Math 249B. Fiber dimension

Let $h : Z \to Y$ be a flat map of finite type between irreducible noetherian schemes, and let η be the generic point of Y. By flatness, the generic point of Z lies over that of Y, and the generic fiber Z_η of finite type over η is irreducible since it is a "localization" of Z. Let $d = \dim Z_\eta \geq 0$, so Z_η has pure dimension d (being irreducible of finite type over a field). We aim to prove that *all* non-empty fibers Z_y have pure dimension d.

As a first step, we reduce to the case when Y is the spectrum of a discrete valuation ring. We pick $y \in Y$ distinct from η such that Z_y is non-empty. By the Krull-Akizuki Theorem, there is a discrete valuation ring $R \subset k(Y)$ local over $\mathscr{O}_{Y,y}$, so the special fiber of Z_R is a scalar extension of Z_y to the residue field of R (which could be a *huge* extension of $k(y)$). Since the property of a non-empty scheme of finite type over a field having a given pure dimension is insensitive to extension of the ground field, it is equivalent to show that Z_R has special fiber of pure dimension d.

By design of R inside $k(Y)$ it is clear that $Z_R \to \mathrm{Spec}(R)$ has the same generic fiber Z_η as for Z over Y, and by R-flatness of Z_R it follows that all generic points of Z_R lie in its generic fiber Z_η. Thus, there is only one such point since the localization Z_η of Z_R is irreducible, so Z_R is irreducible! In this way, the base change to R preserves our hypotheses (at the harmless cost of an extension of the residue field over $k(y)$), so we can replace h with $Z_R \to \mathrm{Spec}(R)$. In other words, we may now assume Y is the spectrum of a discrete valuation ring R and that the special fiber Z_0 is non-empty. Our task is to show that Z_0 has pure dimension d.

Since Z_0 is of finite type over the residue field κ of R, to show it has pure dimension d is a Zariski-local question near each closed point $z \in Z_0$. If some irreducible component Z_0' of Z_0 has dimension distinct from d then we may choose a closed point $z \in Z_0'$ not on any other irreducible component of Z_0 and then pick an affine open neighborhood of z in Z that meets Z_0 inside Z_0'. To get a contradiction we may replace Z with that open neighborhood to arrive at the case when Z_0 is *irreducible* (and Z is affine). In such cases it is enough to show that necessarily $\dim Z_0 = d$.

We may now drop the focus on "pure" dimension d: it is enough to show that a flat affine R-scheme of finite type with irreducible generic fiber of dimension d (so the generic fiber is even of pure dimension d) and non-empty special fiber must have special fiber of dimension d. The following argument due to Artin is taken from [EGA IV_3, 14.3.10]. Since Z is affine, if $n = \dim Z_0$ then we can choose a finite surjection $Z_0 \to \mathbf{A}^n_\kappa$ by the Noether Normalization Theorem. By affineness this lifts to a map $Z \to \mathbf{A}^n_R$ over R. But this map is quasi-finite between the special fibers by design, and rather generally for any map of finite type $W \to V$ between noetherian schemes the locus of points $w \in W$ isolated in their fiber over V (the "quasi-finite locus" of the morphism) is always open. By R-flatness of Z the non-empty special fiber Z_0 cannot be open, so the open quasi-finite locus $\Omega \subset Z$ containing Z_0 must meet Z_η! Since $\Omega_\eta \to \mathbf{A}^n_\eta$ is quasi-finite, the non-empty Ω_η has dimension $\leq n$. But Ω_η is a non-empty open subset of Z_η that has pure dimension d, so $d \leq n$ with equality if and only if $\Omega \to \mathbf{A}^n_R$ is dominant.

Suppose the quasi-finite $\Omega_\eta \to \mathbf{A}^n_\eta$ is not dominant (or equivalently, that $d < n$). Thus, Ω_η factors through a proper closed subset, and so through some hypersurface $(h = 0)$ for $h \in K[t_1, \dots, t_n]$ of positive degree (with $K = \mathrm{Frac}(R)$). We may scale h by a unique integral power of a uniformizer of R so that $h \in R[t_1, \dots, t_n]$ and the reduction h_0 over κ is nonzero (but possibly of lower degree than h over K!). Then the preimage of $(h = 0)$ under $\Omega \to \mathbf{A}^n_R$ is a closed subscheme of Ω that contains Ω_η. But Ω is an open subscheme of the R-flat Z, so it is also R-flat and hence the only closed subscheme of Ω containing Ω_η is Ω. Thus, $\Omega \to \mathbf{A}^n_R$ factors through $(h = 0)$, so on special fibers over $\mathrm{Spec}(R)$ we conclude that the map $\Omega_0 = Z_0 \to \mathbf{A}^n_\kappa$ that is *quasi-finite* factors through $(h_0 = 0)$. But $(h_0 = 0)$ has dimension $n - 1$ when the nonzero h_0 is non-constant and it is empty when h_0 is constant, so $\dim Z_0 \leq n - 1$. This contradicts that $n = \dim Z_0$ with Z_0 non-empty.