Math 249B. Resolving singularities in codimension 3

1. Section 4.26

This handout gives a detailed treatment (a bit more than in deJong's paper) for the final two sections of the proof of the main result (Theorem 4.1) over fields. We begin by fixing any pair (X, Z) satisfying the hypotheses at the end of 4.25: X is a projective variety of dimension $d \geq 3$ over an algebraically closed field k and $Z \subseteq X$ is a reduced closed subset with pure codimension 1 such that:

(1) $Z \cap X^{\rm sm}$ is a normal crossings divisor in $X^{\rm sm}$ (recall this means that locally for the étale topology on $X^{\rm sm}$ the closed subset looks like an intersection of coordinate hyperplanes through the origin in an affine space),

(2) for all $x \in (Z \cap \mathrm{sing}(X))(k)$ there is a k-algebra isomorphism
$$\widehat{\mathscr{O}}_{X,x} \simeq k[\![u, v, t_1, \dots, t_{d-1}]\!]/(uv - t_1 \cdots t_s)$$
with the quotient $\widehat{\mathscr{O}}_{Z,x}$ cut out by the principal ideal generated by $t_1 \cdots t_r$ where $2 \leq s \leq r \leq d-1$ (and r, s may depend on x),

(3) all irreducible components E of $\mathrm{sing}(X)$ are smooth of dimension $d-3$ and all overlaps $E \cap E'$ (in the scheme-theoretic sense) are smooth (possibly not transverse intersections).

Note that if X is smooth then by (1) we have that Z is a normal crossings divisor in X. Hence, if X is smooth then we may apply the handout on passage from normal crossings divisors to *strict* normal crossings divisors (via precise blow-ups) to find a modification $X' \to X$ with smooth X' such that the preimage of Z in X' is a strict normal crossings divisor (i.e., the irreducible components of Z are smooth and are mutually transverse). This final situation is exactly where we want to get to (it is the conclusion of Theorem 4.1), so we may now assume $\mathrm{sing}(X)$ is non-empty. We wish to find a modification $X' \to X$ such that for the preimage Z' of Z in X' we have that (X', Z') satisfies the same axioms as above for the pair (X, Z) but with $\mathrm{sing}(X')$ somehow "simpler" than $\mathrm{sing}(X)$ so that the process eventually ends with no singularities in the total space (and then one final modification brings us to the case desired in Theorem 4.1, namely with Z also a strict normal crossings divisor).

2. Section 4.27

Let E be an irreducible component of $\mathrm{sing}(X)$, and let $\pi : X' = \mathrm{Bl}_E(X) \to X$ be the blow-up. Let $Z' = \pi^{-1}(Z)_{\rm red}$. The answer to our prayers is given by 4.27 in deJong's paper:

Theorem 2.1. *With notation as just defined, the pair (X', Z') satisfies the same axioms as (X, Z) at the outset in* 4.26*, but the number of irreducible components of* $\mathrm{sing}(X')$ *is one less than the number of irreducible components of* $\mathrm{sing}(X)$*. More precisely, the irreducible components of* $\mathrm{sing}(X')$ *are the strict transforms of the irreducible components of* $\mathrm{sing}(X)$ *distinct from E.*

Proof. The idea is similar to 3.4: we do some concrete blow-up calculations in convenient étale charts. As before, we need to exercise some care with respect to keeping track of irreducibility under completion and étale localization, and once again regularity will be the key to handling such issues.

Let $\widetilde{E}$ be an irreducible component of $\mathrm{sing}(X)$ distinct from E (if any exist). There is a natural E-isomorphism between $\mathrm{Bl}_{E \cap \widetilde{E}}(E)$ and the strict transform $\widetilde{E}'$ of $\widetilde{E}$ under the blow-up $\pi : X' \to X$ along E (where $E \cap \widetilde{E}$ is taken in the scheme-theoretic sense, which is regular by our axioms). This relationship between strict transform of a closed subset under a blow-up and a blow-up of the closed subset is explained in detail in Lemma 1.1 in the notes on Nagata compactifications on my web page. The advantage of such a "blow-up" description of $\widetilde{E}'$ in X' is that it shows that this strict transform (which we know to be integral: it is a closure of $\widetilde{E} \cap (X - E)$ in X') is *regular*. Indeed, working locally for the étale topology, the study of $\mathrm{Bl}_{E \cap \widetilde{E}}(E)$ is the same as that of blowing up an affine space along a linear subspace, for which the blow-up is seen to be regular by explicit calculation on blow-up charts.

The upshot is this: all such $\widetilde{E}'$'s are *smooth* codimension-3 subvarieties of X'. Since $X' - \pi^{-1}(E) \simeq X - E$, all singularities of X' away from the $\widetilde{E}'$'s must lie in the exceptional divisor. Thus, our problem is to verify the following three assertions, with $\{E_i\}$ the set of irreducible components of $\mathrm{sing}(X)$:

1. X' is regular at all points of along $\pi^{-1}(E)$ away the reduced union $\cup_{E_i \neq E} E'_i$ (this is the strict transform of the closure of $\mathrm{sing}(X) - E$, even when this complement is empty!),
2. for all $E_i, E_j \neq E$ with $i \neq j$ (if two such exist!), the overlap $E'_i \cap E'_j$ in the scheme-theoretic sense is smooth (or equivalently, regular),
3. At all k-points of $Z' \cap \pi^{-1}(E)$ the formal structure of the pair (X', Z') is as in the second axiom at the outset.

