Math 249C. The $*$-action, based root datum, and automorphism schemes

1. Motivation

Let G be a connected semisimple (not just reductive) group over a field k, and let S be a maximal split k-torus and P a minimal parabolic k-subgroup of G containing S. Let T be a maximal k-torus of P containing S (so T is also a maximal k-torus of G, since P is parabolic in G). Define the notation

$$_k\Phi = \Phi(G, S),\ \ _k\Phi^+ = \Phi(P, S),\ \ \Phi = \Phi(G_{k_s}, T_{k_s}).$$

Choose a Borel k_s-subgroup $B \subset P_{k_s}$ containing T_{k_s} (so $B = P_{k_s}$ if G is quasi-split over k). This amounts to choosing a positive system of roots $\Phi^+ = \Phi(B, T_{k_s})$ for Φ contained inside the parabolic set of roots $\Phi(P_{k_s}, T_{k_s})$ in Φ. We define Δ to be the basis of Φ^+ (so its elements correspond to the nodes of the Dynkin diagram obtained from (G_{k_s}, T_{k_s}, B)), and Δ_0 denotes the set of $a \in \Delta$ for which the restriction $a|_{S_{k_s}} \in \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ is trivial. Let $_k\Delta \subset \mathrm{X}(S) - \{0\} = \mathrm{X}(S_{k_s}) - \{0\}$ denote the restriction of $\Delta - \Delta_0$ along the inclusion $S_{k_s} \hookrightarrow T_{k_s}$, so restriction defines a map

$$\Delta \to {_k\Delta} \cup \{0\}.$$

In class, we defined an action of $\Gamma = \mathrm{Gal}(k_s/k)$ on the set Δ, called the "$*$-action", as follows. There is an evident Γ-action on Φ. For each $\gamma \in \Gamma$, $\gamma(\Phi^+)$ is a positive system of roots for Φ, so there is a unique $w_\gamma \in W(G_{k_s}, T_{k_s})$ such that $w_\gamma(\gamma(\Phi^+)) = \Phi^+$. Considering minimal elements of these positive systems of roots, we see that $w_\gamma(\gamma(\Delta)) = \Delta$. We saw in class that

$$w_{\gamma'\gamma} = w_{\gamma'}\gamma'(w_\gamma),$$

so $\Gamma \times \Delta \to \Delta$ defined by

$$(\gamma, a) \mapsto \gamma * a := w_\gamma(\gamma(a))$$

is an action of Γ on the set Δ. This is visibly continuous, since the action factors through $\mathrm{Gal}(K/k)$ for a finite Galois extension K/k inside k_s that splits T and over which representatives in $N_G(T)(k_s)$ for the elements of the finite group $W = W(G_{k_s}, T_{k_s})$ are defined.

Example 1.1. As was noted in class, if G is quasi-split (i.e., P is a Borel k-subgroup of G) then $w_\gamma = 1$ for all γ. Thus, in the quasi-split case the $*$-action is induced by the natural Γ-action on Φ. The converse is true too: if the $*$-action is induced by the Γ-action on Φ then G is quasi-split.

Indeed, since any nontrivial element of $W(G_{k_s}, T_{k_s})$ moves Φ^+ to another positive system of roots, any two of which are disjoint from each other, in such a situation necessarily $w_\gamma = 1$ for all γ. Thus, $\Phi^+ = w_\gamma(\gamma(\Phi^+)) = \gamma(\Phi^+)$, which is to say that $\Phi(B, T_{k_s})$ is Γ-stable inside Φ. But any parabolic k_s-subgroup of G_{k_s} containing T_{k_s} (such as a Borel k_s-subgroup) is uniquely determined by its associated parabolic set of roots, so B is Γ-stable inside G_{k_s} and hence descends to a Borel k-subgroup of G.

