Notes from Brian Conrad's course on Linear Algebraic Groups at Stanford, Winter 2010

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June 22, 2017

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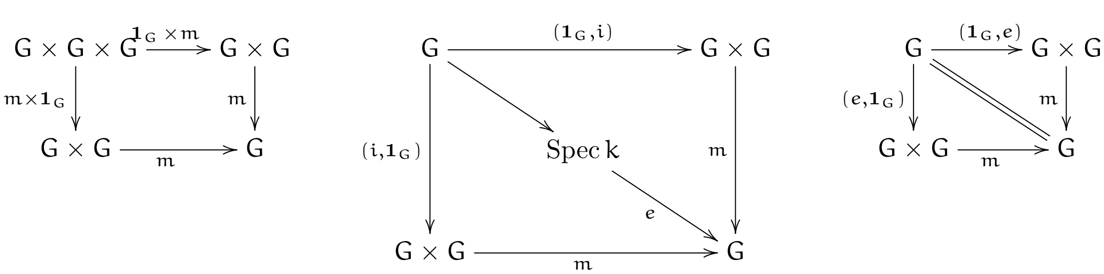
1.1 Some Definitions

Let k be a field.

Definition 1.1.1. A *group variety* over k is a smooth k-scheme G (possibly disconnected!) of finite type, equipped with k-morphisms $m : G \times G \to G, i : G \to G$, and a rational point $e \in G(k)$ satisfying the group axioms, in the sense that the following diagrams commute:



Remark 1.1.2. The rational point e can be regarded as a section of the structure map $G \to \operatorname{Spec} k$. In the third diagram, the morphisms $e : G \to G$ are the "constant maps to the identity", i.e. the compositions $G \to \operatorname{Spec} k \xrightarrow{e} G$.

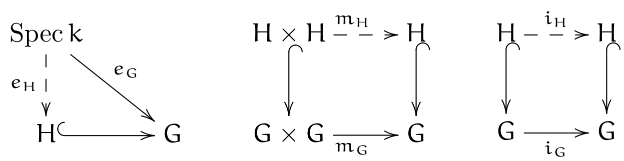
Remark 1.1.3. On Homework 1, it is shown that a connected group variety G over k is [*geometrically connected and*] geometrically irreducible. By some people's usage, this justifies the term "variety" in the name.

Definition 1.1.4. If we drop the smoothness condition in Definition 1.1.1, but keep everything else, then G is called an *algebraic* k*-group scheme*.

Definition 1.1.5. A group variety G over k is called *linear algebraic* if it is affine.

Remark 1.1.6. If G is an algebraic k-group scheme, then one can show that G is affine if and only if it is a k-subgroup scheme (cf. Definition 1.1.7) of GL_n for some n. (See Example 1.4.1 below for the definition of GL_n.) This is special to the case of fields in the sense that it is *not known* over more general rings (e.g., not even over the dual numbers over a field), though it is also true (and useful) over Dedekind domains by a variation on the argument used for fields.

Definition 1.1.7. Let G be a group variety over k, with corresponding multiplication map m_G, inversion map i_G, and identity section e_G. A k-*subgroup* $H \subset G$ is a closed subscheme such that there exist factorizations



By Yoneda's lemma, a k-group scheme is the same as a k-scheme G such that the Yoneda functor $h^G = \mathrm{Hom}_{k-\mathsf{Sch}}(-, G) : \mathsf{Sch}/k \to \mathsf{Sets}$ is equipped with a factorization through the forgetful functor $\mathsf{Groups} \to \mathsf{Sets}$. This is useful! Note that this is the same as the requirement that $G(R)$ is a group, functorially in R, for *all* k-algebras R, not just fields K/k.

By the same reasoning, a closed subscheme $H \subset G$ is a k-subgroup if and only if $H(R) \subset G(R)$ is a[n abstract] subgroup for all k-algebras R.

1.2 Smoothness

Let X be a k-scheme which is [locally] of finite type. Here is one definition of many for what it means for X to be smooth.

Definition 1.2.1. We say that X is *smooth* if and only if $X_{\overline{k}}$[1] is regular, meaning that all of the local rings are regular local rings, which can be checked via the Jacobian criterion.

Remark 1.2.2. For the purpose of checking the Jacobian criterion, one may freely go up and down between algebraically closed fields containing k. For example, one might care about both $X_{\overline{\mathbf{Q}}}$ and $X_{\mathbf{C}}$ in the case $k = \mathbf{Q}$.

Remark 1.2.3. If k happens to be perfect, then smoothness is the same as regularity.

Let G be a k-group scheme [of finite type]. The group $G(\overline{k})$ acts on $G_{\overline{k}}$ by translation. Now G is smooth if and only if the local rings of $G_{\overline{k}}$ are regular, and over an algebraically closed field it is enough to check smoothness at the classical points $G_{\overline{k}}(\overline{k}) = G(\overline{k})$. But by commutative algebra plus the aforementioned "homogeneity", it is thus enough to check that the completed local ring $\widehat{\mathcal{O}}_{G_{\overline{k}},e}$ is regular. On Homework 1, it is shown that it is equivalent to check that $\widehat{\mathcal{O}}_{G_{\overline{k}},e} \simeq \overline{k}[\![x_1, \ldots, x_n]\!]$ is a power series ring. In fact it is also shown there that it is equivalent to do this on the rational level; i.e. G is smooth if and only if $\widehat{\mathcal{O}}_{G,e} \simeq k[\![x_1, \ldots, x_n]\!]$ is a power series ring over k; and also that the latter has a functorial characterization due to Grothendieck. This is very useful for proving smoothness.

[1] Here and throughout we use the convention that for an S-scheme X and a map $T \to S$, X_T denotes the base change $X \times_S T$.

