Math 249B. From normal crossings to strict normal crossings

1. Introduction

Let S be an integral noetherian scheme. Recall that we defined an effective Cartier divisor $D \subseteq S$ to be a *strict normal crossings divisor* if S is regular at each point $d \in D$ and the irreducible components $\{D_i\}_{i \in I}$ of D (with their reduced structure) are regular and have (scheme-theoretic) overlaps $D_J = \cap_{i \in J} D_i$ that are regular with pure codimension $\#J$ for all subsets J of I. (The case $J = \emptyset$ is either a tautology or is to be ignored; take your pick.) Locally, this says that for each $d \in D$ lying on exactly m irreducible components, a local generator $f \in \mathscr{O}_{S,d}$ for the ideal sheaf of D at d has irreducible factorization $f = \prod_i f_i$ in $\mathscr{O}_{S,d}$ with the f_i's pairwise coprime (that is, not unit multiples of each other) and part of a regular system of parameters of $\mathscr{O}_{S,d}$.

We also defined the concept of *normal crossings divisor*: a closed subscheme $D \subseteq S$ is a normal crossings divisor if it pulls back to a strict normal crossings divisor on an étale cover of S (or equivalently, for each $s \in S$ there is an étale map $U \to S$ hitting s so that the pullback of D in U is a strict normal crossings divisor in U). In particular, D must be reduced and have invertible ideal sheaf (that is, it is Cartier). The purpose of this handout is to prove that for any normal crossings divisor $D \subseteq S$ there is an integral noetherian scheme S' and a proper birational map $\varphi : S' \to S$ that is an isomorphism outside of D (and in fact is a blow-up along a coherent ideal whose zero scheme is physically supported in D, but we do not require this fact) such that $\varphi^{-1}(D)_{\mathrm{red}}$ is a strict normal crossings divisor on S'. The reason we wish to prove this result is two-fold: it is a very basic fact for which we do not know a reference, and at the end of deJong's proof he gets a divisor that is merely a normal crossings divisor (due to a local calculation in the étale topology) so we need such a φ to get the "strict normal crossings" aspect of deJong's theorem.

2. Construction of some closed sets

At the end of section 2.4 of deJong's paper it is explained how one carries out the construction of $\varphi : S' \to S$ as above. The explanation involves the étale local notion of "branches", so one really wants a global instrinsic (rather than étale-local) description of the process and for a general noetherian integral scheme (as opposed to a variety over a field) one may pause to question if everything really works out correctly. We wish to prove that everything does indeed work as it should. Given a normal crossings divisor $D \subseteq S$, our goal is to construct a finite sequence of blowings-up

$$\varphi : S_n \to S_{n-1} \to \cdots \to S$$

along closed subschemes supported over D such that $\varphi^{-1}(D)_{\mathrm{red}} \subseteq S_n$ is a strict normal crossings divisor. It then follows from the self-contained (but slightly obscurely proved) Lemma 5.14 in the paper of Raynaud–Gruson (Inv. Math. 13) that such a composite map φ is necessarily a blow-up along a closed subscheme physically supported in D, though the fact that φ is actually a blow-up within D (rather than merely an isomorphism over $S - D$) is not necessary for the proof of deJong's theorems.

For $n \geq 1$, let $Z_n \subseteq D$ be the locus where D has "$\geq n$ analytic branches". Rigorously,

$$Z_n = \{d \in D \,|\, \mathscr{O}_{D,d}^{\mathrm{sh}} \text{ has } \geq n \text{ minimal primes}\}.$$

This is merely a subset of D, and its geometry will be sorted out in a moment. Observe that the formation of Z_n is compatible with étale localization on S. That is, if $S' \to S$ is étale and if D' is the pullback of D to S' and $Z'_n \subseteq D'$ is defined analogously to $Z_n \subseteq D$, then Z'_n is the full preimage of Z_n in S'.

The following lemma isolates some intrinsic properties of strict normal crossings divisors that are local for the étale topology and so are well-posed for normal crossings divisors such as our D:

Lemma 2.1. *Let $D \subseteq S$ and $\{Z_n\}_{n\geq 1}$ be as above.*

1. *The scheme S is regular at all points of D.*
2. *The subset $Z_n \subseteq S$ is Zariski-closed with pure codimension n if it is nonempty; that is, $\inf_{z\in Z_n} \dim \mathscr{O}_{S,z} = n$ if $Z_n \neq \emptyset$.*
3. *Giving Z_n its reduced structure, the regular locus Z_n^{reg} is equal to the open subset $Z_n - Z_{n+1}$ in Z_n, and this is both dense in Z_n as well as of pure codimension n in S.*

Proof. For (1) we may work étale-locally, so the assertion is clear (as it holds for strict normal crossings divisors). As for (2), since the formation of the set Z_n is local for the étale topology and since étale descent theory is effective for descent of reduced closed subschemes it suffices to prove the result étale-locally over S. Hence, for (2) it is enough to treat the case when D is a strict normal crossings divisor. By (1) and EGA IV_4, 5.1.9, it likewise follows that for the proof of (3) it is enough to check it étale-locally on S. Thus, to prove both (2) and (3) we lose no generality in assuming that D is a strict normal crossings divisor in S. Since the Cartier D is reduced, if $\{D_i\}$ is the set of (reduced) irreducible components of D then $D = \sum D_i$ as closed subschemes of S. By the regularity conditions in the definition of a strict normal crossings divisor, for all $d \in D$ the irreducible components of $\mathrm{Spec}\, \mathscr{O}_{D,d}^{\mathrm{sh}}$ are given by $\mathrm{Spec}\, \mathscr{O}_{D_i,d}^{\mathrm{sh}}$ for D_i that contain d. Hence, $Z_n = \cup_{\#J=n} D_J$ for all $n \geq 1$, so Z_n is Zariski-closed. This settles (2).