The $*$-action on the set Δ respects a lot of structure, such as the data encoded in the Dynkin diagram (directed edges and edge multiplicities), and a bit more. To see this, note that by definition, the Γ-action on Φ is induced by the natural Γ-action on $\mathrm{X}(T_{k_s})$. This latter action and its $\mathbf{Z}$-dual permute the sets of absolute roots and coroots, respecting the evaluation pairing between them. The same holds for the action of the absolute Weyl group $W(G_{k_s}, T_{k_s})$. Thus, the $*$-action of any $\gamma \in \Gamma$ also respects these structures, and hence acts through not only an automorphism of the based root system (i.e., the root system equipped with a choice of positive system of roots, or equivalently a choice of basis) – which is to say an automorphism of the Dynkin diagram – but even an automorphism of the *based root datum* for (G_{k_s}, T_{k_s}) (i.e., the root datum equipped with a choice of positive system or roots, or equivalently a choice of basis).

We say that a subset of Δ is $*$-*stable* if it is stable for the above action of Γ on Δ. In this handout, our main aim is to prove two properties of this action:

Theorem 1.2. *The restriction map $\Delta \to {}_k\Delta \cup \{0\}$ has Γ-stable fibers, and for a subset $I \subset \Delta - \Delta_0$ the parabolic set $\Phi(P_{k_s}, T_{k_s}) \cup [I] = \Phi^+ \cup [\Delta_0 \coprod I] \subset \Phi$ is Γ-stable inside Φ if and only if the subset $I \subset \Delta - \Delta_0$ is $*$-stable.*

In the final section of this handout, we explain a more conceptual perspective on the $*$-action that links it up with Galois cohomological considerations to be studied later.

2. Proof of Theorem 1.2

The key point is to show:

Lemma 2.1. *For any $\gamma \in \Gamma$, $w_\gamma \in N_{Z_G(S)}(T)(k_s)/T(k_s)$ inside $N_G(T)(k_s)/T(k_s)$.*

Proof. Let $U = \mathscr{R}_{u,k}(P)$, so $P = Z_G(S) \ltimes U$ due to the minimality of P. Thus, Borel k_s-subgroups of P necessarily contain U and so the set of these corresponds bijectively to the set of Borel k_s-subgroups of $P/U = Z_G(S)$ via "image" and "preimage". In particular, the set of Borel k_s-subgroups of P_{k_s} containing T_{k_s} is in bijective correspondence with the set of Borel k_s-subgroups of $Z_G(S)_{k_s}$ containing T_{k_s}. This bijection is visibly Γ-equivariant, and $W(Z_G(S)_{k_s}, T_{k_s})$ acts (simply) transitively on the set of Borel k_s-subgroups of $W(Z_G(S)_{k_s}, T_{k_s})$. Thus, for the purpose of choosing w_γ we can find a choice inside $N_{Z_G(S)}(T)(k_s)$. ■

Since $\mathrm{Lie}(Z_G(S)) = \mathrm{Lie}(G)^S$, an element of Φ occurs in $\mathrm{Lie}(Z_G(S))_{k_s}$ if and only if S_{k_s} i killed by that absolute root. In other words, the elements of Δ whose 1-dimensional weight space in $\mathrm{Lie}(G)_{k_s}$ occurs inside $\mathrm{Lie}(Z_G(S))_{k_s}$ are exactly the elements of Δ_0. As we noted in class, the minimality of P implies that for $\lambda \in \mathrm{X}_*(S)$ satisfying $P = P_G(\lambda)$, we have $Z_G(S) = Z_G(\lambda) = P_G(\lambda) \cap P_G(-\lambda)$, so

$$\Phi(Z_G(S)_{k_s}, T_{k_s}) = \Phi(P_{k_s}, T_{k_s}) \cap -\Phi(P_{k_s}, T_{k_s}) = [I]$$

for the unique $I \subset \Delta$ such that $\Phi(P_{k_s}, T_{k_s}) = \Phi^+ \cup [I]$. Thus,

$$I = \Delta \cap \Phi(P_{k_s}, T_{k_s}) \cap -\Phi(P_{k_s}, T_{k_s}) = \Delta \cap \Phi(Z_G(S)_{k_s}, T_{k_s}) = \Delta_0.$$

