Math 249B. Hilbert schemes and a "universal" stable marked curve

1. Review of Hilbert schemes

Let $f : Y \to S$ be a proper finitely presented map of schemes. We seek to classify "flat families" of (finitely presented) closed subschemes. To be precise, consider the contravariant Set-valued functor $\underline{\mathrm{Hilb}}_{Y/S}$ on S-schemes that assigns to each S-scheme T the set of finitely presented closed subschemes $Z \subset Y_T$ that are T-flat. (For a closed immersion $j : Z \hookrightarrow Y_T$, it is equivalent to say that j is finitely presented and that Z is finitely presented over T; this avoids any possible confusion about what it means to say "Z is finitely presented". We leave the equivalence of conditions as an exercise; it uses only that Y_T is finitely presented over T.)

To be precise about regarding $\underline{\mathrm{Hilb}}_{Y/S}(T)$ as a set, we observe that Z is determined up to *unique isomorphism* by the specification of its finitely presented quasi-coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_{Y_T}$, subject to the condition that $\mathscr{O}_{Y_T}/\mathscr{I}$ is T-flat. (Alternatively, a closed subscheme is determined uniquely up to unique isomorphism by its associated closed subset of Y_T and the stalks of its structure sheaf as quotients of those of $\mathscr{O}_{Y_T}$ at points of that closed subset.) The assignment $T \rightsquigarrow \underline{\mathrm{Hilb}}_{Y/S}(T)$ is a functor by means of base change: for any $T' \to T$ over S, given any such $j : Z \hookrightarrow Y_T$ we get $j_{T'} : Z_{T'} \hookrightarrow Y_{T'}$ via base change.

One of Grothendieck's first significant results on moduli schemes was to prove the representability of such Hilbert functors under projectivity conditions on Y over S (the name "Hilbert functor/scheme" due to the role of Hilbert polynomials in the result under projective hypotheses, as we will see below). A good reference for the material we review below, including a modern existence proof that incorporates Mumford's simplifications in Grothendieck's original proof, in Chapter 5 in the book *FGA Explained*. (Grothendieck's original proof in his Bourbaki report is very inspiring reading, but it is tough going in places since he omits some details that are not so easy for a beginner to fill in.)

What does it mean to say that $\underline{\mathrm{Hilb}}_{Y/S}$ is representable? Or rather, for an S-scheme $\mathscr{H}$ what does it mean to specify an isomorphism of functors $\xi : \mathrm{Hom}_S(\cdot, \mathscr{H}) \simeq \underline{\mathrm{Hilb}}_{Y/S}$? (That is, not just to say that $\mathscr{H}$ represents $\underline{\mathrm{Hilb}}_{Y/S}$, but *how* it does so.) Evaluating ξ on $\mathscr{H}$ and chasing $\mathrm{id}_{\mathscr{H}}$ gives rise to an element of $\underline{\mathrm{Hilb}}_{Y/S}(\mathscr{H})$, which is to say an $\mathscr{H}$-flat finitely presented closed subscheme $\mathscr{Z} \subset Y \times_S \mathscr{H} =: Y_{\mathscr{H}}$, and by Yoneda-style general nonsense the condition that ξ is an isomorphism says precisely that this $\mathscr{Z}$ is *universal*: for any $T \to S$ and T-flat finitely presented closed subscheme $j : Z \hookrightarrow Y_T$, there exists a *unique* S-map $q : T \to \mathscr{H}$ such that there is a Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & \mathscr{Z} \\ {\scriptstyle j}\big\downarrow & & \big\downarrow \\ Y_T & \xrightarrow[1_Y \times q]{} & Y_{\mathscr{H}} \end{array}$$

We stress that given q, α is *unique* if it exists at all, since α amounts to an isomorphism $Z \simeq Y_T \times_{1\times q, Y_{\mathscr{H}}} \mathscr{Z} = T \times_{q,\mathscr{H}} \mathscr{Z}$ as closed subschemes of $Y_T = T \times_S Y = T \times_{q,\mathscr{H}} Y_{\mathscr{H}}$, any such isomorphism being unique if it exists since Z admits no nontrivial automorphism leaving j invariant. This is the reason that the specification of such q is all that matters (assuming some α exists at all), and is related to the fact that the objects being classified admit no nontrivial automorphisms (i.e., a T-automorphism of Z leaving j invariant must be id_Z); were that not the case then there is no reasonable way one could expect to have a universal structure $\mathscr{Z} \subset Y_{\mathscr{H}}$ classifying everything via solely the data of maps along the base (such as q above).