1.3 Connectedness

"Connectedness is a crutch": it is essential to keep all the complexity of finite group theory from invading the theory of algebraic groups, since any finite group is a (disconnected) algebraic group. More specifically, a marvelous feature of the theory of smooth *connected* affine k-groups is that for a rather large class (the reductive ones) there is a rich classification and structure theory in terms of concrete combinatorial objects; nothing of the sort is available for a comparably broad class of finite groups, for example. (However, remarkably, the structure of most finite simple groups can be understood via the theory of connected reductive groups over finite fields.)

Let G^0 denote the connected component of the identity in a k-group scheme G of finite type.

Proposition 1.3.1. *The open and closed subscheme* G^0 *is a* k*-subgroup of* G*.*

Proof. To see that $G^0 \times G^0 \hookrightarrow G \times G \stackrel{m}{\to} G$ factors through G^0, it suffices by topology to check that $G^0 \times G^0$ is connected. While it's not true *a priori* that a fiber product of connected schemes is connected, this *is* the case for a fiber product (over k) of *geometrically* connected k-schemes, and this is the situation we are in. In the finite type case (which is what we need) this is easily seen by extending scalars to $\overline{k}$ and using linked chains of irreducible components to reduce to the more familiar fact that over an algebraically closed field a direct product of irreducible schemes of finite type is irreducible. For those interested in the hyper-generality without finite type hypotheses, see [EGA, IV_2, 4.5.8].

The case of inversion is handled similarly (in fact, it is much easier), and by construction the rational point e lies in G^0. So we're done. □

On Homework 1, it is shown that $[G(K) : G^0(K)] = [G(\overline{k}) : G^0(\overline{k})]$ for any algebraically closed field K/k.

Remark 1.3.2. The case of orthogonal groups, all of whose geometric connected components turn out to be defined over the ground field, is atypical. It is easy to write down examples of smooth affine k-group schemes whose non-identity connected components over k are not geometrically connected over k (and so there are more connected components over $\overline{k}$ than there are over k). An example is the group scheme μ_5 of 5th roots of unity over $k = \mathbf{Q}$ (which has 5 geometric components, but only two connected components as a $\mathbf{Q}$-scheme).

This is an aspect of a more general phenomenon, whereby G might contain a lot more information than the group of rational points $G(k)$. Indeed, if k is finite then this is certainly the case (except when G is a finite constant k-group). Later, however, we'll see that *over infinite fields*, the set of rational points is often Zariski-dense in smooth connected affine groups (away from certain exceptional situations related to unipotent groups over imperfect fields), so for some proofs it will suffice to study the group of rational points!

1.4 Examples

Example 1.4.1. $\mathrm{GL}_n = \{\det \neq 0\} \subset \mathrm{Mat}_n = \mathbf{A}_k^{n\times n}$, is an open subset of an affine space, hence obviously smooth and connected.

Remark 1.4.2. For $k = \mathbf{R}$, the connectedness of GL_n as an algebraic group has nothing to do with the fact that in the classical topology $\mathrm{GL}_n(\mathbf{R})$ is disconnected.

Remark 1.4.3. We'll develop a Lie algebra theory, so that for a k-group scheme [of finite type] G we get a Lie algebra $\mathrm{Lie}(G)$, and it will be compatible with the analytic theory in the sense that when $k = \mathbf{R}$ we have $\mathrm{Lie}(G) = \mathrm{Lie}(G(\mathbf{R}))$ (and likewise when $k = \mathbf{C}$). This is actually very useful for studying disconnected Lie groups, when they happen to come from connected algebraic groups!

As a functor on k-algebras, GL_n is given by $\mathrm{GL}_n(R) = \mathrm{Aut}_R(R^n)$. It is, of course, an affine scheme: $\mathrm{GL}_n = \mathrm{Spec}\, k[x_{ij}][\frac{1}{\det}]$.

Example 1.4.4. A special case of Example 1.4.1 is GL_1, which gets a special name, $\mathbf{G}_m$, the multiplicative group scheme.

Consider, by contrast, the circle group scheme, $\mathbf{S}^1 = \{x^2 + y^2 = 1\} \subset \mathbf{A}^2$ over an arbitrary field k. The group structure is given by $e = (1, 0)$, the composition law

$$(x, y)(x', y') = (xx' - yy', xy' + yx'),$$

and the inversion $(x,y)^{-1} = (x,-y)$. Note that when $k = \mathbf{R}$, the group $\mathbf{S}^1(\mathbf{R})$ is compact (in the classical topology), unlike $\mathbf{G}_m(\mathbf{R})$. Hence, $\mathbf{S}^1_{\mathbf{R}} \not\simeq \mathbf{G}_m$ as $\mathbf{R}$-groups. However, $\mathbf{S}^1_{\mathbf{C}} \simeq \mathbf{G}_m$ via the maps of group functors $(x,y) \mapsto x+iy$ [note that x and y lie in $\mathbf{C}$-algebras, not just in $\mathbf{R}$-algebras!] and in the other direction $t \mapsto (\frac{1}{2}(t+t^{-1}), \frac{1}{2i}(t-t^{-1}))$.

This example is so important that it gets a special generalization.

Definition 1.4.5. A k-*torus* is a group variety T over k such that $T_{\overline{k}} \simeq \mathbf{G}_m^N$ for some $N \geq 0$. [So in particular, T must be commutative, connected, smooth, etc.]

Example 1.4.6. ["Example A"] Let $\mathrm{SL}_n \subset \mathrm{GL}_n$ be the closed k-subgroup scheme defined by $\det = 1$. There's an algebraic proof that SL_n is connected using that each variable x_{ij} occurs only once in the formula for the determinant; one can try to conclude (with some care) that $\det - 1$ is irreducible. But that's not a good proof. A more robust geometric approach is to use actions and fibrations: SL_n acts transitively on the connected $\mathbf{P}^{n-1}$ with stabilizer that maps onto GL_{n-1} with kernel an affine space; this method will be discussed more generally (and the example of SL_n addressed more fully) in §8.2.