By excellence and regularity, the irreducible $\widetilde{E}'$'s remain regular and irreducible under base change to completion of X at any $x \in E(k)$; here we use that $\mathscr{O}_{X,x}$ has regular formal fibers (and that for a flat map between noetherian schemes with regular fibers, the top is regular if the bottom is regular). Thus, it is enough (check!) to study the analogous situation for an étale chart with the same formal structure as (X, x).

Our problem is now reduced to checking the above three assertions for the following "open variety" situation:

$$X = \mathrm{Spec}\, k[u, v, t_1, \ldots, t_{d-1}]/(uv - t_1 \cdots t_s), \;\; Z = V(t_1 \cdots t_r) \hookrightarrow X$$

with $2 \leq s \leq r \leq d-1$ and $E = V(u, v, t_i, t_j)$ with $1 \leq i < j \leq s$ (where $V(I)$ denotes the zero-scheme in X of an ideal I). Without loss of generality, $i = 1$ and $j = 2$. The blow-up $\mathrm{Bl}_E(X)$ is covered by four Zariski-open subschemes, namely the charts $D_+(u)$, $D_+(v)$, $D_+(t_1)$, and $D_+(t_2)$. By symmetry, it suffices to study the opens $D_+(u)$ and $D_+(t_1)$. We will see by calculation that $D_+(u)$ is regular with $Z' \cap D_+(u)$ a strict normal crossings divisor (which is only enough to infer the normal crossings condition in the original situation, as we are presently only working on an étale chart of the original setup and the strict normal crossings condition is not local for the étale topology whereas the normal crossings condition is). In particular, all of the real work will therefore take place in $D_+(t_1)$. Keep in mind, as we have already seen, that $Z' = \pi^{-1}(Z)_{\mathrm{red}}$.

On the open locus $D_+(u)$ we have relations $v = v'u$, $t_1 = t'_1 u$, and $t_2 = t'_2 u$, so the relation $uv = t_1 \cdots t_s$ is the same as $u^2(v' - t'_1 t'_2 \cdots t_s) = 0$. Killing the u-power torsion therefore gives that that

$$D_+(u) = \mathrm{Spec}(k[u, t'_1, t'_2, t_3, \ldots, t_{d-1}])$$

since this clearly has no nonzero u-power torsion. Note by inspection that this is regular. Also, $t_1 \cdots t_r = u^2 t'_1 t'_2 t_3 \cdots t_r$, so since u is not a zero-divisor on $D_+(u)$ we conclude that the overlap $Z' \cap D_+(u) = \pi^{-1}(Z)_{\mathrm{red}} \cap D_+(u)$ is the integral zero-scheme cut out by $u t'_1 t'_2 t_3 \cdots t_r = 0$. By inspection, $Z' \cap D_+(u)$ is clearly a strict normal crossings divisor in $D_+(u)$.

Now we turn our attention to $D_+(t_1)$, which is a bit more involved. We have the relations $u = u' t_1$, $v = v' t_1$, and $t_2 = t'_2 t_1$, so killing t_1^2-torsion gives

$$D_+(t_1) = \mathrm{Spec}(k[u', v', t_1, t'_2, \ldots t_{d-1}]/(u'v' - t'_2 t_3 \cdots t_s)), \;\; Z' = V(t_1 t'_2 t_3 \cdots t_r)$$

by the same sort of calculation as on $D_+(u)$. Note that the relation cutting out $D_+(t_1)$ does not involve t_1. Thus, the singular locus on $D_+(t_1)$ is the union of the irreducible loci of the following two types:

$$V(u', v', t_2', t_i), \quad V(u', v', t_i, t_j) \tag{2.1}$$

with $3 \leq i \leq s$ and $3 \leq i < j \leq s$ respectively. These are clearly smooth and have smooth scheme-theoretic overlaps. By direct inspection we see that $Z' \cap D_+(t_1)$ has the desired formal structure within $D_+(t_1)$ at all of its k-points lying in $\mathrm{sing}(X')$.

Our final task is to relate the irreducibles in (2.1) to the strict transforms of the irreducible components of $\mathrm{sing}(X)$ distinct from $E = V(u, v, t_1, t_2)$. This is purely a counting problem and only requires us to work generically on irreducible components of singular locus on X and X' (so it is enough to identify work out how these meet either of $D_+(t_1)$ or $D_+(t_2)$; there is no need to work on $D_+(u)$ or $D_+(v)$ because we have seen that X' is smooth on these latter two opens). By consideration of generic points, the map $D_+(t_1) \subseteq X' \to X$ induces a dominant map $V(u', v', t_2', t_i) \to V(u, v, t_2, t_i)$ for $3 \leq i \leq s$, so the closed locus $V(u', v', t_2', t_i)$ in $D_+(t_1)$ is where the strict transform of $V(u, v, t_2, t_i)$ meets $D_+(t_1)$. Likewise, for $3 \leq i < j \leq s$ we get a dominant map $V(u', v', t_i, t_j) \to V(u, v, t_i, t_j)$ that realizes the source as the part of $D_+(t_1)$ meeting the strict transform of the target. There remain the irreducible components $V(u, v, t_1, t_i)$ of $\mathrm{sing}(X)$ with $3 \leq i \leq s$, but since t_2 is a *unit* at the generic point of each of these components we see that these generic points must all lift into $D_+(t_2)$ under $X' \to X$. Hence, we may analyze these strict transforms generically by working in $D_+(t_2)$ exactly as we just worked out where $D_+(t_1)$ meets a dense open in the strict transform of $V(u', v', t_2', t_i)$ (swap the roles of t_1 and t_2). ■