The open complement $Z_n - Z_{n+1}$ in Z_n with its reduced structure is given by

$$Z_n - Z_{n+1} = \coprod_{\#J=n} (D_J - \cup_{i\notin J} D_i),$$

and this is regular with pure codimension n in S if it is nonempty because each $D_J - \cup_{i\notin J} D_i$ is regular with pure codimension n in S if it is nonempty. Hence, $Z_n - Z_{n+1} \subseteq Z_n^{\mathrm{reg}}$. To get the reverse inclusion, pick $d \in Z_n$ and suppose $d \in D_J$ with $\#J = n$. Hence, $\mathrm{Spec}\, \mathscr{O}_{D_J,d} \subseteq \mathrm{Spec}\, \mathscr{O}_{S,d}$ is a closed subscheme of codimension n that is regular. But $Z_n = \cup_{\#J=n} D_J$ set-theoretically, so $\mathrm{Spec}\, \mathscr{O}_{Z_n,d}$ is set-theoretically the union of the $\mathrm{Spec}\, \mathscr{O}_{D_J,d}$'s for all D_J that contain d (with $\#J = n$) and these $\mathrm{Spec}\, \mathscr{O}_{D_J,d}$'s *must* be the irreducible components of $\mathrm{Spec}\, \mathscr{O}_{Z_n,d}$ (as they are irreducible and none contains the other since they all have the same dimension). Hence, if $\mathrm{Spec}\, \mathscr{O}_{Z_n,d}$ is regular then d must lie in a *unique* D_J with $\#J = n$. This also gives the density of $Z_n - Z_{n+1}$ in Z_n. ∎

3. The blow-up process

We are now ready to make some blow-ups. Let m be maximal so that Z_m is nonempty, and consider the strictly increasing chain of closed sets

$$\emptyset \neq Z_m \subsetneq Z_{m-1} \subsetneq \cdots \subsetneq Z_1 = D.$$

By Lemma 2.1, each $Z_i \subseteq S$ is of pure codimension i, the open locus $Z_i - Z_{i-1} \subseteq Z_i$ is dense and equal to Z_i^{reg}, and Z_m is regular. If $m = 1$ then D is regular and we are done. Hence, we shall assume $m > 1$, and the first blow-up we shall study is that of S along Z_m.

Let $S' = \mathrm{Bl}_{Z_m}(S)$. We claim that the reduced preimage D' of D in S' is a normal crossings divisor equal to $D' = \widetilde{D} \cup E$ as closed sets with $\widetilde{D}$ the strict transform of D and E the exceptional divisor of the blow-up; moreover, E is *regular.* To justify this, by working locally for the étale topology we come down to the easily checked fact that in a regular local ring A with $\{f_1, \ldots, f_m\}$ part of a regular system of parameters, in the regular *integral* scheme

$$\mathrm{Spec}\, A[t_1, \ldots, t_n]/(f_j - f_1 t_j)$$

the locally closed subscheme $\{f_{j_0} \neq 0, f_1 = 0\}$ for $j_0 \neq 1$ is empty (yet in $\operatorname{Spec} A$ the locus with the same description is dense in $\{f_1 = 0\}$) whereas the locally closed subscheme $\{f_{j_0} = 0, f_1 \neq 0\}$ in this integral scheme has closure equal to the *regular* Cartier divisor $\{t_{j_0} = 0\}$.

It follows that after some étale local base change on S (to make the normal crossings divisor D becomes a strict normal crossings divisor) the m-fold overlaps of strict transforms of irreducible components of D are empty, so the strict transform $\widetilde{Z}_{m-1}$ of Z_{m-1} is a *disjoint* union of $(m-1)$-fold overlaps of such strict transforms étale locally on S. Hence, $\widetilde{Z}_{m-1}$ is *regular*. Again working étale locally over S to make D a strict normal crossings divisor, on the blow-up S' the $(m+1)$-fold overlaps among irreducible components of D' are empty because they involve at most one irreducible component of the regular exceptional divisor and so at least m of the strict transforms of the irreducible components of D (and we have just seen that such m-fold overlaps in the blow-up are empty).

Now consider the blow-up $S'' = \mathrm{Bl}_{\widetilde{Z}_{m-1}}(S')$. If we work Zariski-locally on S' and use the disjoint union decomposition of $\widetilde{Z}_{m-1}$ into certain $(m-1)$-fold overlaps, the same calculations as above show:

- the reduced preimage $D'' \subseteq S''$ of D' in S' is a normal crossings divisors;
- if $m > 2$, the strict transforms $\widetilde{Z}_{m-2}$ of Z_{m-2} with respect to $S'' \to S$ is regular. (The key point is that $Z_{m-2} - Z_{m-1}$ is a dense open in Z_{m-2} if $m > 2$.)

We can keep iterating this procedure until we arrive at a (composite) blow-up $\overline{S} \to S$ that is an isomorphism over $S - D$ and on which the reduced preimage $\overline{D}$ of D is a normal crossings divisor. In fact, we can arrange that $\overline{D}$ is the union of some regular irreducible components and the strict transform $\widetilde{D}$ of D in $\overline{S}$, with the strict transform $\widetilde{Z}_1$ of Z_1 in $\overline{S}$ also regular. But $\widetilde{Z}_1 = \widetilde{D}$ by construction, so we conclude that $\widetilde{D}$ is a *regular* normal crossings divisor, so its irreducible components are its connected components and thus are regular. Thus, $\overline{D} \subseteq \overline{S}$ is a normal crossings divisor whose irreducible components are regular, so $\overline{D}$ is a strict normal crossings divisor in $\overline{S}$.