For smoothness, we can check the Jacobi criterion at the identity explicitly as follows. We have that $\det(1+X) - 1 = \mathrm{tr}(X) + (\text{higher order terms})$. The Jacobi criterion (for a hypersurface like $\mathrm{SL}_n \subset \mathrm{GL}_n$) says that SL_n is smooth, since the linear part $\mathrm{tr}(X)$ of this expansion is nonzero.

Example 1.4.7. ["Example C"] Let V be a k-vector space of dimension $2n$, and $\langle \cdot, \cdot \rangle$ a symplectic [= nondegenerate alternating bilinear] form on V. For k-algebras R, define

$$G(R) = \{g \in \mathrm{GL}(V)(R) \mid \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V\} \subset \mathrm{GL}(V)(R).$$

By Yoneda this defines a k-group scheme $\mathrm{Sp}(\langle \cdot, \cdot \rangle)$, usually denoted Sp_{2n} since all pairs $(V, \langle \cdot, \cdot \rangle)$ with a given dimension $2n$ are isomorphic.

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2.1 Translations

If G is a group variety over k then for any extension field k'/k the group $G(k')$ acts by translations on $G_{k'}$ – that is, *not* on G itself, only after extending scalars to k'. Concretely, this comes from defining for $g \in G(k') = G_{k'}(k')$ the left-translation-by-g map

$$\ell_g : G_{k'} \stackrel{x \mapsto (g,x)}{\to} G_{k'} \times G_{k'} \stackrel{m_{k'}}{\to} G_{k'}.$$

But of course there's no reason this only works for extension *fields*. Rather, for any k-algebra R, we obtain an action of the group $G(R)$ on the R-scheme G_R in a similar fashion. There are a few entirely equivalent ways to think about this, as we now explain.

First, an R-point $g \in G(R)$ is the same as a k-map $\mathrm{Spec}\, R \stackrel{g}{\to} G$, which is the same as an R-map $\widetilde{g} : \mathrm{Spec}\, R \to G_R$. Then ℓ_g is $G_R \stackrel{(\widetilde{g}, \mathbf{1}_{G_R})}{\to} G_R \times G_R \stackrel{m_R}{\to} G_R$, where the $\widetilde{g}$ in the first factor means the "constant map" at $\widetilde{g}$, i.e. the composition $G_R \to \mathrm{Spec}\, R \stackrel{\widetilde{g}}{\to} G_R$.

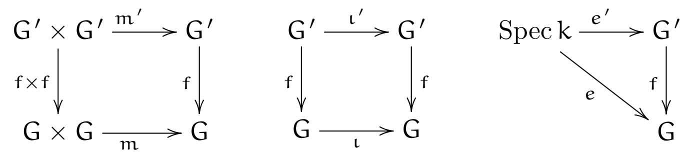
Equivalently, we can think of this as the R-map obtained by base-change from the k-map

$$G_R = \mathrm{Spec}\, R \underset{\mathrm{Spec}\, k}{\times} G \stackrel{(g, \mathbf{1}_G)}{\to} G \times G \stackrel{m}{\to} G.$$

Equivalently, and perhaps most elegantly, we can think about this functorially (via Yoneda) as the R-map corresponding to the map of functors on R-algebras defined by left-translation-by-g_B in the group $G(B)$ for any R-algebra B, where g_B is the base change of the R-point g to B. (Note that we are not requiring G to be affine, so we are implicitly using the elementary fact that to define a map of schemes it is enough to know the corresponding map between the functors they define on affine schemes. Often the restriction to evaluation on affines is just a notational or psychological convenience, but sometimes it is genuinely useful.)

2.2 Homomorphisms

Definition 2.2.1. A k-*homomorphism* $G' \to G$ of k-group schemes is a k-morphism such that the following diagrams commute:



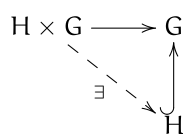
Equivalently, by Yoneda's lemma, it is a k-map such that for all k-algebras R, the corresponding map of sets $f_R : G'(R) \to G(R)$ is a group homomorphism.

Example 2.2.2. The determinant is a k-homomorphism $\det : \mathrm{GL}_n \to \mathbf{G}_m$ for any n. This is easiest to see via the functorial characterization. The corresponding map of affine coordinate rings is $k[t, t^{-1}] \to k[x_{ij}][\frac{1}{\det}]$ given by $t \mapsto \det$.

Remark 2.2.3. For k-group *varieties* (i.e. smoothness is crucial, connectedness is not) a k-homomorphism $f : G' \to G$ is the same as a k-map such that the induced map on geometric points $G'(\overline{k}) \to G(\overline{k})$ is a group homomorphism. This is because to check that two maps of smooth varieties agree (or for that matter, that a map of smooth varieties factors through a smooth locally closed subvariety of the target) it is sufficient to check on geometric points.

2.3 Normal subgroups

Definition 2.3.1. A (closed) k-subgroup $H \subset G$ is called *normal* if the conjugation map $(h, g) \mapsto ghg^{-1}$: $H \times G \to G$ factors through the closed immersion $H \hookrightarrow G$:



Since $H \hookrightarrow G$ is a closed immersion, if this factorization exists it is automatically unique.

Equivalent reformulations of this definition include (1) that $H(R) \lhd G(R)$ for all k-algebras R; and (2) in the smooth situation (cf. Remark 2.2.3 above), that $H(\overline{k}) \lhd G(\overline{k})$.