We conclude that $\Phi(Z_G(S)_{k_s}, T_{k_s}) = [\Delta_0]$. This says that Δ_0 is the basis of the positive system of roots for $(Z_G(S)_{k_s}, T_{k_s})$ associated to the Borel k_s-subgroup of $Z_G(S)_{k_s} = P_{k_s}/U_{k_s}$ whose preimage in P_{k_s} is the Borel k_s-subgroup $B \subset P_{k_s}$. Hence, the Weyl group $W(Z_G(S)_{k_s}, T_{k_s})$ is generated by the reflections r_a for $a \in \Delta_0$. In view of the Lemma above, we conclude that $w_\gamma = r_{a_1} \cdots r_{a_m}$ for a sequence $a_1, \dots, a_m \in \Delta_0$. For $a \in \Delta_0$ and $x \in \mathrm{X}(T_{k_s})$,

$$r_a(x) = x - \langle x, a^\vee \rangle a \in x + \mathbf{Z}\Delta_0.$$

Since moreover $r_a(\Delta_0) \in \mathbf{Z}\Delta_0$, it follows that $r_a(x+\mathbf{Z}\Delta_0) = x+\mathbf{Z}\Delta_0$. Thus, $w_\gamma(x) \in x+\mathbf{Z}\Delta_0$ for any such x, so $\gamma * a = w_\gamma(\gamma(a)) \in \gamma(a) + \mathbf{Z}\Delta_0$. Restricting to S_{k_s} kills Δ_0, so using the triviality of the natural Γ-action on $\mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ implies that

$$(\gamma * a)|_{S_{k_s}} = \gamma(a)|_{S_{k_s}} = a|_{S_{k_s}}.$$

This proves the Γ-stability of the fibers of $\Delta \to {}_k\Delta \cup \{0\}$.

It remains to show for $I \subset \Delta - \Delta_0$ that $\Phi(P_{k_s}, T_{k_s}) \cup [I]$ is Γ-stable inside Φ if and only if I is $*$-stable inside Δ. (Keep in mind that $\Phi(P_{k_s}, T_{k_s}) = \Phi^+ \cup [\Delta_0]$.) It suffices to show that under the natural Γ-action on Φ,

$$\gamma(\Phi(P_{k_s}, T_{k_s}) \cup [I]) = \Phi(P_{k_s}, T_{k_s}) \cup [\gamma * I]$$

for all $\gamma \in \Gamma$, where $\gamma * I$ denotes the image of I under the $*$-action of $\gamma \in \Gamma$. Clearly $\gamma(\Phi(P_{k_s}, T_{k_s})) = \Phi(P_{k_s}, T_{k_s})$, so

$$\gamma(\Phi(P_{k_s}, T_{k_s}) \cup [I]) = \Phi(P_{k_s}, T_{k_s}) \cup [\gamma(I)] = \Phi^+ \cup [\Delta_0 \cup \gamma(I)]$$

(where $[J]$ denotes the set of absolute roots that are $\mathbf{Z}$-linear combinations of a given set J of such roots). We want this to coincide with $\Phi(P_{k_s}, T_{k_s}) \cup [\gamma * I] = \Phi^+ \cup [\Delta_0 \cup \gamma * I]$. Thus, it suffices to show that the sets $[\Delta_0 \cup \gamma(I)]$ and $[\Delta_0 \cup \gamma * I]$ coincide. But $\gamma * I = w_\gamma(\gamma(I)) \subset \gamma(I) + \mathbf{Z}\Delta_0$, so we are done.

3. Another interpretation

The last item we address in this handout is a conceptual interpretation of the $*$-action of Γ on the based root datum. This requires a preliminary digression to discuss the automorphism functor of a connected semisimple group.

