Math 249B. Blow-up of semistable curves over a regular base

Let R be a regular local ring of positive dimension d, $\{t_j\}$ part of a regular system of parameters, and $\{n_j\}$ positive integers such that $n_1 \geq 2$. Define $B = R[u,v]/(uv - \prod_{j=1}^r t_j^{n_j})$, and $I = (u, v, t_1)$ (so $B/I = R/t_1$ is regular local of dimension $d-1$). Our aim is to show that $\mathrm{Bl}_I(B)$ is an open semistable R-curve. The case $d = 1$ has been done already (our earlier work over discrete valuation rings), and we now wish to carry out such a calculation for general $d > 0$. Our only aim is to prove "open semistable" and not to keep track of any irreducible components in the non-regular locus or any function fields of such components, so the problem is solely one of examining the blow-up charts $D_+(u), D_+(v), D_+(t_1)$ to check that each of these affine open subschemes of the blow-up is an open semistable R-curve. Since u and v play entirely symmetric roles in the setup, the task is really just to study $D_+(u)$ and $D_+(t_1)$.

For $D_+(u)$, we introduce new variables v' and t_1' subject to the relations $v = v'u$ and $t_1 = t_1'u$. We need to determine the quotient of the R-algebra

$$R[u,v,v',t_1']/(uv - \prod t_j^{n_j}, v - v'u, t_1 - t_1'u) = R[u,v',t_1'](u^2v' = (t_1')^{n_1}u^{n_1}t_2^{n_2}\cdots t_r^{n_r}, t_1 - t_1'u)$$

modulo its u^∞-torsion. Since $n_1 \geq 2$ we can cancel the factor of u^2 in the final relation to obtain the relation

$$v' = (t_1')^{n_1}u^{n_1-2}t_2^{n_2}\cdots t_r^{n_r}$$

that permits us to remove v'. That leaves us with the quotient

$$R[u,v',t_1']/(t_1 - t_1'u)$$

which is visibly a regular domain (in particular, vanishing u^∞-torsion) and R-semistable.

Next, for $D_+(t)$ we introduce variables u'', v'' and relations $u = u''t_1$, $v = v''t_1$. This amounts to finding the quotient of the R-algebra

$$R[u,v,u'',v'']/(uv - \prod t_j^{n_j}, u - u''t_1, v - v''t_1) = R[u'',v'']/(u''v''t_1^2 - \prod t_j^{n_j})$$

modulo its t_1^∞-torsion. Since $n_1 \geq 2$ we can cancel a factor of t_1^2 from the final relation, leaving us with

$$R[u'',v'']/(u''v'' - t_1^{n_1-2}\prod_{j=2}^r t_j^{n_j}).$$

This is visibly R-semistable.