Example 2.3.2. $\mathrm{SL}_n \lhd \mathrm{GL}_n$.

2.4 Further discussion of examples

Example 2.4.1 (C_n). We return to the consideration of symplectic groups, Example "C", or more properly C_n, from Example 1.4.7 above.

Let (V, B) be a symplectic space, so V is a finite-dimensional nonzero vector space over k, and $B : V \times V \to k$ is a nondegenerate alternating bilinear form. This forces $\dim V = 2n$ to be even, and with appropriately chosen coordinates the matrix of the bilinear form B is given in $n \times n$ blocks by $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. The **symplectic group** Sp_{2n} is defined (using coordinates) the functor of points $g \in \mathrm{Mat}_{2n}$ such that

$$g^t \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] g = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right].$$

If we write such g in the block form $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for $a, b, c, d \in \mathrm{Mat}_n$ then it is the same to impose the conditions:

$$a^t c = c^t a, \ b^t d = d^t b, \ a^t d - c^t b = \mathrm{id}_n.$$

Since the condition on $g \in \mathrm{GL}_{2n} \subset \mathrm{Mat}_{2n}$ is obviously Zariski-closed, and since the functorial description (as the subfunctor of GL_{2n} preserving B) makes it clear that Sp_{2n} is a subgroup, it is clear that $\mathrm{Sp}_{2n} \subset \mathrm{GL}_{2n}$ is a (closed) k-subgroup.

The natural questions about this group are not necessarily easy to answer. Is Sp_{2n} smooth? Yes, but the Jacobian criterion is difficult, or at least prohibitively laborious and inelegant, to verify directly. In §8.3 an elegant functorial method will be used to address it. Is Sp_{2n} connected? In §8.3 we will see that it is. The classical approach to this involved constructing explicit curves (families of symplectomorphisms) connecting an arbitrary group element to the identity, using the theory of so-called "transvections".

Example 2.4.2 (B_n and D_n). Suppose that the characteristic of k is different from 2. (Fear not, we will revisit this with a better characteristic-free approach later. Characteristic 2 should never be ignored, as otherwise one cannot have a truly adequate version over rings.) Consider a *nondegenerate* quadratic space (V, q) over k. Here V is a nonzero finite-dimensional vector space over k, and q is a nondegenerate quadratic form on V, which means (because $\mathrm{char}(k) \neq 2$) that the corresponding bilinear form B_q gives a perfect pairing $V \times V \to k$. When we later address a characteristic-free notion of non-degeneracy for quadratic spaces that works uniformly even over any ring (including $\mathbf{Z}/4\mathbf{Z}$ and $\mathbf{Z}$ in which 2 may be a nonzero nilpotent or a non-unit that is not a zero-divisor), the smoothness of the projective quadric $(q = 0)$ will be the right perspective for defining non-degeneracy.

The **orthogonal group** of q is defined to be

$$\mathrm{O}(q) = \{g \in \mathrm{GL}(V)[\simeq \mathrm{GL}_n] : q(gv) = q(v) \text{ for all } v \in V\}.$$

Functorially,

$$\mathrm{O}(q)(R) = \{g \in \mathrm{Aut}_R(V_R) : q(gv) = q(v) \text{ for all } v \in V_R\}.$$

It is easy to see, for example using coordinates, that preserving the quadratic form is a Zariski-closed condition on g. An equivalent condition (since 2 is a unit in the base ring k) is

$$\mathrm{O}(q)(R) = \{g \in \mathrm{Aut}_R(V_R) : B_q(gv, gw) = B_q(v, w) \text{ for all } v, w \in V_R\}.$$

Or equivalently

$$\mathrm{O}(q) = \{g \in \mathrm{GL}_n : g^t[B_q]g = [B_q]\},$$

where $[B_q]$ is your favorite matrix for the quadratic form B_q, after identifying V with k^n. The last description lets one write down a lot of explicit quadratic equations which cut out $\mathrm{O}(q)$ from the affine space Mat_n. So, the moral is, $\mathrm{O}(q) \subset \mathrm{GL}_n$ is a closed k-subgroup scheme.

Is $\mathrm{O}(q)$ smooth? Again, the answer is yes, and again checking the Jacobian criterion directly is probably not the way one wants to see this. Is $\mathrm{O}(q)$ connected? As the case of $\mathrm{O}_n = \mathrm{O}(k^n, \sum x_i^2)$ over the reals indicates, the answer is No. What may be more surprising is that $\mathrm{O}(q)$ always has exactly two connected components, for any q over any field k. (And in fact everything we have said works fine even in characteristic 2, provided one works to define the notion of nondegeneracy appropriately, which is not so obvious in this case.) We define the **special orthogonal group** to be $\mathrm{SO}(q) = \mathrm{O}(q)^0$. (This is the "wrong" definition in characteristic 2 when n is odd, as in such cases it turns out that $\mathrm{O}(q) = \mathrm{SO}(q) \times \mu_2$ as group schemes, with $\mathrm{SO}(q) = \mathrm{O}(q)_{\mathrm{red}}$ a smooth closed subgroup. In particular, $\mathrm{O}(q)$ is *connected* in such cases!)

Remark 2.4.3. This affords a good example of the phenomenon alluded to in Remark 1.4.3. Algebraically, $\mathrm{SO}(q)$ is of course connected; but the Lie group $\mathrm{SO}(q)(\mathbf{R})$ is disconnected whenever the quadratic form q has mixed signature [$\#\pi_0(\mathrm{SO}(q)(\mathbf{R}))$ in fact depends only on the signature of q, since the signature classifies quadratic forms uniquely over the reals]. Hence, Lie-algebra methods *can* be used to answer *algebraic* questions about $\mathrm{SO}(q)$ which would have seemed to give information only about $\mathrm{SO}(q)(\mathbf{R})^0$ in the classical Lie group setting.