Remark 1.1. Grothendieck proved a remarkable functorial characterization [EGA IV_3, 8.14.2] for when a map of schemes is locally of finite presentation (though no such criterion is known for being locally of finite type beyond the locally noetherian case, nor is there any known functorial criterion for being quasi-compact). This is not logically necessary here, but it explains why a representing object for the Hilbert functor must be locally of finite presentation over S if it exists at all.

As we will see, representing objects for Hilbert functors tend to be a countably infinite disjoint union of clopen subschemes, due to certain discrete invariants not controlled by the definition of the Hilbert functor (much as when considering line bundles on relative proper curves one needs to fix the fiberwise degree to cut out a bounded part of the relative Picard scheme).

To get a handle on the task of representing $\underline{\mathrm{Hilb}}_{Y/S}$, Grothendieck chopped this up into more manageable parts via the crutch of an auxiliary relatively ample line bundle. To be precise, we now put ourselves in the projective setting: suppose there exists a closed immersion $i : Y \hookrightarrow \mathbf{P}^n_S$ and define $\mathscr{L} = i^*(\mathscr{O}(1))$. This line bundle $\mathscr{L}$ is "S-ample" (or "relatively ample") on Y in any of the following three senses that are *equivalent* for any proper finitely presented map $Y \to S$:

(i) there exists an open affine cover $\{U_\alpha = \mathrm{Spec}(A_\alpha)\}$ of S such that $\mathscr{L}|_{Y_{U_\alpha}}$ is ample on the A_α-proper Y_{U_α} for all α;
(ii) for all open affine $U = \mathrm{Spec}\, A \subset Y$, the restriction $\mathscr{L}|_{Y_U}$ is ample on the A-proper Y_U;
(iii) for all $s \in S$, the line bundle $\mathscr{L}_s$ on the $k(s)$-proper Y_s is ample.

The equivalence of (i) and (ii) is elementary, but the equivalence with (iii) lies much deeper and is remarkable since there are no flatness hypotheses on Y over S (so one cannot access any base-change theorems for coherent cohomology, say even if S were locally noetherian); see [EGA IV_3, 9.6.4] for this amazing equivalence.

Let's now retain $\mathscr{L}$ and forget about i (which we won't need): we suppose there is given an S-ample $\mathscr{L}$ on Y in the above three equivalent senses (and we do not assume that $\mathscr{L}$ is globally generated by a finite set of global sections, as happens when $\mathscr{L}$ arises via pullback of $\mathscr{O}(1)$ as above). To decompose $\underline{\mathrm{Hilb}}_{Y/S}$ into a disjoint union of "clopen subfunctors" via such an $\mathscr{L}$, we need two results on Euler characteristics. The first is Corollary 1(b) in §5 of Chapter II of Mumford's book *Abelian Varieties* (proved there for a locally noetherian base; the general case is reduced to this by some limit techniques):

Theorem 1.2. *Let $Z \to T$ be a proper finitely presented map, and $\mathscr{F}$ a finitely presented quasi-coherent sheaf on Z that is T-flat. Then the function on T defined by $t \mapsto \chi(\mathscr{F}_t) := \sum(-1)^i h^i(Z_t, \mathscr{F}_t)$ is locally constant for the Zariski topology. In particular, for varying $n \in \mathbf{Z}$ the pairwise disjoint subsets $\{t \in T \,|\, \chi(\mathscr{F}_t) = n\}$ of T are clopen.*

This is a striking fact, since the individual terms $h^i(Z_t, \mathscr{F}_t) = \dim_{k(t)} \mathrm{H}^i(Z_t, \mathscr{F}_t)$ are merely upper semi-continuous as t varies in T. The next result concerns the Euler characteristic of varying powers of a fixed line bundle:

Theorem 1.3. *If Z is a proper scheme over a field and $\mathscr{L}$ is an invertible sheaf on Z then for any coherent sheaf $\mathscr{F}$ on Z the function $n \mapsto \chi(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ is a polynomial in $\mathbf{Q}[n]$ with degree at most* $\dim Z$.

Proof. In the projective case the result is addressed in many textbooks on algebraic geometry. The general case is deduced from the projective case via Chow's Lemma, as follows. By Grothendieck's Unscrewing Lemma [EGA III_1, 3.1.2], to show that all coherent $\mathscr{F}$ satisfy the desired conclusion it suffices to show that for every irreducible closed subset $Z' \subset Z$ equipped with the reduced structure there exists some coherent $\mathscr{O}_{Z'}$-module $\mathscr{G}$ with generic fiber of dimension 1 such that $\mathscr{G}$ viewed as an $\mathscr{O}_Z$-module satisfies the desired conclusion. We will show that $\mathscr{G} = \mathscr{O}_{Z'}$ does the job.

