Math 249C. Reductive centralizer

1. Motivation

Let G be a connected reductive group over a field k. Let T be a maximal k-torus and let M be a closed k-subgroup scheme of T. (The case of interest to us is $M = \ker(a)$ for a nontrivial character $a : S \to \mathbf{G}_{\mathrm{m}}$ on a k-split subtorus $S \subseteq T$.) Consider the scheme-theoretic centralizer $Z_G(M)$. For example, if $g \in T(k)$ and M is the Zariski-closure of $g^{\mathbf{Z}}$ in T then M is a smooth (possibly disconnected) closed k-subgroup of T and $Z_G(M) = Z_G(g)$.

In general $Z_G(M)$ is smooth. Indeed, to check this we may assume $k = \overline{k}$, so M is a "split" group of multiplicative type, and then we can verify the infinitesimal smoothness criterion for $Z_G(M)$ by using the complete reducibility of k-linear representations of split multiplicative-type k-group schemes. This calculation is a special case of Proposition A.8.10(2) in "Pseudo-reductive groups", applied to the M-action on G via conjugation (the proof of the general case in A.8.10(2) simplifies a lot under our present hypotheses that the base is a field and the group G is affine, as the interested reader can check).

Note that the preceding smoothness argument applies to any closed k-subgroup scheme M of multiplicative type inside G *without* assuming that M occurs inside a maximal k-torus of G. It is a genuine constraint on M that it occurs inside a maximal k-torus:

Example 1.1. For $n \geq 3$, let q_n denote the standard "split" quadratic form ($x_1x_2+\cdots+x_{n-1}x_n$ for n even, and $x_0^2 + q_{n-1}(x_1, \ldots, x_{n-1})$ for n odd). Let G be the split connected semisimple group $\mathrm{SO}_n = \mathrm{SO}(q_n) \subset \mathrm{SL}_n$. Consider the k-subgroup

$$M' = \{(\zeta_1, \ldots, \zeta_n) \in \mu_2^n \mid \prod \zeta_j = 1\} \simeq \mu_2^{n-1}$$

inside G. The maximal tori of $G_{\overline{k}}$ have dimension $\lfloor n/2 \rfloor$, and so have 2-torsion equal to $\mu_2^{\lfloor n/2 \rfloor}$. Since $\lfloor n/2 \rfloor < n-1$ for $n \geq 3$, M' is not contained in any k-torus of G.

Remark 1.2. Although we don't require it, the special case that $\mathrm{char}(k) = p > 0$ and $M = \mu_p$ makes an appearance in the classical theory in the sense that for a nonzero element X in the line $\mathrm{Lie}(M) \subset \mathrm{Lie}(G)$, Proposition A.8.10(3) in "Pseudo-reductive groups" shows that the smooth closed k-subgroup $Z_G(M)$ equals the group denoted $Z_G(X)$ in the classical theory (see 9.1 in Borel's textbook on linear algebraic groups).

It is an important fact in the classical theory that $Z_G(M)^0$ is *reductive* when M is smooth with cyclic étale component group or when $M = \mu_p$ with $\mathrm{char}(k) = p > 0$. The former case immediately reduces to $Z_G(g)$ for $g \in T(k)$, and the latter case can be expressed in the form of $Z_G(X)$ as explained above. In Borel's textbook, the reductivity of $Z_G(M)^0$ for such M is proved in 13.19.

The goal of this handout is to carry out a generalization of the classical reductivity argument in our scheme-theoretic framework, proving that $Z_G(M)^0$ is reductive for *any* multiplicative type k-subgroup M of T. In the special case that M is smooth and connected, hence a torus, this is a ubiquitous fact in the theory of connected reductive groups. Our aim is to remove connectedness and smoothness hypotheses on M.

Remark 1.3. It is natural to wonder if the reductivity of $Z_G(M)^0$ requires that assumption (that we have seen need not always hold for multiplicative type subgroups of split connected semisimple groups) that M occurs inside a maximal torus of G. That is, if M is *any* closed k-subgroup scheme of multiplicative type inside G then is the smooth connected k-subgroup $Z_G(M)^0$ reductive? The answer is affirmative, but our technique of proof (which uses the structure of root groups relative to $\Phi(G_{k_s}, T_{k_s})$) is not applicable without the crutch of a maximal torus containing M.