Remark 2.4.4. The behavior of SO_n turns out to be qualitatively different depending on the parity of n, as we shall see. In the ABCDEFG classification, the examples SO_{2n+1} and SO_{2n} correspond to B_n and D_n respectively (with $n \geq 2$, say).

2.5 How far is a general smooth connected algebraic group from being either affine (i.e. linear algebraic) or projective (i.e. an abelian variety)?

Although we'll never need it in this course, for cultural awareness we wish to mention two important theorems towards answering the question in the heading of this section.

Theorem 2.5.1 (Chevalley)**.** *If* k *is perfect, then every connected* k*-group variety (N.B.: smooth!) fits into a unique short exact sequence (a notion to be defined later in the course)*

$$1 \to \mathsf{H} \to \mathsf{G} \to \mathsf{A} \to 1$$

where H *is linear algebraic and* A *is an abelian variety.*

Define, for a smooth connected k-group G of finite type, the affinization to be $\mathsf{G}^{\mathrm{aff}} = \mathrm{Spec}\, \mathcal{O}(\mathsf{G})$,

Theorem 2.5.2 (Anti-Chevalley)**.** *For a smooth connected* k*-group* G *of finite type,* $\mathsf{G}^{\mathrm{aff}} = \mathrm{Spec}\, \mathcal{O}(\mathsf{G})$ *is smooth (in particular, finite type!) and admits a unique* k*-group structure such that the natural map* $\mathsf{G} \to \mathsf{G}^{\mathrm{aff}}$ *is a surjective homomorphism. Moreover, there is an exact sequence*

$$1 \to \mathsf{Z} \to \mathsf{G} \to \mathsf{G}^{\mathrm{aff}} \to 1$$

where Z *is smooth, connected, and central in* G*. Moreover, if the characteristic of* k *is positive, then* Z *is semi-abelian: it fits into an exact sequence*

$$1 \to \mathsf{T} \to \mathsf{Z} \to \mathsf{A} \to 1$$

where T *is a torus and* A *is an abelian variety.*

The Anti-Chevalley theorem is quite amazing, and deserves to be more widely known (though its proof *uses* the Chevalley theorem over $\overline{\mathsf{k}}$; see [CGP, Theorem A.3.9] and references therein). It is actually more "useful" than the Chevalley theorem when $\mathrm{char}(\mathsf{k}) > 0$, even for perfect k, because the commutative (even central) term appears on the left, which is super-handy for studying degree-1 cohomology with coefficients in G (since degree-2 cohomology is most convenient with commutative coefficients).

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Here's an important example of a linear algebraic group: $\mathbf{G}_a = \mathbf{A}^1$, the additive group scheme, where "multiplication" is ordinary addition. It represents the forgetful functor from rings (or k-algebras, if we are working over a field) to abelian groups.

An important note is that $\mathbf{G}_a \not\simeq \mathbf{G}_m$ as k-schemes, let alone k-groups. This is in contrast to the fact that in the classical topology, $(\mathbf{R}^\times)^0 \simeq \mathbf{R}$ as Lie groups, via logarithm and exponential; that isomorphism is actually more of a nuisance than useful in Lie theory.

It is an important fact that $\mathbf{G}_a$ and $\mathbf{G}_m$ are the only 1-dimensional connected (smooth) linear algebraic k-groups when k is algebraically closed; we'll prove this later (building on Homework exercises).

3.1 How do linear algebraic groups arise in nature?

Here is a nice motivating example.

Example 3.1.1. Consider any abstract group Γ and a representation $\rho : \Gamma \to \mathrm{GL}_n(\mathsf{k})$ in a finite dimensional vector space over a field. (This could be the monodromy representation for a fundamental group arising from a local system of finite-dimensional k-vector spaces, for instance. Or an ℓ-adic representation of a Galois group or of the étale fundamental group of a connected noetherian scheme.)

Inside the group variety GL_n there is a group of rational points $\mathrm{GL}_n(\mathsf{k})$; inside *this* lies the image $\rho(\Gamma)$ of our representation. Let $\mathcal{G} \subset \mathrm{GL}_n$ be the Zariski-closure of $\rho(\Gamma)$; it is some closed k-subscheme of GL_n. It turns out that $\mathcal{G}$ is a smooth closed k-subgroup of GL_n, as we will discuss below.

3.2 Zariski closures of subgroups

To prove the claim at the end of Example 3.1.1, we will show something stronger:

Theorem 3.2.1. *Let* G *be any* k*-group variety and* $\Sigma \subset \mathsf{G}(\mathsf{k})$ *any subgroup of the group of rational points. Then the Zariski closure* $\mathsf{Z}_\Sigma \subset \mathsf{G}$ *of* Σ *is a smooth closed* k*-subgroup, and for any field extension* k'/k*, in fact the closed subscheme* $(\mathsf{Z}_\Sigma)_{\mathsf{k}'} \subset \mathsf{G}_{\mathsf{k}'}$ *is the Zariski closure of* $\Sigma \subset \mathsf{G}(\mathsf{k}') = \mathsf{G}_{\mathsf{k}'}(\mathsf{k}') \subset \mathsf{G}_{\mathsf{k}'}$.

Remark 3.2.2. Returning to Example 3.1.1, $\mathcal{G}$ contains $\rho(\Gamma)$ as a Zariski-dense subset, so in fact $\mathcal{G}$ is "controlled" by Γ. For example, this will imply that the irreducibility or complete reducibilty of the representation ρ can be studied in the algebraic category by looking at the group variety $\mathcal{G}$. Note that $\mathcal{G}$ might be highly disconnected, but at least the (geometric) component group will be finite. So representation theory questions over infinite fields can, in principle, be reduced to questions about connected algebraic groups and finite (étale) component groups, which might be more tractable.

