

1. INTRODUCTION

In the proof of Lemma 4.11, there is a genericity property whose proof deJong omits from his paper and which is essential for getting a smooth and geometrically connected fiber in the normal case. We first have to review the basic setup. Let X be d -dimensional closed subvariety in some projective space \mathbf{P}^N with $1 \leq d < N$ (over an algebraically closed base field k), and let $Z \subseteq X$ be a reduced closed subscheme with pure dimension $d - 1$. We form a finite map $\pi : X \rightarrow \mathbf{P}^d$ by composition of $N - d$ linear projections away from points, and it is arranged to be generically étale with $\pi|_Z$ birational onto its image. Explicitly, $\pi = \pi_L$ is projection away from an $(N - d - 1)$ -plane $L \subseteq \mathbf{P}^N$ disjoint from X , where L is the projective span of independent k -points $\{p_1, \dots, p_{N-d}\}$ where p_1 is generic in \mathbf{P}^N , p_2 lifts a generic choice of k -point in the hyperplane target of projection away from p_1 , etc. (We will prove below that L 's arising in this way are generic in the Grassmann variety of $(N - d - 1)$ -planes in \mathbf{P}^N ; this is *not* a tautology, even though each p_j is chosen generically within a projective space depending on the previous p_i 's!)

Geometrically, if L' is a fixed d -plane in \mathbf{P}^N disjoint from the $(N - d - 1)$ -plane L then the projection $\mathbf{P}^N - L \rightarrow L'$ away from L sends a k -point z (such as $z \in X$) to the unique point where L' meets the $(N - d)$ -plane spanned by L and z . In coordinate-free and choice-free language (avoiding the non-canonical L'), if $\mathbf{P}^N = \mathbf{P}(V) := \text{Proj}(\text{Sym}(V))$ for an $(N + 1)$ -dimensional vector space V then $L = \mathbf{P}(V/W)$ for a $(d + 1)$ -dimensional subspace W and π is the restriction to X of the map $\mathbf{P}(V) - \mathbf{P}(V/W) \rightarrow \mathbf{P}(W)$ that carries the homothety class of a nonzero linear form $\ell \in V^\vee$ not vanishing in W (that is, not in $(V/W)^\vee$) to the homothety class of its nonzero restriction $\ell|_W$. This coordinate-free and choice-free description admits a functorial description over k -schemes (justifying algebraicity without needing to use coordinates), and this viewpoint will be required below when we work in a relative situation over some Grassmannians.

[We are of course using the contravariant Grothendieck convention for projective bundles and Grassmannians, so for example the projective space $\mathbf{P}(V)$ classifies families of hyperplanes in V , or equivalently isomorphism classes of families of 1-dimensional quotients of V . This is “dual” to the classical covariant viewpoint, but Grothendieck’s version is more convenient when working over base schemes as we shall have to do below. A very nice introduction to his perspective on these matters, as well as to the functorial picture and *construction* of Grassmannians and flag schemes in general, is Grothendieck’s elegant write-up in the complex-analytic category in Exposé 12 of volume 13 of the Seminaire Cartan collection that you can find in the on-campus math library (since I made sure that those volumes remain in the on-campus collection after the massive down-sizing when the math library moved into the “Old Chem” building). The arguments there carry over to the case of schemes verbatim.]

Having made the finite (necessarily surjective) map $\pi : X \rightarrow \mathbf{P}^d$, we then chose a generic k -point $p \in \mathbf{P}^d - \pi(Z)$ such that π is étale over an open neighborhood of p and the finite projection $\text{pr}_p : \pi(Z) \rightarrow \mathbf{P}^{d-1}$ to the “hyperplane” of lines through p is generically étale. In coordinate-free language, if $\mathbf{P}^N = \mathbf{P}(V)$ and $L = \mathbf{P}(V/W)$ then π has target $\mathbf{P}(W)$ and the point $p \in \mathbf{P}(W)$ is $\mathbf{P}(W/W_0)$ for a unique d -dimensional hyperplane $W_0 \subseteq W$. The projection pr_p has target given by the projective $(d - 1)$ -space $\mathbf{P}(W_0)$. We introduced the incidence scheme

$$X' = \{(x, \ell) \in X \times \mathbf{P}(W_0) \mid x \in \pi^{-1}(\ell) = \ell \cap X\}$$

