Math 249B. Isomorphisms of quadratic spaces

1. Introduction

Let A be a local ring, with residue field k, and $Q, Q' : A^n \rightrightarrows A$ two residually non-degenerate quadratic forms in n variables over A such that their reductions $q, q' : k^n \rightrightarrows k$ are isometric. (That is, there exists a linear automorphism L_0 of k^n such that $q \circ L_0 = q'$.)

Subject to the mild hypothesis $\mathrm{char}(k) \neq 2$ *when n is odd*, we aim to prove that if A is henselian (e.g., complete local noetherian, or the henselization of any local ring) then Q and Q' are isometric; i.e., we seek to build a linear automorphism L of A^n such that $Q \circ L = Q'$.

Remark 1.1. Let's show that the parity condition on n when $\mathrm{char}(k) = 2$ cannot be removed. (We will only care about the case of even n, in fact $n = 2$, so the reader who doesn't care about residue characteristic 2 is welcome to skip this.) Suppose $\mathrm{char}(k) = 2$ and $n = 2m + 1$ with $m \geq 0$. Let

$$Q = x_0^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m}, \quad Q' = ux_0^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m}$$

for $u \in A^\times$. The reductions (k^n, q) and (k^n, q') have 1-dimensional defect spaces each coinciding with the line ke_0. (Recall that for a non-degenerate quadratic space (V, q) of odd dimension $2m+1$ over a field of characteristic 2, the symmetric bilinear form $B_q(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$ is alternating and has defect space $V^\perp := \{v \in V \,|, B_q(v, \cdot) = 0\}$ equal to a line, with the induced alternating form $\overline{B}_q$ on the $2m$-dimensional $V/V^\perp$ non-degenerate; i.e., symplectic.)

The restrictions of q and q' to their respective defect lines are x_0^2 and $\overline{u}x_0^2$ respectively, so the existence of a residual isometry forces the reduction $\overline{u} \in k^\times$ to be a square in k. Since $\mathrm{char}(k) = 2$, the reduction of u being a square does not generally imply that u is a square in $A^\times$ (even if A is a complete discrete valuation ring). Although it is clear by inspection that u being a square in $A^\times$ is sufficient for Q and Q' to be isometric, it isn't evident in general if this is *necessary* for Q and Q' to be isometric (in which case we would have a genuine obstruction, showing that the parity condition on n for residue characteristic 2 cannot be avoided).

In case A is an $\mathbf{F}_2$-algebra then necessity is obvious because in such cases the intrinsic *defect modules* for Q and Q' respectively each coincide with the subbundle Ae_0 on which they respectively restrict to x_0^2 and ux_0^2, so we can argue as we did over k. Thus, by taking A to be a local henselian $\mathbf{F}_2$-algebra and $u \in A^\times$ a non-square unit whose reduction is 1 (e.g., $A = \kappa[\![t]\!]$ for a field κ of characteristic 2 and $u = 1 + t$) we get the desired examples of non-isometric Q and Q' that are residually non-degenerate and residually isometric for all odd $n = 2m + 1$.

In fact, by using more serious input through the structure of "odd" orthogonal group schemes over rings one can show over any local ring A that the condition of u being a square is always necessary for Q and Q' to be isometric; see Remark 2.1. Thus, we also get mixed-characteristic examples with A any 2-adic integer ring, by taking u to be $1 + \pi$ for a uniformizer π (visibly a non-square precisely because if u were a square then it would have to have a square root that is a 1-unit but the square of any 1-unit is 1 mod π^2 since $\pi | 2$ in A).