Arguing by noetherian induction on the scheme Z, we may assume that for every proper closed subscheme $Z' \subset Z$ that is reduced and irreducible, every coherent $\mathscr{O}_{Z'}$-module satisfies the desired conclusion when viewed as an $\mathscr{O}_Z$-module. In particular, by the Unscrewing Lemma, we are done if Z is non-reduced or reducible (as the sufficient criterion above only involves closed subschemes of reduced irreducible components of Z_{red}), so we may assume Z is integral, and again via the Unscewing Lemma we know that for every proper closed subscheme $Z' \subset Z$ (not necessarily irreducible!) and coherent $\mathscr{O}_{Z'}$-module $\mathscr{G}$ the function $n \mapsto \chi(\mathscr{G} \otimes \mathscr{L}^{\otimes n})$ is in $\mathbf{Q}[n]$ with degree at most $\dim Z' \le \dim Z$. By the Unscrewing Lemma one more time, it is now only necessary to exhibit a single coherent $\mathscr{O}_Z$-module $\mathscr{F}$ with generic stalk of rank 1 such that $\chi(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ is in $\mathbf{Q}[n]$ with degree at most $\dim Z$. We will show that $\mathscr{F} = \mathscr{O}_Z$ works.

By Chow's Lemma, there exists a proper birational map $\pi : Y \to Z$ with Y projective and integral. Thus, $\mathscr{O}_Z \to \pi_*(\mathscr{O}_Y)$ is an inclusion whose cokernel $\mathscr{Q}$ is supported on a proper closed subscheme of Z (not necessarily irreducible!), so $n \mapsto \chi(\mathscr{Q} \otimes \mathscr{L}^{\otimes n})$ belongs in $\mathbf{Q}[n]$ with degree $< \dim Z$. Since

$$\chi(\mathscr{O}_Z \otimes \mathscr{L}^{\otimes n}) = \chi(\pi_*(\mathscr{O}_Y) \otimes \mathscr{L}^{\otimes n}) - \chi(\mathscr{Q} \otimes \mathscr{L}^{\otimes n}),$$

it suffices to show that $n \mapsto \chi(\pi_*(\mathscr{O}_Y) \otimes \mathscr{L}^{\otimes n})$ belongs to $\mathbf{Q}[n]$ with degree at most $\dim Z$.

Each higher direct image $\mathrm{R}^j\pi_*(\mathscr{O}_Y)$ is coherent on Z with support on a proper closed subscheme for $j > 0$, so $n \mapsto \chi(\mathrm{R}^j\pi_*(\mathscr{O}_Y) \otimes \mathscr{L}^{\otimes n})$ belongs to $\mathbf{Q}[n]$ with degree at most $\dim Z$ for $j > 0$. Since $\mathrm{R}^j\pi_*(\mathscr{O}_Y) \otimes \mathscr{L}^{\otimes n} = \mathrm{R}^j\pi_*(\pi^*\mathscr{L}^{\otimes n})$ for all $j \ge 0$, by the Leray spectral sequence

$$\mathrm{H}^i(Z, \mathrm{R}^j\pi_*(\cdot)) \Rightarrow \mathrm{H}^{i+j}(Y, \cdot)$$

applied to $\pi^*\mathscr{L}^{\otimes n}$ we have

$$\chi(Y, \pi^*\mathscr{L}^{\otimes n}) = \sum_j (-1)^j \chi(Z, \mathrm{R}^j\pi_*(\mathscr{O}_Y) \otimes \mathscr{L}^{\otimes n})$$

with the left side and all terms for $j > 0$ on the right side belonging to $\mathbf{Q}[n]$ with degree at most $\dim Z = \dim Y$. (We have used the settled projective case to handle the left side.) Thus, the term in this final sum for $j = 0$ satisfies the same property, and that is exactly what we needed. ■

Let's now return to our initial setup with the proper finitely presented $Y \to S$ equipped with an S-ample line bundle $\mathscr{L}$, and a T-flat finitely presented closed subscheme $Z \subset Y_T$ for an S-scheme T. Note in particular that $\mathscr{L}|_Z$ is T-flat. Thus, for $n \in \mathbf{Z}$ the function $t \mapsto \chi((\mathscr{L}|_Z)_t^{\otimes n})$ is locally constant on T. But this function of n is a polynomial of degree at most $\dim Z_t$, so it is determined by its values at the $1 + \dim Z_t$ integers $0 \le n \le \dim Z_t$. Since $t \mapsto \dim Z_t$ is locally constant on T (as $Z \to T$ is proper, flat, and finitely presented), for each $\Phi \in \mathbf{Q}[n]$ the pairwise disjoint sets

$$T_\Phi = \{t \in T \mid \chi((\mathscr{L}|_Z)_t^{\otimes n}) = \Phi(n) \text{ for all } n \in \mathbf{Z}\}$$

are all open (maybe likely empty!), so all are open and closed; note that T_Φ depends on the specification of $Z \subset Y_T$, despite the notation.