(points of $\mathbf{P}(W_0) = \mathbf{P}^{d-1}$ are identified with lines ℓ in $\mathbf{P}(W) = \mathbf{P}^d$ through p), and we considered the projection

$$f : X' \rightarrow \mathbf{P}(W_0) = \mathbf{P}^{d-1}.$$

For a point ℓ in the target (considered as a line through p in $\mathbf{P}(W) = \mathbf{P}^d$), the scheme-theoretic fiber $f^{-1}(\{\ell\})$ over the “point” $\ell \in \mathbf{P}(W_0)$ is the same as the scheme-theoretic preimage $\pi^{-1}(\ell)$ of the line $\ell \subseteq \mathbf{P}(W)$ under the finite surjection $\pi : X \rightarrow \mathbf{P}(W) = \mathbf{P}^d$. Let $H \subseteq \mathbf{P}(V) = \mathbf{P}^N$ be the $(N-d+1)$ -plane that is the closure of the subvariety of homothety classes of linear forms on V with a specified nonzero restriction to W_0 (corresponding to ℓ as a point of $\mathbf{P}(W_0)$), or more geometrically let H be the projective span the independent points p_1, \dots, p_{N-d} and two points lifting a pair of independent points on the line $\ell \subseteq \mathbf{P}^d$. The scheme-theoretic fiber $f^{-1}(\{\ell\}) = \pi^{-1}(\ell)$ is precisely the linear slice $X \cap H$.

The question is this: as we vary through the permissible choices of $(N-d-1)$ -plane $L = \mathbf{P}(V/W)$, and then choices of the point $p = \mathbf{P}(W/W_0)$ in the projective space $\mathbf{P}(W)$ (which depends on $L = \mathbf{P}(V/W) \subseteq \mathbf{P}(V)$), are the resulting $(N-d+1)$ -planes H generic? That is, does this collection of H 's at least exhaust some non-empty *open* set in the associated Grassmannian $\text{Gr}_d(V)$ of codimension- d subspaces in V (i.e., codimension- $(d+1)$ planes in $\mathbf{P}(V)$)? We need an affirmative answer because the Bertini theorem ensures that, with respect to the Zariski topology in $\text{Gr}_d(V)$, a generic $(N-d+1)$ -plane H has scheme-theoretic intersection $X \cap H$ that is a smooth and (geometrically) irreducible curve.

Intuitively, it must be the case that as we generically vary $L = \mathbf{P}(V/W)$ and *then* vary the point $p = \mathbf{P}(W/W_0) \in \mathbf{P}(W)$ subject to the various “generic” restrictions imposed upon it (namely, it lies outside of $\pi_L(Z) \subseteq \mathbf{P}(W)$, the projection $\text{pr}_p : \pi_L(Z) \rightarrow \mathbf{P}(W_0)$ is generically étale, and π_L is étale over an open neighborhood of p), the resulting $(N-d+1)$ -planes $H \in \mathbf{P}(V)$ “spanned” by L and a generic choice of line in $\mathbf{P}(W)$ through p should at least exhaust a non-empty open set in $\text{Gr}_d(V)$. If you reflect on the matter, you’ll see that there is something to be proved here because we have iterated genericity conditions: the conditions on the point p that lies in the projective space $\mathbf{P}(W)$ which *depends* on L , and the way that L was built up through iterated generic choices of k -points with each *depending on the preceding* generic choice. In the end we wish to sweep out at least a dense open locus in the Grassmannian $\text{Gr}_d(V)$ of H 's in $\mathbf{P}(V)$.

It is probably the case that if you make artful non-canonical choices of planes in \mathbf{P}^N to realize all projective spaces inside \mathbf{P}^N then you can see the desired genericity by working entirely within \mathbf{P}^N . However, at least for me the clearest way to do this rigorously is to avoid non-canonical choices and work directly with universal bundles over suitable Grassmannians. This is the approach that we shall take below.