Lemma 3.2.3. *If* k *is algebraically closed, then a reduced* k*-group scheme* G *(of finite type) is smooth.*

Proof. Over a perfect field, such as the algebraically closed field k, any reduced scheme of finite type is smooth on a dense open set. The proof of this standard fact from algebraic geometry rests upon finding a "separating transcendence basis" $\{t_i\}$ for the function field K of (each component of) the scheme, so that K is separable over $k(t_1, \dots, t_n)$; by the primitive element theorem, the smoothness situation is now amenable to study via the Jacobian criterion.

So let $U \subset G$ be a smooth, dense open subscheme. Choose a point $g \in G(k)$ and $u \in U(k)$ [there are plenty of points, because k is algebraically closed]. Look at the translation map $\ell_{gu^{-1}} : G \to G$, which is an isomorphism sending $u \mapsto g$. Hence $\widehat{\mathcal{O}}_u \simeq \widehat{\mathcal{O}}_g$; the left side is a power series ring, since U is smooth, so the right side is one too. This was one of our criteria for showing that G is smooth at g.

The point is that since $k = \overline{k}$, the $G(k)$-translates of U cover G. Indeed, $\bigcup_{\gamma \in G(k)} \ell_\gamma(U)$ is open in G and by the above it contains all of $G(k)$. For any variety over an algebraically closed field, if an open set contains all the old-fashioned (rational) points $G(k)$, it must be the whole variety G. So we are done. $\square$

Proposition 3.2.4. *Let* X *be a* k*-scheme (locally) of finite type, and* $\Sigma \subset X(k) \subset X$ *any collection of rational points. Define* $Z_{\Sigma,k}$ *to be the Zariski closure of* Σ *in* X*. Then the following hold:*

(i) The scheme $Z_{\Sigma,k}$ *is geometrically reduced over* k*.*

(ii) The formation of $Z_{\Sigma,k}$ *is compatible with base change, in the sense that* $(Z_{\Sigma,k})_{k'} = Z_{\Sigma,k'}$ *inside* $X_{k'}$*, where on the right side we view* $\Sigma \subset X(k) \subset X_{k'}(k')$*.*

(iii) The formation of $Z_{\Sigma,k}$ *is compatible with products, in the sense that for any other pair* (X', Σ') *over* k*, we have* $Z_{\Sigma\times\Sigma',k} = Z_{\Sigma,k} \times Z_{\Sigma',k}$ *as closed subschemes of* $X \times X'$*.*

(iv) The formation of $Z_{\Sigma,k}$ *is functorial in the pair* (X, Σ)*, in the sense that if* $f : X_1 \to X_2$ *takes* Σ_1 *to* Σ_2*, then it takes the subscheme* $Z_{\Sigma_1} \subset X_1$ *to* $Z_{\Sigma_2} \subset X_2$*.*

Proof of Theorem 3.2.1 *from Proposition* 3.2.4 (*Sketch*). Apply the proposition to $X = G$, our k-group variety, and $\Sigma \subset G(k)$ our subgroup of rational points. Compatibility with base change is part (ii) of the proposition. We use the functoriality of the closure construction [part (iv)] and compatibility with products [part (iii)], with respect to the multiplication, inversion, and identity maps to see that we actually have a subgroup scheme. To get smoothness, we use part (i) of the proposition, along with Lemma 3.2.3. $\square$

Proof of Proposition 3.2.4. The proof will be in a different order than the formulation of the result.

(iv) Consider a map $f : X_1 \to X_2$ such that on k-points, f sends $\Sigma_1 \subset X_1(k)$ to $\Sigma_2 \subset X_2(k)$. We would like to show that $Z_{\Sigma_1} \hookrightarrow X_1 \to X_2$ factors through $Z_{\Sigma_2} \hookrightarrow X_2$. Form the fiber product $f^{-1}(Z_{\Sigma_2}) = X_1 \times_{X_2} Z_{\Sigma_2}$; it is a closed subscheme of X_1, and by hypothesis it contains Σ_1. Hence the scheme-theoretic closure Z_{Σ_1} must be contained in this preimage, which is equivalent to what we wanted.

(i) and (ii) Note that the formation of Zariski-closures is Zariski-local, in the sense that it commutes with passage to open subsets. (This is just a fact from topology.) So we can assume without loss of generality that X is affine, equal to $\operatorname{Spec} A$ for some k-algebra A, and thus the set of rational points Σ is a collection of maps $\{\sigma : A \to k\}$. Denote the kernel of σ by I_σ, which is a maximal – and in particular, a radical – ideal of A. The subscheme Z_Σ corresponds to the ideal $\bigcap_{\sigma\in\Sigma} I_\sigma$, the smallest ideal of A whose zero locus contains Σ. This is an intersection of radical ideals, so it is radical. Hence Z_Σ is reduced.

Geometric reducedness (i.e., assertion (i)) will follow immediately one we know compatibility with base change (i.e., assertion (ii)). For this, note that in $A_{k'} = A \otimes_k k'$, the ideal I'_σ of the point σ viewed as a point of $X_{k'} = \operatorname{Spec} A_{k'}$ is simply $I_\sigma \otimes_k k'$. This is because $I'_\sigma = \ker(\sigma_{k'} : A_{k'} \to k') = I_\sigma \otimes_k k'$ (the latter equality because change of field is a flat base extension). So we are reduced to a problem in linear algebra: let V be a k-vector space (e.g., A) and $\{V_i\}$ a collection of k-subspaces of V (e.g. $\{I_\sigma : \sigma \in \Sigma\}$); then we want $(\bigcap V_i) \otimes_k k' = \bigcap (V_i \otimes k')$ inside $V \otimes_k k'$. To see why this equality holds, forget the multiplicative structure of k' and instead consider any k-vector space W in place of k'. We claim that $(\bigcap V_i) \otimes W = \bigcap (V_i \otimes W)$ inside $V \otimes W$. Any element of $V \otimes W$ lies in $V \otimes W'$ for some $W' \subset W$ finite-dimensional, and W' is a direct summand of W, so we are reduced to the easy case of finite-dimensional W.