If we define $\underline{\mathrm{Hilb}}_{Y/S}^{\Phi}(T) = \{Z \in \underline{\mathrm{Hilb}}_{Y/S}(T) \mid \chi((\mathscr{L}|_Z)_t^{\otimes n} = \Phi(n) \text{ for all } n \in \mathbf{Z}, t \in T\}$ then when $\underline{\mathrm{Hilb}}_{Y/S}$ is represented by some universal pair $(\mathscr{H}, \mathscr{Z})$ it follows easily (check!) that each clopen subscheme $\mathscr{H}_\Phi$ (defined via $\mathscr{Z} \subset Y_{\mathscr{H}}$) equipped with the restriction $\mathscr{Z}|_{\mathscr{H}_\Phi}$ represents $\underline{\mathrm{Hilb}}_{Y/S}^{\Phi}$. More importantly, the general clopen decomposition $\coprod_\Phi T_\Phi = T$ attached to any $Z \in \underline{\mathrm{Hilb}}_{Y/S}(T)$ implies (check!) that if there exist representing pairs $(\mathscr{H}_\Phi, \mathscr{Z}_\Phi)$ for $\underline{\mathrm{Hilb}}_{Y/S}^{\Phi}$ for all Φ (some such $\mathscr{H}_\Phi$ may be empty!) then

$$(\coprod_\Phi \mathscr{H}_\Phi, \coprod_\Phi \mathscr{Z}_\Phi)$$

represents $\underline{\mathrm{Hilb}}_{Y/S}$.

Remark 1.4. Even for the concrete case $\underline{\mathrm{Hilb}}_{\mathbf{P}^N_{\mathbf{Z}}/\mathbf{Z}}$, it is a rather non-trivial matter to determine for which Φ the subfunctor $\underline{\mathrm{Hilb}}^{\Phi}_{\mathbf{P}^N_{\mathbf{Z}}/\mathbf{Z}}$ is non-empty (equivalently, has a point valued in some algebraically closed field). Of course, a necessary condition is that Φ is a "numeric polynomial", meaning that $\Phi(n) \in \mathbf{Z}$ for all $n \in \mathbf{Z}$; these are precisely the $\mathbf{Z}$-linear combinations of the polynomials $n \mapsto \binom{n}{j}$ for $j \geq 0$.

The real theorem on this matter is:

Theorem 1.5 (Grothendieck)**.** *For any proper finitely presented $Y \to S$ and S-ample $\mathscr{L}$ on Y, each $\underline{\mathrm{Hilb}}^{\Phi}_{Y/S}$ is represented by a proper and finitely presented S-scheme $\mathrm{Hilb}^{\Phi}_{Y/S}$ that is equipped with an S-ample line bundle. Hence, $\underline{\mathrm{Hilb}}_{Y/S}$ is represented by the scheme $\coprod_{\Phi} \mathrm{Hilb}^{\Phi}_{Y/S}$.*

The schemes $\mathrm{Hilb}^{\Phi}_{Y/S}$ are constructed as locally closed subschemes of Grassmannians Zariski-locally on S (then shown to be closed subschemes of such via the valuative criterion and not via the initial construction). A basic technical role for the T-flatness hypothesis on Z is that for all $T' \to T$ the natural surjection of $\mathscr{O}_{Y_{T'}}$-modules

$$(\mathscr{I}_Z)_{T'} \to \mathscr{I}_{Z_{T'}}$$

is an isomorphism (since $0 \to \mathscr{I}_Z \to \mathscr{O}_{Y_T} \to \mathscr{O}_Z \to 0$ retains short-exactness after any base change on T due to T-flatness of $\mathscr{O}_Z$).

As motivation for some considerations with marked stable curves, it is instructive to consider a refinement of the Hilbert functor:

Definition 1.6. Define $\underline{\mathrm{Hilb}}^{\mathrm{sec}}_{Y/S}$ to be the contravariant functor that assigns to any S-scheme T the set of pairs (Z, σ) for $Z \in \underline{\mathrm{Hilb}}_{Y/S}(T)$ and $\sigma \in Z(T)$.