2. GENERICITY OF L

As a first step toward genericity of H 's, we want to show that the L 's obtained in the initial part of the construction exhaust a Zariski-open locus in the variety $\text{Gr}_{d+1}(V)$ of codimension $(d+1)$ -planes in the $(N+1)$ -dimensional vector space V (which is to say, codimension- d planes in $\mathbf{P}(V)$). That is, although we built our L in successive stages, what matters for our purposes is to have a dense open locus of such L 's. For any codimension- d plane $L = \mathbf{P}(V/W) \subseteq \mathbf{P}(V)$ disjoint from X such that the projection $\mathbf{P}(V) - L \rightarrow \mathbf{P}(W)$ has (necessarily finite and) generically étale restriction π_L to X , by choosing $N-d$ independent points p_1, \dots, p_{N-d} in L we can realize π_L as a successive composite of projections away from the p_j 's. Hence, it is equivalent to prove:

Theorem 2.1. *Let k be a field and V an $(N+1)$ -dimensional k -vector space. Let $X \subseteq \mathbf{P}(V)$ be a generically smooth closed subscheme with pure dimension $d < N$. Let $\mathbf{G} = \text{Gr}_{d+1}(V)$ be the Grassmann variety of codimension- d planes in $\mathbf{P}(V)$. The locus of “points” $L = \mathbf{P}(V/W) \in \mathbf{G}$ such that L is disjoint from X and $\pi_L : X \rightarrow \mathbf{P}(W)$ is generically étale is dense and open.*

In the final statement of the theorem it has to be understood that we are working with arbitrary field-valued points of the Grassmannian (not just k -points), and we are engaging in the usual abuse of notation by not explicitly writing in the base change to the field in which the point takes its values. (Of course, it is equivalent to work with points taking values in a fixed algebraically closed extension of k .)

Proof. We already know that this locus has a geometric point (and in fact has lots of geometric points). Since \mathbf{G} is irreducible, the problem is therefore just to prove that this locus is open. By definition, \mathbf{G} classifies isomorphism classes of quotient bundles of $V_S := V \otimes_k \mathcal{O}_S$ with rank $(N + 1) - (d + 1)$, which is to say subbundles of V_S with rank $d + 1$ (for varying k -schemes S). In particular, there is a universal subbundle $\mathcal{E} \hookrightarrow V_{\mathbf{G}}$ of rank $d + 1$ over \mathbf{G} , and so we get a “universal” $(N - d - 1)$ -plane $\mathbf{P}(V_{\mathbf{G}}/\mathcal{E}) \hookrightarrow \mathbf{P}(V_{\mathbf{G}}) = \mathbf{P}(V) \times_k \mathbf{G}$ over \mathbf{G} . Consider the associated “universal projection” morphism

$$\Pi : \mathbf{P}(V_{\mathbf{G}}) - \mathbf{P}(V_{\mathbf{G}}/\mathcal{E}) \rightarrow \mathbf{P}(\mathcal{E})$$

over \mathbf{G} that is functorially defined as follows: to a rank- $(d + 1)$ subbundle $\mathcal{F} \subseteq V_S$ and hyperplane bundle $\mathcal{H} \subseteq V_S$ fiberwise not containing \mathcal{F} (so \mathcal{F} corresponds to a k -map $\phi : S \rightarrow \mathbf{G}$ and \mathcal{H} corresponds to a \mathbf{G} -map $S \rightarrow \mathbf{P}(V_{\mathbf{G}})$) we associate the hyperplane bundle $\mathcal{H} \cap \mathcal{F} \subseteq \mathcal{F}$. Over a geometric point $\text{Spec } K \rightarrow \mathbf{G}$ corresponding to a $(d + 1)$ -dimensional subspace $W \subseteq V_K = V \otimes_k K$, the pullback of Π is the usual projection $\mathbf{P}(V_K) - \mathbf{P}(V_K/W) \rightarrow \mathbf{P}(W)$ as considered earlier (given by restriction of homothety classes of linear functionals).

On geometric points, consider the condition on \mathbf{G} that a codimension- d plane $\mathbf{P}(W)$ is disjoint from X . This is a *dense open* condition (stronger than merely containing a dense open set). Indeed, the overlap $X_{\mathbf{G}} \cap \mathbf{P}(V_{\mathbf{G}}/\mathcal{E})$ of closed subschemes in $\mathbf{P}(V_{\mathbf{G}}) = \mathbf{P}(V) \times_k \mathbf{G}$ has closed image in \mathbf{G} by properness of projective bundles, and this closed set is not all of \mathbf{G} because Bertini’s theorem provides (many) codimension- $(d + 1)$ planes in $\mathbf{P}(V)$ that do not meet the pure d -dimensional closed subscheme $X \subseteq \mathbf{P}(V)$. The complement of this closed set is a dense open $\mathbf{G}^0 \subseteq \mathbf{G}$, and it is precisely the locus of interest. All we have done is obtain an openness aspect in a Bertini theorem.

