Math 249B. Residue field and blow-up

Let R be a discrete valuation ring with fraction field K, residue field k, and uniformizer π. Let X be a semistable curve over R with smooth generic fiber; we make no properness or geometric connectedness hypotheses. Let $x_0 \in X_k$ be a non-smooth point in the special fiber, so $k(x_0)/k$ is a finite separable extension. We want to analyze the blow-up $X' := \mathrm{Bl}_{\{x_0\}}(X)$. Obviously $X'_K = X_K$ is K-smooth, and to analyze the local structure near closed points over x_0 in its special fiber over R (such as to find non-regular points and determine their residue fields over $k(x_0)$) we want to reduce to the case when $k(x_0) = k$. The aim of this handout is to explain the passage to the case $k(x_0) = k$ at the cost of replacing R with a suitable local-étale extension.

The finite separable extension $k(x_0)/k$ is primitive, so if $f \in k[T]$ is the minimal polynomial for a primitive element and $F \in R[T]$ is a monic lift of f then $R_1 = R[T]/(F)$ is a finite flat R-algebra such that $R_1/(\pi) = k[T]/(f) = k(x_0)$. Hence, R_1 is a discrete valuation ring that is finite étale over R with residue field $k(x_0)$. Let $X_1 = X \otimes_R R_1$ and $Z = \mathrm{Spec}(k(x_0) \otimes_k k(x_0)) \subset (X_1)_{k(x_0)}$, so X_1 is a semistable curve over R_1 (with smooth generic fiber) and the compatibility of blow-up with flat base change gives

$$X' \otimes_R R_1 = \mathrm{Bl}_Z(X_1).$$

Note that the diagonal point x_1 of Z is clopen in Z, so the local structure of $\mathrm{Bl}_Z(X_1)$ at points over $x_1 \in Z$ is identical to that of $X'_1 := \mathrm{Bl}_{x_1}(X_1)$.

Our aim is to determine the local structure of X' at its closed points over x_0: regularity (or not), semistability, and its residue field at such points that are not regular (if any). We want to relate this to an analogous calculation for a blow-up of X_1 at x_1, the advantage of the latter being that $k(x_1) = k(x_0)$ is the residue field of R_1. Due to the definition of $x_1 \in Z$ as a diagonal point, every closed point $x' \in X'$ over $x_0 \in X$ is the image of a unique closed point $\widetilde{x}' \in X' \otimes_R R_1$ over $x_1 \in Z$, and we have $\widehat{\mathscr{O}}_{X',x'} = \widehat{\mathscr{O}}_{X' \otimes_R R_1, \widetilde{x}'}$. Thus, to show that X' is semistable over R at all closed points over $x_0 \in X$ (the semistability being obvious elsewhere, since X is semistable over R) it suffices to show the same for X'_1 at all closed points over $x_1 \in X_1$, and the following invariants agree at corresponding points: the residue field (over $k(x_0) = k(x_1)$) and the measure of irregularity. Hence, we conclude that to show when $n_{x_0} \geq 2$ that X' is always semistable over R and is regular at all points over x_0 when $n_{x_0} = 2, 3$ but non-regular at exactly one point over x_0 when $n_{x_0} \geq 4$, with residue field $k(x_0)$ at that point and measure of irregularity $n_{x_0} - 2$ at that point, it suffices to do the same for X'_1 over R_1 at its points over $x_1 \in X_1$. In this way, we may replace (X, x_0, R) with (X_1, x_1, R_1) to reduce to the case $k(x_0) = k$.

Having reduced our task to the case $k(x_0) = k$, we next want to show that it suffices to treat the case

$$(Y, y_0) := (\mathrm{Spec}(R[u,v]/(Q(u,v) - \pi^n)), \overline{0})$$

with $n \geq 2$ and Q some (unknown) residually non-degenerate quadratic form over R. For the "Main Example" of semistable curves with smooth generic fibers over a reduced local noetherian ring A to illustrate the Structure Theorem for ordinary double points (no henselian hypothesis on A!), the refinement of the Structure Theorem for the case of residue field $A/\mathfrak{m}_A$ at the double point provides a residually non-degenerate quadratic form $Q(u,v)$ over R such that there is a common *residually trivial* pointed étale neighborhood (Y', y'_0) of (X, x_0) and $(\mathrm{Spec}(R[u,v]/(Q - \pi^n)), \overline{0})$ with $n = n_{x_0} \geq 2$. By shrinking Y' around y'_0 to arrange that y'_0 is the *unique* point of Y' over $x_0 \in X$ and over $y_0 \in Y$, so scheme-theoretically

$$Y' \times_X \{x_0\} = \{y'_0\} = Y' \times_Y \{y_0\}.$$

Hence, the compatibility of blow-up with respect to flat (e.g., étale!) base change provides *cartesian* sqaures

$$\begin{array}{ccccc} \mathrm{Bl}_{x_0}(X) & \longleftarrow & \mathrm{Bl}_{y'_0}(Y') & \longrightarrow & \mathrm{Bl}_{y_0}(Y) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & Y' & \longrightarrow & Y \end{array}$$

so the étale-local nature of regularity *and* of the "measure of irregularity" on semistable curves with smooth generic fibers imply that to show $X' := \mathrm{Bl}_{x_0}(X)$ is semistable over R with the desired regularity and residue field properties at its closed points over x_0 it is suffices to check the same for $\mathrm{Bl}_{y_0}(Y)$ over R.

Finally, we have to study how the situation for $\mathrm{Bl}_{y_0}(Y)$ over R interacts with local étale extension $R \to R'$. Note that $Y \otimes_R R'$ has a unique point over y_0, and $\mathrm{Bl}_{y_0}(Y) \otimes_R R'$ is the blow-up of $Y \otimes_R R'$ at that corresponding point. Hence, if after such extension regularity holds then it held in the first place, and likewise if there is a *unique* non-regular point on the blow-up of $Y \otimes_R R'$ and the residue field at that point agrees with that of R' then the same holds for $\mathrm{Bl}_{y_0}(Y)$ relative to R. (Here we are using that $k(y_0) = k$.) The measure of irregularity is also unchanged by computing on an étale cover, so we see that it suffices to work with $Y \otimes_R R'$ and the unique point on it over y_0. By taking R' big enough we can arrange that Q is residually split, so $Q_{R^{\mathrm{h}}}$ is isometric to uv (see the handout on quadratic forms). But then over some big enough local-étale extension of R we see that Q becomes isometric to uv. Taking that to be R' thereby reduces us to the case $Q = uv$. In other words, we have *finally* reduced to the "model case"

$$(\mathrm{Spec}(R[u,v]/(uv - \pi^n)), \overline{0}).$$