There is an evident forgetful map $\underline{\mathrm{Hilb}}^{\mathrm{sec}}_{Y/S} \to \underline{\mathrm{Hilb}}_{Y/S}$ defined by $(Z, \sigma) \mapsto Z$. Thus, if $\underline{\mathrm{Hilb}}^{\mathrm{sec}}_{Y/S}$ is to be representable, then a representing object naturally lives over the Hilbert scheme $\mathrm{Hilb}_{Y/S}$ (when the latter exists). In fact, it is a very specific object over the Hilbert scheme: the universal flat family! More precisely:

Proposition 1.7. *Assume that $\underline{\mathrm{Hilb}}_{Y/S}$ is represented by a pair $(\mathscr{H}, \mathscr{Z})$. Then $\underline{\mathrm{Hilb}}^{\mathrm{sec}}_{Y/S}$ is represented by the S-scheme $\mathscr{Z}$.*

Proof. Consider any S-scheme T and pair $(Z, \sigma) \in \underline{\mathrm{Hilb}}^{\mathrm{sec}}_{Y/S}(T)$. The data of $Z \subset Y_T$ is classified by a unique S-map $f : T \to \mathscr{H}$, which is to say that $Z = \mathscr{Z} \times_{\mathscr{H}, f} T$ inside $Y_{\mathscr{H}} \times_{\mathscr{H}, f} T = Y_T$. Then the section $\sigma : T \to Z$ as a T-morphism amounts to a map $F : T \to \mathscr{Z}$ over the classifying S-map $f : T \to \mathscr{H}$ for Z. Clearly F is an S-map, and conversely given any S-map $F : T \to \mathscr{Z}$ we may define an S-map $f : T \to \mathscr{H}$ by composing F with the structure map $\mathscr{Z} \to \mathscr{H}$, with such f defining an element $Z \in \underline{\mathrm{Hilb}}_{Y/S}(T)$ via pullback of $\mathscr{Z}$ along f. In this way we see that specifying $(Z, \sigma) \in \underline{\mathrm{Hilb}}^{\mathrm{sec}}_{Y/S}(T)$ is *exactly* the same as specifying an S-map $F : T \to \mathscr{Z}$, and this is all functorial in T. ■

It is a very instructive exercise to bootstrap the preceding argument to show that for any $n > 0$, the n-fold fiber power

$$\mathscr{Z} \times_{\mathscr{H}} \mathscr{Z} \times_{\mathscr{H}} \cdots \times_{\mathscr{H}} \mathscr{Z}$$

represents the functor $\underline{\mathrm{Hilb}}^{\mathrm{sec},n}_{Y/S}$ assigning to any S-scheme T the set of data $(Z; \sigma_1, \dots, \sigma_n)$ for $Z \in \underline{\mathrm{Hilb}}_{Y/S}(T)$ and $\sigma_1, \dots, \sigma_n \in Z(T)$. (Note that the sections are given as an *ordered* n-tuple;

this corresponds to making maps into the fiber product that has n ordered copies of $\mathscr{Z}$.) The open complement of the closed union of the pr_{ij}-pullback of the diagonal $\mathscr{Z} \subset \mathscr{Z} \times_{\mathscr{H}} \mathscr{Z}$ for all $1 \leq i < j \leq n$ represents the subfunctor of $\underline{\mathrm{Hilb}}^{\mathrm{sec},n}_{Y/S}$ corresponding to *pairwise disjoint* sections.

Example 1.8. Over a representing scheme $\mathscr{H}_n$ for $\underline{\mathrm{Hilb}}^{\mathrm{sec},n}_{Y/S}$, the condition that fibers be geometrically connected and semistable with (arithmetic) genus g and the sections σ_i be pairwise disjoint is represented by an open subscheme $\mathscr{U}_n \subset \mathscr{H}_n$ over which we have a "universal genus-g curve" $(\mathscr{C}; s_1, \dots, s_n)$ inside Y with n disjoint markings. Imposing the further condition on the functor that each σ_i lands inside the relative smooth locus for the given flat family of curves is represented by the open subscheme of $\mathscr{U}_n$ complementary to the union of the closed subsets $s_i^{-1}(\mathscr{C} - \mathscr{C}^{\mathrm{sm}})$.