Now over \mathbf{G}^0 we have the inclusion $X_{\mathbf{G}^0} \subseteq \mathbf{P}(V_{\mathbf{G}^0}) - \mathbf{P}(V_{\mathbf{G}^0}/\mathcal{E}|_{\mathbf{G}^0})$, and so by restriction of Π we arrive at the “universal” π_L :

$$\pi^{\text{univ}} : X_{\mathbf{G}^0} \rightarrow \mathbf{P}(\mathcal{E}|_{\mathbf{G}^0}) = \mathbf{P}(\mathcal{E})_{\mathbf{G}^0};$$

as we vary through geometric points of \mathbf{G}^0 , this map gives precisely the projections π_L from X away from codimension- $(d + 1)$ planes in $\mathbf{P}(V)$ that do not touch X . Since all maps π_L are finite, the map π^{univ} is finite on fibers over geometric points of \mathbf{G}^0 , so π^{univ} is quasi-finite. It is also a \mathbf{G}^0 -map between proper \mathbf{G}^0 -schemes, so it is proper. Hence, π^{univ} is a finite map.

Letting X^{sm} be the dense open k -smooth locus of X , the quasi-finite restriction $\pi^0 : X_{\mathbf{G}^0}^{\text{sm}} \rightarrow \mathbf{P}(\mathcal{E})_{\mathbf{G}^0}$ of π^{univ} between flat \mathbf{G}^0 -schemes of finite type is fiberwise quasi-finite between smooth algebraic schemes of the same pure dimension. By the miracle flatness theorem (23.1 in Matsumura’s CRT) such fibral restrictions are therefore flat, and so by the fibral flatness criterion it follows that the restriction of π^{univ} to $X_{\mathbf{G}^0}^{\text{sm}}$ is *flat*. Hence, the open étale locus U of π^0 meets each fiber over a geometric point $L \in \mathbf{G}^0$ in the étale locus of $\pi_L|_{X^{\text{sm}}}$. I claim that there is a dense open in the irreducible \mathbf{G}^0 over which U is fiberwise dense, and so this would provide a dense open locus of L ’s for which π_L is generically étale on X .

It is harmless (check!) to make a finite extension on k , so we may assume that the connected components of X^{sm} are geometrically connected (and so geometrically irreducible). In this case, if $\{X_j^{\text{sm}}\}$ is the set of such components then the problem is to find a non-empty open $\mathbf{U}_j \subseteq \mathbf{G}^0$ over which the open set $U \cap (X_j^{\text{sm}})_{\mathbf{G}^0}$ meets all (irreducible!) fibers of $(X_j^{\text{sm}})_{\mathbf{G}^0}$. Indeed, such \mathbf{U}_j ’s

in the irreducible \mathbf{G}^0 must be dense and so they would have a common non-empty overlap that is the desired dense open locus of L 's in \mathbf{G}^0 over which U is fiberwise dense. To find the U_j 's, first use the *definition* of \mathbf{G}^0 and deJong's 2.11 to find geometric points L of \mathbf{G}^0 for which $\pi_L|_{X_j^{\text{sm}}}$ is generically étale. Thus, $U \cap (X_j^{\text{sm}})_{\mathbf{G}^0}$ is a *non-empty* open set and hence its image under the *smooth* projection $(X_j^{\text{sm}})_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$ is a non-empty *open* set U_j that does the job. \blacksquare

To summarize, we have proved that in the Grassmannian \mathbf{G} , the condition on a geometric point $L = \mathbf{P}(V/W)$ that $L \cap X = \emptyset$ and $\pi_L : X \rightarrow \mathbf{P}(W)$ is generically étale is a dense *open* set. (This is stronger than just saying it contains a dense open set.) It remains to show that the further requirement that π_L carries Z birationally onto its image in $\mathbf{P}(V)$ is a generic condition. Consider the universal case (using notation as in the previous proof)

$$\pi_Z^{\text{univ}} = \pi^{\text{univ}}|_{Z_{\mathbf{G}^0}} : Z_{\mathbf{G}^0} \rightarrow \mathbf{P}(\mathcal{E}|_{\mathbf{G}^0}).$$

This map is certain finite (as it is proper and has finite restriction to geometric fibers over \mathbf{G}^0), and we seek a dense open locus in \mathbf{G}^0 over which it is birational onto its image on (geometric) fibers.