(iii) One inclusion of the desired equality $Z_{\Sigma\times\Sigma'} = Z_\Sigma \times Z_{\Sigma'}$ is easy. Since $\Sigma \times \Sigma'$ projects down to Σ (resp. Σ') in the first (resp. second) factor, which is contained in Z_Σ (resp. $Z_{\Sigma'}$), we automatically get an inclusion $\Sigma \times \Sigma' \subset Z_\Sigma \times Z_{\Sigma'}$. A product of closed subschemes is a closed subscheme of the product. Hence the Zariski closure $Z_{\Sigma\times\Sigma'}$ is certainly contained in $Z_\Sigma \times Z_{\Sigma'}$.

For the other direction, we first apply (i) and extend scalars, so we can assume k is algebraically closed, and $Z_\Sigma, Z_{\Sigma'}$ are reduced. Hence their product is reduced as well [since a product of reduced schemes of finite type over an *algebraically closed* field is reduced; e.g., this follows from consideration of separating transcendence bases of function fields of irreducible components]. So it's enough to compare rational points. We want to prove $Z_\Sigma(k) \times Z_{\Sigma'}(k) \subset Z_{\Sigma\times\Sigma'}(k)$. This we do by a standard symmetry trick. First choose $\sigma \in \Sigma$. We claim that $\sigma \times Z_{\Sigma'}(k) \subset Z_{\Sigma\times\Sigma'}(k)$. Why does this hold? Well, $\sigma \times \Sigma' \subset \Sigma \times \Sigma' \subset Z_{\Sigma\times\Sigma'}$, and the closure of $\sigma \times \Sigma'$ in $\sigma \times X' \simeq X'$ is $\sigma \times Z_{\Sigma'}$. But $\sigma \times X'$ is closed in $X \times X'$. Hence the closure of $\sigma \times \Sigma'$ in $X \times X'$ is $\sigma \times Z_{\Sigma'}$, which is therefore contained in the closed subscheme $Z_{\Sigma\times\Sigma'}$, so the intermediate claim is proved. Now choose $z \in Z_{\Sigma'}(k)$. By analogous reasoning, we see that $Z_\Sigma(k) \times z \subset Z_{\Sigma\times\Sigma'}(k)$. The original claim follows.

□

4 January 11 (Substitute lecture by A. Venkatesh)

4.1 Tori

Definition 4.1.1. A k-*split torus* T is a k-group isomorphic (over k) to $\mathbf{G}_m^r$ for some r.

Recall that a torus is a k-group such that $T_{\bar{k}}$ is $\bar{k}$-split. (Definition 1.4.5.)

Example 4.1.2. The special orthogonal group $\mathrm{SO}(x^2+y^2) = \{\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) \mid a^2+b^2=1\}$ is an $\mathbf{R}$-torus under the isomorphism $\mathrm{SO}(x^2+y^2)_{\mathbf{C}} \to \mathbf{G}_m$ given by $\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) \mapsto a+ib$. But it is not $\mathbf{R}$-split (cf. Example 1.4.4).

Definition 4.1.3. The *character group* $X(T)$ or $X^*(T)$ of a k-split torus T is $\mathrm{Hom}_k(T, \mathbf{G}_m)$. The *cocharacter group* $X_*(T)$ is $\mathrm{Hom}_k(\mathbf{G}_m, T)$.

(If T is not k-split then the "right" notions are not the group homomorphisms over k, but rather over $\bar{k}$, or as we shall see, over k_s.) Since (by Homework 1) $\mathrm{End}_k(\mathbf{G}_m) = \mathbf{Z}$ under the identification $(t \mapsto t^n) \leftrightarrow n$, we get several facts, assuming T is k-split:

1. The groups of characters and cocharacters $X^*(T)$ and $X_*(T)$ are both finite free $\mathbf{Z}$-modules, isomorphic to $\mathbf{Z}^r$ if $T \simeq \mathbf{G}_m^r$.
2. We have a perfect pairing $X_*(T) \times X^*(T) \to \mathrm{Hom}_k(\mathbf{G}_m, \mathbf{G}_m) = \mathbf{Z}$ via composition. Explicitly, if $T = \mathbf{G}_m^r$ we identify $X^*(T)$ with $\mathbf{Z}^r$ via $(n_1,\ldots,n_r) \mapsto [(t_1,\ldots,t_r) \mapsto \prod t_i^{n_i}]$, and $X_*(T)$ with $\mathbf{Z}^r$ via $(m_1,\ldots,m_r) \mapsto [t \mapsto (t^{m_1},\ldots,t^{m_r})]$. Under these identifications, the pairing is defined by

$$\langle (m_1,\ldots,m_r),(n_1,\ldots,n_r)\rangle = [t \mapsto t^{\sum m_i n_i}] \leftrightarrow \sum m_i n_i \in \mathbf{Z} \simeq \mathrm{End}_k(\mathbf{G}_m).$$

Remark 4.1.4. In the non-split case we have a functor $T \mapsto X^*(T_{k_s})$ from algebraic k-tori to finite free $\mathbf{Z}$-modules equipped with a discrete action by $\mathrm{Gal}(k_s/k)$. The theorem will be that this is an eqiuvalence of categories.