Rather generally, consider any finite type affine k-group scheme H such that the representation theory of $H_{\overline{k}}$ is completely reducible. For any action by H on a connected reductive k-group G, the schematic centralizer G^H is smooth with reductive identity component. This result lies *much* deeper than the case "$H \subseteq T$ acting through conjugation" treated below, and a proof is given in Proposition A.8.12 in "Pseudo-reductive groups". The proof rests on a remarkable necessary and sufficient reductivity criterion for smooth connected k-subgroups G' of G independently due to Borel and Richardson: G' is reductive if and only if G/G' is affine. (Borel's proof rests on the general apparatus of étale cohomology, and Richardson's proof rests on the work of Haboush and Mumford in geometric invariant theory).

2. Reductivity

To prove the reductivity of $Z_G(M)^0$ we may and do assume $k = \overline{k}$. Suppose to the contrary that $U = \mathscr{R}_u(Z_G(M)^0)$ is nontrivial, so $\mathrm{Lie}(U)$ is a nonzero representation space for T through its adjoint action on the smooth connected group $Z_G(M)^0$. This representation space cannot support the trivial weight, since $\mathfrak{g}^T = \mathrm{Lie}(T)$ by reductivity of G and $\mathrm{Lie}(T) \cap \mathrm{Lie}(U) = \mathrm{Lie}(T \cap U) = 0$ (as $T \cap U$ is a multiplicative type subgroup scheme of the unipotent U, so it has to be trivial since $\mathfrak{G}_{\mathrm{a}}$ contains no nontrivial multiplicative type closed subgroup scheme). Thus, for some $a \in \Phi(G,T)$ the 1-dimensional weight space $\mathfrak{g}_a$ occurs inside $\mathrm{Lie}(U)$.

Let $H = Z_G(T_a \cdot M)^0$ where $T_a = (\ker a)^0_{\mathrm{red}}$, so H is smooth and connected inside $Z_G(M)^0$. In particular, $U \cap H$ is a normal subgroup scheme of H. Note that since T normalizes U (by working inside $Z_G(M)^0$ in which U is normal), the schematic centralizer U^{T_a} is smooth. But $U \cap H = U^{T_a}$ and this has Lie algebra $\mathrm{Lie}(U)^{T_a} \supseteq \mathfrak{g}_a \neq 0$, so $(U \cap H)^0$ is a nontrivial smooth connected unipotent subgroup of H that is normal. In other words, by replacing M with $T_a \cap M$ we may assume that $T_a \subseteq M$ without losing the hypothesis that H is not reductive.

But $H \subset Z_G(T_a)$ and $Z_G(T_a)$ is an almost direct product of the torus T_a and the rank-1 connected semisimple group $H' := \mathscr{D}(Z_G(T_a)) = \langle U_a, U_{-a} \rangle$ that is either SL_2 or PGL_2 and meets T in the diagonal torus D. Since $T_a \subseteq M$, by writing $T = T_a \cdot D$ we have $M = T_a \cdot \mu$ for $\mu = D \cap M$. Thus, $Z_G(M)^0 = T_a \cdot Z_{H'}(\mu)^0$ as an almost direct product of smooth connected k-groups, so the failure of reductivity for $Z_G(M)^0$ forces the failure for $Z_{H'}(\mu)^0$.

To get a contradiction, we're now reduced to checking for H' equal to either SL_2 or PGL_2 and any closed k-subgroup scheme μ of the diagonal $D = \mathbf{G}_{\mathrm{m}}$ that $Z_{H'}(\mu))^0$ is reductive. The cases $\mu = 1, D$ are trivial, so we can assume $\mu = \mu_n$ for some $n > 1$. Since $\mathrm{Lie}(Z_{H'}(\mu)^0) = \mathrm{Lie}(Z_{H'}(\mu)) = \mathrm{Lie}(H')^{\mu}$, if $H' = \mathrm{PGL}_2$ then $\mathrm{Lie}(H')^{\mu} = \mathrm{Lie}(D)$. Hence, in such cases the inclusion $D \subset Z_{H'}(\mu)^0$ between smooth connected groups is an equality on Lie algebras, so it is an equality of k-groups. Suppose instead that $H' = \mathrm{SL}_2$. If $\mu = \mu_2$ then $Z_{H'}(\mu) = H'$ and we are done, so we may assume $\mu = \mu_n$ with $n > 2$. Thus, squaring on μ_n is nontrivial, so it is easy to check that $\mathrm{Lie}(H')^{\mu} = \mathrm{Lie}(D)$, and hence once again $D = Z_{H'}(\mu)^0$ by Lie algebra considerations.