2. Putting a stable marked curve into standard form

Let S be a scheme, and $f : X \to S$ a proper finitely presented flat map whose geometric fibers are semistable connected curves with arithmetic genus g. Let $\sigma_1, \dots, \sigma_n \in X^{\mathrm{sm}}(S)$ be pairwise disjoint sections supported in the smooth locus. Let $\mathscr{L}_{X/S} = \omega_{X/S}(\sum \sigma_i)$ be the canonically associated invertible sheaf with fiber-degree $2g-2+n$ which we assume to be *positive*. (Despite the notation, $\mathscr{L}_{X/S}$ depends on the σ_i's and not just on X over S.) We noted in class (building on arguments of Deligne–Mumford and Knudsen) that the geometric fibers are stable if and only if $\mathscr{L}_{X/S}$ is ample on all fibers X_s, and we now assume that to be the case.

By adapting cohomological vanishing arguments in the proof of Theorem 1.2 of the paper of Deligne–Mumford (for $n = 0$ and $g \geq 2$), if $m \geq 2$ then $\mathrm{H}^1(X_s, \mathscr{L}^{\otimes m}_{X_s/s})$ vanishes for $m \geq 2$ and $\mathscr{L}^{\otimes m}_{X/S}$ is fiberwise very ample for $m \geq 4$. (The reason we require $m \geq 4$ for fibral very ampleness whereas Deligne–Mumford only require $m \geq 3$ is to handle certain degenerate cases with irreducible geometric fibers when $g \leq 1$.) The polynomial function

$$m \mapsto \chi(\mathscr{L}^{\otimes m}_{X_s/s})$$

has degree 1 with linear coefficient equal to the degree $2g-2+n$ and its value at $m = 0$ is $\chi(\mathscr{O}_{X_s}) = 1 - g$, so

$$\chi(\mathscr{L}^{\otimes m}_{X_s/s}) = (2g-2+n)m + (1-g).$$

Hence, if $m \geq 2$ then $h^0(X_s, \mathscr{L}^{\otimes m}_{X_s/s}) = (2g-2+n)m + (1-g)$.

Defining $\Phi_{g,n}(t) = (2g-2+n)t + (1-g) \in \mathbf{Q}[t]$, by Grothendieck's base change theorems for coherent cohomology we see that if $m \geq 2$ then $\mathscr{E}_m := f_*(\mathscr{L}^{\otimes m}_{X/S})$ is a vector bundle on S whose formation commutes with any base change and has fiber-rank $\Phi_{g,n}(m)$. Moreover, by the same base-change methods, the fibral very ampleness for $m \geq 4$ implies that the natural map $f^*(\mathscr{E}_m) \to \mathscr{L}^{\otimes m}_{X/S}$ is *surjective*. This quotient map is classified by an S-morphism

$$j_m : X \to \mathbf{P}(\mathscr{E}_m)$$

to a projective-space bundle, and the formation of j_m commutes with any base change on S. Thus, if $m \geq 4$ then j_m is a closed immersion on fibers over S, so the proper S-morphism j_m is quasi-finite and hence finite (by Zariski's Main Theorem). Thus, we can apply Nakayama's Lemma to deduce that j_m is a closed immersion.

Now fix $m = 4$ for specificity, and let $j = j_4$ and $\mathscr{E} = \mathscr{E}_4$ with rank

$$N := N(g,n) = 4(2g-2+n) + (1-g);$$

this is a closed immersion of X into the $\mathbf{P}^{N-1}$-bundle $\mathbf{P}(\mathscr{E})$ with fibers having Hilbert polynomial $\Phi(t) = \Phi_{g,n}(4t)$. In particular, if we Zariski-localize on S then we can also *choose* a trivialization

$$\mathscr{O}_S^{\oplus N} \simeq \mathscr{E}$$

and thereby identify j with a closed immersion $X \hookrightarrow \mathbf{P}^{N-1}$. Actually, we can get by with something *slightly less*: we can instead Zariski-localize so that there is merely an isomorphism $\varphi : \mathbf{P}(\mathscr{E}) \simeq \mathbf{P}^{N-1}$ of S-schemes (not necessarily arising from an isomorphism φ of vector bundles: $\mathrm{GL}_N(A) \to \mathrm{PGL}_N(A)$ is not surjective for non-local rings A in general!). Our aim in this handout is to reverse the entire process, by recovering a "universal" such stable n-pointed curve equipped with an appropriate projective embedding.