2. Smoothness of an Isom-scheme

Now we take up the task of proving for henselian local A that Q and Q' are isometric when they are residually isometric, provided that n is even when $\mathrm{char}(k) = 2$. First we dispose of a boring case: $n = 1$. Suppose $n = 1$, so $Q = ax_0^2$ and $Q' = a'x_0^2$ for units $a, a' \in A^\times$ such that a'/a has reduction in $k^\times$ that is a square (by the residual isometry hypothesis). For odd n we are assuming $\mathrm{char}(k) \neq 2$, so a'/a is therefore a square in A by the henselian condition (i.e., $Y^2 - a'/a \in A[Y]$

has a simple residual root, and thus a root in A) and so Q and Q' are isometric. Thus, we now assume $n \geq 2$.

The key idea is to introduce an A-scheme classifying isometries and prove this scheme is *smooth* (which will crucially use the hypothesis that n is even when $\mathrm{char}(k) = 2$); it will then follow via the henselian condition on A that even the given residual isometry $q' \simeq q$ can be lifted to an isometry $Q' \simeq Q$ over A.

Inside the A-group scheme GL_n, the condition on a point L that $Q' = Q \circ L$ is an explicit (albeit nasty) finite system of universal polynomial conditions over A on the matrix entries of L (depending on Q and Q' over A). This defines a finitely presented closed subscheme $I \subset \mathrm{GL}_n$ representing the functor on A-algebras

$$C \rightsquigarrow \mathrm{Isom}((C^n, Q'_C), (C^n, Q_C)).$$

We are given that $I(k)$ is non-empty. Thus, if I is A-smooth then $I(A) \to I(k)$ is surjective for henselian A, so we would be done. Our aim then is to prove that I is A-smooth. The beauty of this idea is that as a property of an explicit finitely presented A-scheme it will be sufficient to check this after suitable fpqc scalar extensions on A that would otherwise seem to lose all contact with our actual problem of interest over A.

We saw in class (with $n \geq 2$) that the residual non-degeneracy of Q implies that the projective quadric $(Q = 0) \subset \mathbf{P}^{n-1}_A$ is A-smooth with relative dimension $n - 2 \geq 0$, so we now forget about the assumptions that A is local and henselian (so in particular we drop the hypothesis involving a residual isometry!) and instead allow A to be any ring whatsoever but *assume* two things:

(i) the projective quadrics $(Q = 0), (Q' = 0) \subset \mathbf{P}^{n-1}_A$ are A-smooth with relative dimension $n - 2$,
(ii) if n is odd then A is a $\mathbf{Z}[1/2]$-algebra.

Under these assumptions, we shall prove that the Isom-scheme I is A-smooth. It is only at the end of the argument that (ii) will be used.

As discussed in Lemma 1.3 of the handout "Orthogonal group schemes" in my course Algebraic Groups I (largely refering to a self-contained concrete calculation in [SGA7, XII, Prop. 1.2]), the smoothness of relative dimension $n - 2$ for the projective quadrics $(Q = 0)$ and $(Q' = 0)$ in $\mathbf{P}^{n-1}_A$ ensures that fppf-locally on $\mathrm{Spec}(A)$, each of Q and Q' becomes isometric to the "standard" fiberwise non-degenerate quadratic form in n variables, namely $Q_{2m} := x_1x_2 + \cdots + x_{2m-1}x_{2m}$ when $n = 2m$ and $Q_{2m+1} := x_0^2 + Q_{2m}$ when $n = 2m+1$. Since the A-smoothness of the Isom-scheme I is an fppf-local problem over $\mathrm{Spec}(A)$, by making a suitable fppf-affine base change on A (!) we can assume $Q = Q' = Q_n$! Thus, our Isom-scheme I becomes the A-group scheme $\mathrm{O}_{n,A} \subset \mathrm{GL}_{n,A}$ representing the functor of isometric automorphisms of Q_n. This A-group scheme is the scalar extension of the corresponding one over $\mathbf{Z}$, so our task has reduced to that of proving the smoothness of O_n over $\mathbf{Z}$ for even n and over $\mathbf{Z}[1/2]$ for odd n. (Recall that we assume A is a $\mathbf{Z}[1/2]$-algebra when n is odd.)