For any connected semisimple k-group H, there is a smooth affine *automorphism scheme* $\mathrm{Aut}_{H/k}$ that classifies automorphisms of H over arbitrary k-algebras. More specifically, this represents the functor $\underline{\mathrm{Aut}}_{H/k}$ that assigns to any k-algebra A the group of A-group automorphism $H_A \simeq H_A$. In other words, there is a "universal" $\mathrm{Aut}_{H/k}$-group scheme automorphism

$$F : H \times \mathrm{Aut}_{H/k} \simeq H \times \mathrm{Aut}_{H/k}$$

such that for any k-algebra A and A-group automorphism $f : H_A \simeq H_A$ there is a unique k-map $\mathrm{Spec}\, A \to \mathrm{Aut}_{H/k}$ along which the pullback of F is f. For example, if K is an extension field of k then $\mathrm{Aut}_{H/k}(K) = \mathrm{Aut}_{K\text{-gp}}(H_K)$ functorially in K.

The construction of k-group $\mathrm{Aut}_{H/k}$ involves Galois descent applied to a construction in the "split" case (which we will review shortly). There is a large part of the automorphism scheme that we can describe in concrete terms, as follows. The action of H on itself via conjugation makes Z_H act trivially on H, so it factors through an action of $H^{\mathrm{ad}} = H/Z_H$ on H. Hence, $H^{\mathrm{ad}}(k)$ naturally acts on the k-group H. Beware that $H^{\mathrm{ad}}(k)$ may be *larger* than $H(k)/Z_H(k)$, due to cohomological obstructions in $\mathrm{H}^1(k, Z_H)$ (fppf abelian cohomology if Z_H isn't smooth).

Example 3.1. For $H = \mathrm{SL}_n$, the group $H^{\mathrm{ad}} = \mathrm{SL}_n/\mu_n$ is identified with PGL_n. Thus, we get an action of $\mathrm{PGL}_n(k)$ on SL_n. Since $\mathrm{H}^1(k, \mathbf{G}_{\mathrm{m}}) = 1$ whereas $\mathrm{H}^1(k, \mu_n) = k^\times/(k^\times)^n \neq 1$ in general, $\mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^\times$ whereas $\mathrm{SL}_n(k)/\mu_n(k)$ is generally a proper normal subgroup of $\mathrm{PGL}_n(k)$ whose cokernel $k^\times/(k^\times)^n$ (which is often infinite!) encodes the determinant modulo nth powers. This action of $\mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^\times$ on SL_n is induced by the usual conjugation action of $\mathrm{GL}_n(k)$ on SL_n.

The action of H^{ad} on H corresponds to a map of fppf group sheaves $H^{\mathrm{ad}} \to \underline{\mathrm{Aut}}_{H/k}$. This map has trivial kernel: any point h' of H^{ad} valued in a k-algebra arises from a point h of H fppf-locally on the base, so h' vanishes in $\mathrm{Aut}_{H/k}$ precisely when h-conjugation is trivial, which is to say h is a point of Z_H, or in other words $h' = 1$ as desired. The subgroup functor H^{ad} inside $\underline{\mathrm{Aut}}_{H/k}$ is normal, by the same computation in ordinary group theory according to which the inner automorphism group of an abstract group is preserved under conjugation inside the full automorphism group.

We refer to points of the k-subgroup H^{ad} (valued in k-algebras or k-schemes) as the *inner automorphisms* of H. Beware that since $H(k) \to H^{\mathrm{ad}}(k)$ may fail to be surjective, there may be elements of $H^{\mathrm{ad}}(k)$ whose effect on $H(k)$ is *not* conjugation by any element of $H(k)$ (and so does not give an inner automorphism of $H(k)$ in the sense of abstract group theory). For example, SL_n has automorphisms arising from $\mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^\times$, and we call these "inner" automorphisms of SL_n but the diagonal elements $\mathrm{diag}(c, 1, 1, \ldots, 1) \bmod k^\times \in \mathrm{PGL}_n(k)$ act on SL_n in a manner that does *not* arise from any $\mathrm{SL}_n(k)$-conjugation when $c \neq 1$.

Theorem 3.2. *The functor $\underline{\mathrm{Aut}}_{H/k}$ is represented by a smooth affine k-group with identity component H^{ad}.*

Proof. Since the automorphism functor is a sheaf for the étale topology (even the fpqc topology), and étale descent (even fpqc descent) is effect for affine schemes, it suffices for the proof of representability to first apply a finite separable extension of k. The same goes for showing that H^{ad} is the identity component of the representing object. Hence, we may and do assume $H = H_0$ is a split connected semisimple k-group, say with a split maximal k-torus S.