3. Hilbert scheme considerations

In the above setup, (X, φ) gives us a map

$$h : S \to \mathrm{Hilb}^{\Phi}_{\mathbf{P}^{N-1}_{\mathbf{Z}}/\mathbf{Z}}$$

which classifies the closed immersion $j : X \hookrightarrow \mathbf{P}^{N-1}$. Let $U \subset \mathrm{Hilb}^{\Phi}_{\mathbf{P}^{N-1}_{\mathbf{Z}}/\mathbf{Z}}$ be the open subscheme over which the universal family has fibers that are connected semi-stable curves with arithmetic genus g, and let $Z \to U$ be the universal such curve inside $\mathbf{P}^{N-1}_U$. The map h visibly factors through U, and the data of the ordered n-tuple of sections σ_i lifts $h : S \to U$ to a map $\widetilde{h} : S \to Z^n$ (fiber power over U). The locus in Z^n where the n sections over U are pairwise disjoint and supported inside Z^{sm} is an open subset $\Omega \subset Z^n$, and $\widetilde{h} : S \to Z^n$ lands inside Ω.

Over Ω there is a "universal semistable curve" C inside $\mathbf{P}^{N-1}_{\Omega}$, and as we discussed in class it is an open condition on Ω that $\mathscr{L}_{C/\Omega}$ has ample fibers. Thus, there is an open subscheme $H \subset \Omega$ over which there is a *universal* stable n-pointed genus-g curve $(C; \xi_1, \dots, \xi_n)$ *equipped with* the data of a specified closed immersion $\iota : C \hookrightarrow \mathbf{P}^{N-1}_H$. That is, $(C; \xi_1, \dots, \xi_n; i)$ represents the functor of stable n-pointed genus-g curves equipped with a closed immersion of the curve into $\mathbf{P}^{N-1}$. Letting $f : C \to H$ be the universal structure morphism, the vector bundle $\mathscr{F} := f_*(\mathscr{L}_{C/H}^{\otimes 4})$ of rank N has *nothing* to do with ι, and in particular $\mathscr{F}$ might not admit a trivialization over H.

Consider the Isom-scheme $I = \mathrm{Isom}(\mathbf{P}(\mathscr{F}), \mathbf{P}^{N-1})$ over H. This is a Zariski-torsor for the automorphism scheme PGL_N of $\mathbf{P}^{N-1}$ (as we see by working Zariski-locally over H where $\mathscr{F}$ admits a trivialization), and by the very definition of I the pullback $\mathbf{P}(\mathscr{F})_I$ over I is equipped with a tautological isomorphism φ to $\mathbf{P}^{N-1}$. Hence, by composing the I-pullback of $j_4 : C \hookrightarrow \mathbf{P}(\mathscr{F})$ with φ, C_I is equipped with a canonical closed immersion $\iota' : C_I \hookrightarrow \mathbf{P}^{N-1}_I$. But this closed immersion has nothing to do with ι_I!

Finally, consider the condition on an I-scheme T that on the pullback C_T, the two closed immersions $C_T \to \mathbf{P}^{N-1}_T$ arising from ι_I and ι' coincide. If this condition is represented by a closed subscheme $B \subset I$ (i.e., $T \to I$ factors through B if and only if the pullbacks of ι_I and ι' over T coincide) then B represents the functor assigning to any scheme S the set of isomorphism classes of stable n-pointed genus-g curves $(X; \sigma_1, \dots, \sigma_n)$ *equipped with* an isomorphism $\varphi : \mathbf{P}(f_*(\mathscr{L}_{X/S}^{\otimes 4})) \simeq \mathbf{P}^{N-1}_S$ where $f : X \to S$ is the structure morphism and $N = N(g, n)$ as above. Here, by "isomorphism" between such data

$$(X; \sigma_1, \dots, \sigma_n, \varphi), \quad (X', \sigma'_1, \dots, \sigma'_n, \varphi')$$

we mean an isomorphism

$$\theta : (X; \sigma_1, \dots, \sigma_n) \simeq (X', \sigma'_1, \dots, \sigma'_n)$$

such that the *induced isomorphism*

$$f_*(\mathscr{L}_{X/S}^{\otimes 4}) \simeq f'_*(\mathscr{L}_{X'/S}^{\otimes 4})$$

has projectivization which carries φ to φ'. Note the crucial fact such combined data have *no non-trivial automorphisms* (so the task of determining an isomorphism class is actually *Zariski-local* on the base). To see this latter rigidity property, consider an automorphism α of $(X; \sigma_1, \ldots, \sigma_n)$ whose effect on $\mathbf{P} := \mathbf{P}(f_*(\mathscr{L}_{X/S}^{\otimes 4}))$ intertwines the isomorphism φ with itself. The effect of α on $\mathbf{P}$ must then be the identity map, yet its effect on $\mathbf{P}$ recovers α on $X \subset \mathbf{P}$ due to the *canonicity* of the inclusion of X into $\mathbf{P}$ (depending on the ordered n-tuples of σ_i's that is respected by α).

