Math 249B. Avoiding Gabber's Theorem

Let S be a noetherian scheme (e.g., the spectrum of a field) and let $\mathscr{M} \to S$ be a proper Deligne–Mumford stack of finite type. Let X be a reduced and irreducible separated S-scheme of finite type, and $U \subset X$ a dense open subscheme equipped with an S-morphism $h : U \to \mathscr{M}$. We wish to explain how to extend this across X at the cost of a generically étale alteration of X.

That is, we seek a generically étale alteration $q : X' \to X$ from a quasi-projective S-scheme X' and a morphism $X' \to \mathscr{M}$ whose restriction over the dense open $q^{-1}(U)$ coincides with (in the sense of being isomorphic to) the morphism $h \circ q : q^{-1}(U) \to \mathscr{M}$. This bypasses the need to appeal to a theorem of Gabber on certain normalizations over $\overline{\mathscr{M}}_{g,n}$ being schemes (as used in de Jong's alterations paper). By Chow's Lemma for schemes applied to the separated finite-type $X \to S$, we can compose back with a preliminary birational proper map $Y \twoheadrightarrow X$ for Y quasi-projective over S and rename Y as X to arrange that X is quasi-projective over S; this will be used at the end.

The key ingredient is Chow's Lemma for DM stacks, whose statement we now recall (and which was not documented in the published literature at the time of deJong's 1996 alterations paper; it was announced as Theorem 4.12 in Deligne and Mumford's 1969 paper but no details were ever published by them)

Theorem 0.1. *Let $\mathscr{X}$ be a Deligne–Mumford stack of finite presentation over a noetherian scheme S. There exists a surjective proper morphism $Z \to \mathscr{X}$ from a quasi-projective S-scheme Z such that the map is étale on a dense open subscheme of Z.*

This is proved in 16.6.1 of the book by Laumon and Moret-Bailly (where DM stacks are required to be quasi-separated by definition, so in their terminology "finite type" is the same as "finite presentation" for a DM stack over a noetherian base). To apply this, consider the morphism $(1, h) : U \to X \times_S \mathscr{M}$ to a DM stack *proper* over X. This factors through $U \times_S \mathscr{M}$, and this factored map $j : U \to U \times_S \mathscr{M}$ is a base change of the diagonal $\Delta_{\mathscr{M}/S}$ that is proper since $\mathscr{M}$ is S-separated. Since $\mathscr{M}$ is a DM-stack, its proper diagonal over S is finite and more specifically is "an immersion up to étale maps" (as one sees via an étale scheme cover of $\mathscr{M}$). The same then holds for j.

Let $\mathscr{X}$ be the schematic image of $(1, h)$, a reduced and irreducible closed substack of $X \times_S \mathscr{M}$. In particular, $\mathscr{X}$ is X-proper (since $\mathscr{M}$ is S-proper) and its underlying space over U coincides with the image of the proper j, so the stacky generic point of $\mathscr{X}$ is *étale* over the generic point of X.

Now we apply Chow's Lemma as stated above to the proper $\mathscr{X} \to X$ to get a reduced and irreducible quasi-projective X-scheme X' along with a surjective proper X-morphism $X' \to \mathscr{X}$ that is generically étale. Restricting over the generic point of $\mathscr{X}$ and using that $X' \to \mathscr{X}$ is generically étale, the function field of X' is finite separable over the function field of X. Hence, $q : X' \to X$ is a generically étale *alteration*. By design the morphism $X' \to \mathscr{X} \to \mathscr{M}$ has restriction over the preimage of U that coincides with $q^{-1}(U) \to U \xrightarrow{h} \mathscr{M}$. Since $X' \to X$ is quasi-projective, and we arranged for $X \to S$ to be quasi-projective by the initial step with Chow's Lemma for schemes, clearly $X' \to S$ is quasi-projective.