As a warm-up, we explain the structure of the group $\mathrm{Aut}(H_0)$ of k-valued points of the automorphism functor, modifying any k-group automorphism of H_0 by an element of $H_0^{\mathrm{ad}}(k)$ so all that remains is an automorphism of the based root datum.

Since S is the preimage of its split maximal k-torus image $S^{\mathrm{ad}} = S/Z_{H_0}$ in H_0^{ad}, and $H_0^{\mathrm{ad}}(k)$ acts transitively on the set of split maximal k-tori of H_0^{ad}, for any k-automorphism f of H_0 we may compose f with the action of a suitable element of $H_0^{\mathrm{ad}}(k)$ to arrange that f preserves S. Choose a Borel k-subgroup B_0 containing S. This is also the preimage of its Borel k-subgroup image $B_0^{\mathrm{ad}} = B_0/Z_{H_0}$ in H_0^{ad}. Since $H_0^{\mathrm{ad}}(k)$ acts transitively on the set of *pairs* (B_0, S) consisting of a Borel k-subgroup and a split maximal k-torus contained inside that Borel subgroup, by adjusting our $H_0^{\mathrm{ad}}(k)$-modification of f we can even arrange that f preserves the pair (B_0, S) inside H_0.

Let Δ be the basis of Φ associated to $\Phi^+ := \Phi(B, S)$. The root groups of (H_0, S) map isomorphically onto those of (H_0^{ad}, S). More specifically, $S \ltimes \prod_{a \in \Phi^+} U_a \simeq B_0$ via multiplication (for any fixed choice of enumeration of Φ^+), and likewise $S^{\mathrm{ad}} \ltimes \prod_{a \in \Phi^+} U_a \simeq B_0^{\mathrm{ad}}$. The "adjoint torus" S^{ad} has character group with basis Δ, and more specifically $S^{\mathrm{ad}} \simeq \prod_{a \in \Delta} \mathbf{G}_{\mathrm{m}}$ via $s \mapsto (a(s))_{a \in \Delta}$. Since $\mathrm{X}(f)$ preserves Φ^+, it must permute the set Δ of minimal elements. If σ_f denotes this permutation, then on the root groups associated to these simple positive roots the effect of f must be determined by isomorphisms $U_a \simeq U_{\sigma_f(a)}$ for $a \in \Delta$. Thus, if we *choose* an isomorphism $U_a \simeq \mathbf{G}_{\mathrm{a}}$ for each such a, or equivalently choose a basis of $\mathrm{Lie}(U_a) = \mathfrak{g}_a$ (this is called a *pinning* of (H_0, B_0, S)) then the isomorphisms $U_a \simeq U_{\sigma_f(a)}$ are identified with automorphisms of $\mathbf{G}_{\mathrm{a}}$, which is to say $k^\times$-scalings.

Consequently, if we adjust f further by a suitable element of $S^{\mathrm{ad}}(k)$ then we can cancel out the effect of those multipliers and leave only the combinatorial data of the permutation σ_f, which "is" an automorphism of $\mathrm{X}(S)$ that preserves Φ^+ and whose dual preserves the associated positive system of coroots. This automorphism respects pairings among simple positive roots and coroots, so it is an automorphism of the based root datum.

Let E_0 be the automorphism group of the based root datum (viewed as a subgroup of the automorphism group of the Dynkin diagram). In the simply connected or adjoint cases, E_0 coincides with the group of all diagram automorphisms because the $\mathbf{Z}$-structure is determined by the root system: $\mathrm{X}(S) = \mathbf{Z}\Phi$ in the adjoint case and $\mathrm{X}_*(S) = \mathbf{Z}\Phi^\vee$ in the simply connected case. In general the based root datum has a smaller automorphism group than does the Dynkin diagram.