Consider the natural map of sheaves $\mathcal{O}_{\mathbf{P}(\mathcal{E}|_{\mathbf{G}^0})} \rightarrow \pi_{Z_*}^{\text{univ}}(\mathcal{O}_{Z_{\mathbf{G}^0}})$. Since π_Z^{univ} is *finite* (hence affine), the formation of this map of sheaves commutes with any base change on $\mathbf{P}(\mathcal{E}|_{\mathbf{G}^0})$, so it commutes with any base change on \mathbf{G}^0 . Thus, the formation of the *open* locus $U \subseteq \mathbf{P}(\mathcal{E}|_{\mathbf{G}^0})$ over which this sheaf map is surjective (use Nakayama for openness) is compatible with any base change on \mathbf{G}^0 . Now by deJong's 2.11 and the definition of \mathbf{G}^0 , there exists a geometric point $L \in \mathbf{G}^0(k)$ such that the open L -fiber $U_L \subseteq \mathbf{P}(\mathcal{E}|_L)$ contains a dense open in $\pi_L(Z)$, or equivalently (by finiteness of π_L and equidimensionality of Z) that the open $(\pi_Z^{\text{univ}})^{-1}(U) \subseteq Z_{\mathbf{G}^0}$ is dense in the L -fiber. We seek a non-empty open in the irreducible \mathbf{G}^0 over which $(\pi_Z^{\text{univ}})^{-1}(U)$ is dense in all fibers of $Z_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$.

Let $\{Z_j\}$ be the (reduced) irreducible components of the equidimensional Z over the algebraically closed field k , so by flatness of $(Z_j)_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$ the restriction $(\pi_Z^{\text{univ}})^{-1}(U) \cap (Z_j)_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$ to an open subset is also flat. This map is therefore open, but its image hits the geometric point L mentioned above and so its image is a *nonempty* open set. By varying j we get finitely many nonempty open sets in the irreducible \mathbf{G}^0 , so they meet in a common dense open over which we have the desired fibral density in fibers of $Z_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$.

This completes the proof that for a generic $(N-d+1)$ -plane $L = \mathbf{P}(V/W)$ in $\mathbf{P}(V) = \mathbf{P}^N$ we have $L \cap X = \emptyset$, $\pi_L : X \rightarrow \mathbf{P}(W) = \mathbf{P}^d$ is generically étale, and the restriction $\pi_L : Z \rightarrow \mathbf{P}(W) = \mathbf{P}^d$ is birational onto its image. In fact, the above proof gives a stronger property than genericity that will be essential in what follows: this collection of conditions on L is an *open* condition. The only aspect of such openness that has not been explained above is the generically étale property for π_L (given $L \cap X = \emptyset$; that is, $L \in \mathbf{G}^0$). But the finite π_L is generically flat for any L disjoint from X , so generic étaleness is equivalent to generic unramifiedness. This latter condition is precisely the generic vanishing of the coherent sheaf $\Omega_{X/\mathbf{P}(W)}^1$, and since X is (geometrically) irreducible it is equivalent that such vanishing occur *somewhere* on X . By Nakayama, it is equivalent to have a vanishing fiber at some point of X , and so by compatibility of formation of Ω^1 with respect to base change on the target we get the desired openness of this condition on L via Nakayama's Lemma. (Form the open set in $X_{\mathbf{G}^0}$ where its Ω^1 with respect to π^{univ} vanishes, and use the open image of this under the flat map $X_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$.)

To summarize: not only have we proved the genericity of L in the Grassmannian $\mathbf{G} = \mathbf{Gr}_{d+1}(V)$, but we have shown that the conditions of interest on the codimension- d plane L in $\mathbf{P}(V)$ cut out exactly an open set $\mathbf{G}' \subseteq \mathbf{G}$.

3. GENERICITY OF H

Consider a codimension- d plane $L = \mathbf{P}(V/W) \subseteq \mathbf{P}(V)$ satisfying the preceding conditions. For each such L , consider points $p = \mathbf{P}(W/W_0) \in \mathbf{P}(W) - \pi_L(Z)$ such that (i) the finite generically étale map $\pi_L : X \rightarrow \mathbf{P}(W)$ is étale over an open neighborhood of p (that is, p lies outside of the closed set given by the π_L -image of the closed non-étale locus of π_L in the irreducible X), and (ii) the finite restriction $\text{pr}_p : \pi_L(Z) \rightarrow \mathbf{P}(W_0)$ is generically étale. To p we associate the codimension- $(d-1)$ plane $H = \mathbf{P}(V/W_0)$ in $\mathbf{P}(V)$, and we want to prove that as we vary through all possible $L = \mathbf{P}(V/W)$ as above and the allowed $p \in \mathbf{P}(W)$, the resulting subspaces $H = \mathbf{P}(V/W_0)$ in $\mathbf{P}(V)$ sweep out at least a dense open in the corresponding Grassmannian G for V .