We have now nearly proved the following theorem:

Theorem 3.1. *Fix $n, g \geq 0$ such that $2g - 2 + n > 0$. Let $N = N(g, n)$ and $\Phi(t) = \Phi_{g,n}(4t)$ as defined above. The functor of isomorphism classes of stable n-pointed genus-g curves $(X; \sigma_1, \ldots, \sigma_n)$ equipped with an isomorphism (or "trivialization")*

$$\varphi : \mathbf{P}(f_*(\mathscr{L}_{X/S}^{\otimes 4})) \simeq \mathbf{P}_S^{N-1}$$

is represented by a locally closed subscheme $\mathscr{M}_{g,n}^{\mathrm{triv}}$ inside a PGL_N-bundle over an open subscheme of the n-fold fiber power of the universal flat family over the Hilbert scheme $\mathrm{Hilb}^{\Phi}_{\mathbf{P}_{\mathbf{Z}}^{N-1}/\mathbf{Z}}$.

It can be shown that the natural map from $\mathscr{M}_{g,n}^{\mathrm{triv}}$ to the n-fold fiber power of the universal flat family over the Hilbert scheme is a locally closed immersion. We will not need this.

Proof. It remains to prove a general fact that has nothing to do with stable curves: if X, Y are closed subschemes of a projective space over a scheme T (in the case of interest Y is itself a projective space over T) with X flat over T, and if $f, h : X \rightrightarrows Y$ are two T-morphisms, then the condition on a T-scheme T' that $f_{T'} = h_{T'}$ is represented by a finitely presented closed subscheme of T.

One of the standard applications of the existence of Hilbert schemes is the existence of Hom-schemes by viewing a morphism $U \to V$ between finitely presented projective schemes over a base B as a graph inside $U \times_B V$, which is to say a finitely presented closed subscheme of $U \times_B V$ on which pr_1 is an isomorphism. Such a graph is flat over B when U is flat over B, so the place to look for a Hom-scheme is inside $\mathrm{Hilb}_{(U \times_B V)/B}$. More specifically, as is explained in Chapter 5 on Hilbert schemes in the book "FGA Explained", the functor $T' \mapsto \mathrm{Hom}_{T'}(X_{T'}, Y_{T'})$ is represented by a countable disjoint union H of finitely presented quasi-projective T-schemes (over which there is a "universal morphism" $X_H \to Y_H$). In particular, H is T-separated and the maps f and h correspond to a pair of T-morphisms $T \rightrightarrows H$, which is to say a section $j : T \to H \times_T H$ (which is a closed immersion since H is T-separated). The condition of equality $f_{T'} = h_{T'}$ after base change along a map $k : T' \to T$ is precisely the condition that $j \circ k : T' \to H \times_T H$ factors through the diagonal, which is to say that k factors through the closed subscheme $j^{-1}(\Delta_{H/T})$. Hence, this latter closed subscheme of T does the job. ■

The "difference" between the moduli scheme built in the above theorem and the dream of a universal stable n-pointed genus-g curve is the specification of φ. To remove φ from the picture, first note that there is a natural action of PGL_N on $\mathscr{M}_{g,n}^{\mathrm{triv}}$ by composing φ with the PGL_N-action on $\mathbf{P}^{N-1}$. But this action is not a *free* action because two φ's for the same stable n-pointed genus-g curve can be related through the action of a non-trivial automorphism of the stable pointed curve. In other words, it is exactly the presence of nontrivial (though finite!) automorphism groups for some stable curves which prevents the action from being free.

The work of Artin on moduli problems provides an appropriate notion of "quotient stack" $\mathscr{M}_{g,n}^{\mathrm{triv}}/\mathrm{PGL}_N$ that is a-priori an Artin stack. Due to the properties of automorphism schemes of stable n-pointed genus-g curves over algebraically closed fields as recorded in class, Artin's work can be used to show that this stack is even Deligne-Mumford (informally, this means that the stack admits an étale cover by a scheme, a property that Deligne and Mumford proved directly in their paper via "slicing arguments" since at that time the notion of Artin stack didn't exist and the notion of DM stack was only then being introduced).

We will sketch in class why this Deligne–Mumford stack is smooth and proper over $\mathbf{Z}$. Were the automorphism groups at geometric points actually trivial then it could be further concluded that this DM stack is an algebraic space, and there are useful sufficient criteria to prove that an algebraic space is a scheme. However, it is not an algebraic space due to the presence of non-trivial (though étale) automorphism schemes at geometric points, and in particular it is not a scheme.