The preceding arguments provide a left exact sequence of groups

$$1 \to H_0^{\mathrm{ad}}(k) \to \mathrm{Aut}(H_0) \to E_0.$$

For each $\sigma \in E_0$, the Isomorphism Theorem provides the existence of a unique k-automorphism f_σ of (H_0, B_0, S) which leaves the chosen pinning invariant and whose effect on the based root datum is σ. In this way, we get an isomorphism of abstract groups

$$H_0^{\mathrm{ad}}(k) \rtimes E_0 \simeq \mathrm{Aut}(H_0)$$

depending on the choice of (B_0, S) and pinning. The preceding calculations work without change over any extension K of k, using $((B_0)_K, S_K)$ and the pinning over K that is the base change of the chosen one over k. This provides isomorphisms

$$H_0^{\mathrm{ad}}(K) \rtimes E_0 \simeq \mathrm{Aut}_K((H_0)_K)$$

functorally in K, and thereby motivates the expectation that the k-group $H_0^{\mathrm{ad}} \rtimes E_0$ should represent the automorphism functor $\underline{\mathrm{Aut}}_{H_0/k}$ of H_0.

More specifically, by using normal subgroup functor inclusion of H_0^{ad} into $\underline{\mathrm{Aut}}_{H_0/k}$ and the inclusion $E_0 \hookrightarrow \mathrm{Aut}(H_0)$ specified above (depending on (B_0, S) and the pinning), we get a map of group functors

$$H_0^{\mathrm{ad}} \rtimes E_0 \to \underline{\mathrm{Aut}}_{H_0/k}$$

(where the semidirect product structure uses the normality of H_0^{ad} inside $\underline{\mathrm{Aut}}_{H_0/k}$). This has been shown to be bijective on field-valued points over k, and in general is a subgroup functor inclusion (as it suffices to check this on geometric points). Thus, for a given k-algebra A and $f \in \underline{\mathrm{Aut}}_{H_0/k}(A) = \mathrm{Aut}_A((H_0)_A)$, to show f arises from an A-point of $H_0^{\mathrm{ad}} \rtimes E_0$ it suffices to do so étale-locally on $\mathrm{Spec}\, A$.

A version of the Isomorphism Theorem and *étale-local* conjugacy theorems for Borel subgroups and maximal tori in reductive group schemes over rings (not just over fields) makes it possible to push through the above arguments étale-locally on any k-algebra, not just over fields. This ensures that our map of group functors is an isomorphism on points valued in all k-algebras, not just fields. By construction, H_0^{ad} is the identity component of $\mathrm{Aut}_{H_0/k}$. ■

In general we have a short exact sequence of k-groups

$$1 \to H^{\mathrm{ad}} \to \mathrm{Aut}_{H/k} \to E \to 1$$

for a finite étale k-group E. This need not split in general (e.g., the quotient map can fail to be surjective on k-points), but when $H = H_0$ is split then we saw in the preceding argument that E is constant and there *is* a canonical semi-direct product splitting upon choosing (B_0, S) and a pinning. Any two choices of pinning for a given (B_0, S) are related through the action of $S^{\mathrm{ad}}(k) = (S/Z_{H_0})(k)$, so in the split case the identification of $E(k)$ with the automorphism group of the based root datum is intrinsic to the triple (H_0, B_0, S).

Now we bring in the link with the $*$-action. Returning to our original connected semisimple k-group G, let G_0 be the split k-form of G (in other words, the unique split connected semisimple k-group whose root datum is the same as that of G_{k_s}). More specifically, if T_0 is a split maximal k-torus of G_0 then (G_0, T_0) is a k-form of (G, T) (they become isomorphic over k_s). Consider the exact sequence