Consider the flag variety F classifying pairs $W_0 \subseteq W$ consisting of a $(d+1)$ -dimensional subspace $W \subseteq V$ and a hyperplane W_0 in W . This is a smooth projective variety, and it is equipped with two natural forgetful maps $F \rightarrow G$ and $F \rightarrow \mathbf{G}$ given by $(W_0, W) \mapsto W_0$ and $(W_0, W) \mapsto W$ respectively. Under the surjection $F \rightarrow G$ between irreducible varieties, the image of any nonempty open set contains a dense open. (In fact, this map is smooth, so the image of an open is open.) It is therefore sufficient to prove genericity in F for the set of pairs (W_0, W) as above over a nonempty open set in \mathbf{G} . We shall of course consider the part of F lying over the dense open $\mathbf{G}' \subseteq \mathbf{G}$ consisting of *precisely* the subspaces $L = \mathbf{P}(V/W)$ as above. Recall again the conditions on $p = \mathbf{P}(W/W_0)$: if $B_L \subseteq X$ denotes the closed nowhere dense non-étale locus (“branch scheme”) for the finite generically étale projection $\pi_L : X \rightarrow \mathbf{P}(W)$, then we want p to lie in $L - \pi_L(Z \cup B_L)$ and that $\text{pr}_p : \pi_L(Z) \rightarrow \mathbf{P}(W_0) = \mathbf{P}^{d-1}$ is generically étale. Note that since the finite $\pi_L|_Z$ is generically étale (it is even birational, by the conditions on L), the hypothesis on pr_p is equivalent to requiring that $\text{pr}_p \circ \pi_L : Z \rightarrow \mathbf{P}(W_0)$ is generically étale.

Aside from the condition that $p = \mathbf{P}(W/W_0)$ not lie in $\pi_L(B_L)$, the other requirements on $(W_0, W) \in F$ over the dense open $\mathbf{G}' \subseteq \mathbf{G}$ are that it lies in the part of F sitting over the dense open $G' \subseteq G$ defined with respect to Z exactly as the dense open $\mathbf{G}' \subseteq \mathbf{G}$ was defined with respect to X . (The openness of G' requires a moment’s reflection, since Z is reducible whereas X is irreducible. We were careful in Theorem 2.1 to not impose irreducibility hypotheses so that it would carry over to Z in the present setting. For the other aspects it is straightforward to allow reducibility without losing openness or density.) It remains to check genericity in F for the locus cut out by the condition that $p = \mathbf{P}(W/W_0) \notin \pi_L(B_L)$ with $L = \mathbf{P}(V/W) \in \mathbf{G}'$. This locus is not empty because we have exhibited such L and p at the outset (the definitions of \mathbf{G}' and G' did *not* involve ambiguous “genericity” restrictions!). It is therefore enough to prove abstractly that this locus is open in F .

Let $F^0 \subseteq F$ be the open preimage of $\mathbf{G}' \subseteq \mathbf{G}$, so the finite map

$$\pi_{F^0}^{\text{univ}} : X_{F^0} \rightarrow \mathbf{P}(V_{F^0}/\mathcal{E}_{F^0})$$

over F^0 has open étale locus that is dense in all fibers over F^0 . The closed complement of this open set has closed image in $\mathbf{P}(V_{F^0}/\mathcal{E}_{F^0})$, so the desired openness result follows from the rather general:

Lemma 3.1. *Let $\mathcal{G}' \subseteq \mathcal{G}$ be a hyperplane bundle in a vector bundle \mathcal{G} over a scheme S , and let $T \subseteq \mathbf{P}(\mathcal{G})$ be a closed subscheme. Let $j : S = \mathbf{P}(\mathcal{G}/\mathcal{G}') \hookrightarrow \mathbf{P}(\mathcal{G})$ be the canonical section. The condition on S -schemes that the pullback of S not meet the pullback of T is represented by an open subscheme of S .*

Proof. The open locus $S - j^{-1}(T)$ does the job. ■