It is shown by a concrete equation-counting argument in Proposition 2.3 of the handout "Orthogonal group schemes" from my course Algebraic Groups I that if $n = 2m$ with $m \geq 1$ then the orthogonal group scheme O_n is $\mathbf{Z}$-smooth and in fact (see also Corollary 2.4 of loc. cit.) is an extension of the constant group $\mathbf{Z}/(2)$ by a smooth affine group scheme SO_n with connected fibers of dimension $n(n-1)/2$ (this latter group scheme is *not* defined to be $\mathrm{O}_n \cap \mathrm{SL}_n$ because for even n this gives the wrong group in characteristic 2; the Dickson morphism defined in §1 of "Orthogonal group schemes" gives a unified definition over $\mathbf{Z}$ for even n, recovering the usual notion over $\mathbf{Z}[1/2]$ by Corollary 2.5 of loc. cit.).

In Proposition 3.5 of that same handout, it is shown that if $n = 2m + 1$ then $\mathrm{O}_n = \mu_2 \times \mathrm{SO}_n$ as $\mathbf{Z}$-group schemes for the $\mathbf{Z}$-group scheme $\mathrm{SO}_n := \mathrm{O}_n \cap \mathrm{SL}_n$ that is shown to be smooth with connected fibers of dimension $n(n-1)/2$. Thus, over $\mathbf{Z}[1/2]$ we recover the desired smoothness for O_n for odd n!

Remark 2.1. The description $\mathrm{O}_{2m+1} = \mu_2 \times \mathrm{SO}_{2m+1}$ over $\mathbf{Z}$ allows us to fill in the loose end in Remark 1.1 concerning the necessity of u being a square (for those who didn't completely ignore Remark 1.1, which we don't logically need anyway). Namely, the Isom-scheme $I = \mathrm{Isom}(Q', Q)$ that we want to have an A-point is a left torsor for $\mathrm{O}(Q) = \mathrm{O}_{2m+1} = \mu_2 \times \mathrm{SO}_{2m+1}$ for the fppf topology on A. To get from Q to Q' over the fppf cover $A' = A[T]/(T^2 - u)$ of A, we can apply the diagonal operation on A'^n given by

$$\mathrm{diag}(\sqrt{u}, \sqrt{u}, 1/\sqrt{u}, \sqrt{u}, 1/\sqrt{u}, \ldots, \ldots \sqrt{u}, 1/\sqrt{u}).$$

Thus, the resulting fppf 1-cocycle over $A' \otimes_A A'$ is given by

$$\mathrm{diag}(\zeta, \zeta, 1/\zeta, \ldots, \zeta, 1/\zeta)$$

for $\zeta = \sqrt{u} \otimes (1/\sqrt{u})$. But $\zeta^2 = u \otimes (1/u) = 1$, so $\zeta \in \mu_2(A' \otimes_A A')$ and hence $1/\zeta = \zeta$.

We conclude that the fppf descent datum describing I as an O_{2m+1}-torsor comes from $\mathrm{H}^1(A, \mu_2)$. Since $\mathrm{O}_{2m+1} = \mu_2 \times \mathrm{SO}_{2m+1}$, this O_{2m+1}-torsor is trivial if and only if the class in $\mathrm{H}^1(A, \mu_2)$ is trivial (since a coboundary splitting the cocycle over some fppf cover of A refining A' can be projected to its μ_2-factor). By fppf Kummer theory, the class in $\mathrm{H}^1(A, \mu_2)$ represented by $\sqrt{u} \otimes 1/\sqrt{u} \in (A' \otimes_A A')^\times$ corresponds to the class of $u \in A^\times/(A^\times)^2$ under the natural injective map $A^\times/(A^\times)^2 \to \mathrm{H}^1(A, \mu_2)$. Hence, $I(A)$ is non-empty if and only if u is a square in A. This is exactly the necessity claimed in Remark 1.1.