$$1 \to G_0^{\mathrm{ad}} \to \mathrm{Aut}_{G_0/k} \to E \to 1.$$

We have seen that E is a constant k-group, and $E(k)$ is identified with the automorphism group of the based root datum associated to a choice of Borel k-subgroup B_0 of G_0 containing T_0. The set of bases of the root datum is a principal homogeneous space for $W(G_0, T_0)$, and the set of choices of B_0 containing T_0 is compatibly a principal homogeneous space for $W(G_0, T_0)$. The simply transitive Weyl group action canonically identifies the automorphism groups of the based root data arising from all choices of basis. If we change the choice of B_0 containing T_0 then the effect is to identify $E(k)$ with the automorphism group of $R(G_0, T_0)$ equipped with another basis, and this respects the sense in which the automorphism groups of the based root data arising from all choices of basis are compatibly identified with each other (for a fixed choice of T_0). Moreover, all choices of T_0 are permuted by the action of $H_0^{\mathrm{ad}}(k)$. Hence, the identification between $E(k)$ and the automorphism group of the based root datum is *canonical* (independent of (B_0, T_0)).

As we shall see in our later discussion of Galois cohomology, the isomorphism class of G as a k-group is classified by an element $[G] \in \mathrm{H}^1(k, \mathrm{Aut}_{G_0/k})$. This class has an image in $\mathrm{H}^1(k, E)$. Since E is a *constant* k-group, this latter cohomology set is identified with the quotient set $E(k) \backslash \mathrm{Hom}(\Gamma, E(k))$ of the pointed set of continuous homomorphisms $\Gamma = \mathrm{Gal}(k_s/k) \to E(k)$ modulo the effect of $E(k)$-conjugation. Hence, $[G]$ gives rise to a conjugacy class of continuous homomorphisms $\Gamma \to E(k)$.

Since $(G_0, T_0, B_0)_{k_s} \simeq (G_{k_s}, T_{k_s}, B)$, so we have $R(G_0, T_0) \simeq R(G_{k_s}, T_{k_s})$ respecting choices of a basis of each, but this isomorphism of based root data is not canonically attached to G alone, the k-isomorphism class of G gives rise to an intrinsic conjugacy class of continuous homomorphisms from Γ into the automorphism group of the based root datum. What could this conjugacy class be? It is the class of the $*$-action! This is left to the reader as an instructive exercise with the definition of the $*$-action.

The upshot is that the $*$-action records precisely the information of the finite étale component group of the automorphism scheme of G. This is a description of the $*$-action that is intrinsic to G, without reference to T or $B \subset G_{k_s}$ containing T_{k_s} (or a pinning thereof).

Remark 3.3. There is another viewpoint one can take: a continuous Γ-action on a finite set is a finite étale k-scheme, so the $*$-action gives rise to a finite étale k-scheme whose set of k_s-points is identified with the set of nodes of the Dynkin diagram of (G_{k_s}, T_{k_s}, B). Note that the $*$-action preserves the structure of the diagram (directed edges and edge multiplicities), and this structure can be encoded in terms of (i) specifying a subset of $\Delta \times \Delta$ away from the diagonal (directed edges that are not loops) and (ii) a map from that subset to $\{1, 2, 3\}$ (edge multiplicity).

To summarize, the $*$-action defines a finite étale k-scheme $\mathrm{Dyn}(G)$ and a finite étale closed subscheme $\mathrm{DirEdge}(G) \subset \mathrm{Dyn}(G) \times \mathrm{Dyn}(G)$ disjoint from the diagonal along with a map from $\mathrm{DirEdge}(G)$ to the constant k-scheme $\{1, 2, 3\}$ (and an identification of this structure on k_s-points with the Dynkin diagram). Actually, the $*$-action is a bit finer, since it respects information related to the root datum and not just the root system (which is all that is "known" through the diagram).

In SGA3, Exp. XXIV, §3, the notion of the finite étale *scheme of Dynkin diagrams* is defined for semisimple group schemes over a general (non-empty) base scheme S. This is a finite étale S-scheme D equipped with a finite étale closed subscheme of $D \times D$ disjoint from the diagonal and a map from that closed subscheme to the constant scheme $\{1, 2, 3\}_S$ (satisfying some axioms which ensure it arises from an actual Dynkin diagram on geometric fibers). Working over the field k and applying this to G, we recover $\mathrm{Dyn}(G)$ with its additional structure built above via the $*$-action.