (Note that $k(\eta')$ may not be separable over $k(\eta)$, so an argument is needed to ensure R' is R-finite; this is why we kept track of excellence for R.) Thus, R' is a semi-local Dedekind domain.

By the valuative criterion for properness, adapted to the case of Dedekind domains, the given map $\eta' \to Y$ over $\mathrm{Spec}(R)$ extends uniquely to an R-map $h : \mathrm{Spec}(R') \to Y$. This latter map between integral schemes is an isomorphism between generic fibers over R, so it is dominant, and it is also proper between it is a map between proper R-schemes. But a dominant proper map is surjective, so h is surjective. Hence, Y_0 is the image of the special fiber of $\mathrm{Spec}(R')$ over $\mathrm{Spec}(R)$. The R-finiteness of R' then ensures that Y_0 consists of only finitely many points, as desired. ■