Example 4.1.5. Returning to Example 4.1.2, i.e. $\mathrm{SO}(x^2+y^2)$ over $k=\mathbf{R}$, the corresponding abelian group is just $\mathbf{Z}$ with the unique nontrivial action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})=\mathbf{Z}/(2)$.

Remark 4.1.6. Once the right definitions of character and cocharacter groups are given for a general k-torus T, it will be the case that all elements of these groups are defined (as homomorphisms to or from $\mathbf{G}_m$) over any extension of k that splits T.

4.2 Maximal split tori

The big theorem about maximal split tori is the following (for which we will later discuss the proof in some important special cases that we need in developing the basic theory).

Theorem 4.2.1. *If G is any smooth connected linear algebraic k-group, all maximal k-split tori $T \subset G$ are $G(k)$-conjugate.*

Remark 4.2.2. (1) It might well be the case that any k-split torus $T \subset G$ is trivial; for example, this is true for $G = \mathrm{SO}(x^2+y^2+z^2)$ over $\mathbf{R}$.

(2) The theorem is false if "maximal k-tori" were to replace "maximal k-split tori". And note that it is trivial (by dimension reasons) that maximal k-split tori always exist; in contrast, there may be no k-split maximal k-tori (once again, $\mathrm{SO}(x^2+y^2+z^2)$ over $\mathbf{R}$).

Example 4.2.3. Over $k=\mathbf{R}$, both $\mathrm{SO}(x^2+y^2)$ and the standard diagonal torus $\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\}$ are maximal $\mathbf{R}$-tori inside SL_2. But they're not $\mathrm{SL}_2(\mathbf{R})$-conjugate, or even $\mathbf{R}$-isomorphic.

Example 4.2.4. Let $G = \mathrm{GL}_n$ over a field k. Then any separable extension E/k of degree n gives rise to a "Weil restriction" maximal torus $\mathrm{R}_{E/k}\mathbf{G}_m$ inside GL_n. (At the level of k-points, this consists of elements of $E^\times$ acting on a k-basis for E by multiplication.)

Example 4.2.5. Let $G = \mathrm{SL}_n$. Let $T = \{\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mid \prod t_i = 1\} \subset G$ be the standard torus. The claim is that T is maximal as a k-torus in G and that all maximal k-split tori in G are $G(k)$-conjugate to T.

We will not prove the conjugacy assertion now, but it will follow from the fact that any commuting set of diagonalizable matrices can be simultaneously diagonalized, plus the fact that if $T' \subset \mathrm{SL}_n$ is a torus then $T'(k)$ consists of semisimple elements. But let's at least now prove that this diagonal T is maximal in SL_n:

Proof of maximality. Note that it suffices to check this after extending scalars to an infinite field. Under this assumption, the claim follows from the fact that for any k-algebra R, we have

$$Z_{G(R)}(T_R) = T(R). \qquad (\star)$$

Granting $(\star)$, in fact it follows that $T \subset G$ is a maximal commutative subgroup scheme, and hence *a fortiori* a maximal torus.

To see $(\star)$ simply compute

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} [g_{ij}] \begin{pmatrix} t_1^{-1} & & \\ & \ddots & \\ & & t_n^{-1} \end{pmatrix} = [t_i t_j^{-1} g_{ij}].$$

By the infinitude of k, we thus see that if $g \in Z_{G(R)}(T_R)$ then $g_{ij} = 0$ for $i \neq j$. □

Remark 4.2.6. There are plenty of other maximal commutative k-subgroups (not tori) non-conjugate to T. As an exercise, find a commutative $(n-1)$-dimensional smooth connected unipotent subgroup of SL_n that is a maximal smooth commutative k-subgroup. Note that this is "as far as possible from being a torus" since it is unipotent. (The abstract notion of "unipotence" and its contrast with tori will be developed later in the course. For present purposes with this example, just think in terms of matrices.)

Example 4.2.7. Let $G = \mathrm{Sp}_{2n}$. Let U be an n-dimensional k-vector space, and U^* its dual. Denote the pairing $U \times U^* \to k$ by $[\cdot, \cdot]$. Define $W = U \oplus U^*$. It has a standard symplectic form ω defined by

$$\omega((x, x^*), (y, y^*)) = [x, y^*] - [y, x^*].$$

Thus we can realize G as $\mathrm{Aut}(W, \omega) = \mathrm{Sp}(W)$. There's a map $\varphi : \mathrm{GL}(U) \to \mathrm{Sp}(W)$ given by $g \mapsto g \oplus (g^t)^{-1}$. This is easily seen to be a closed immersion. The claim about maximal tori in G is that for $T' \subset \mathrm{GL}(U)$ a maximal k-split torus, the image $\varphi(T') \subset \mathrm{Sp}(W)$ is maximal k-split, and any such is $G(k)$-conjugate to $\varphi(T)$, where T is the standard maximal k-split torus in $\mathrm{GL}(U)$. In particular, the dimension of the maximal k-split tori in $G = \mathrm{Sp}_{2n}$ is n.

The proof of maximality is as before: show that $Z_{G(R)}(\varphi(T)_R) = \varphi(T)(R)$. The proof that all maximal k-split tori are $G(k)$-conjugate to $\varphi(T)$ can in fact be reduced to the corresponding claim for $\mathrm{GL}(U)$. This is just an example, so we do not address rigorous details at the present time; we'll come back to this more broadly in the sequel course as well.

4.3 Building up groups from tori

Given any connected *reductive*[2] k-group G, choose a maximal k-split torus T. One can construct the following data:

- A finite subset of *roots* $\Phi \subset X^*(T)$,
- A finite subset of *coroots* $\Phi^\vee \subset X_*(T)$,
- A bijection $\alpha \mapsto \alpha^\vee$, mapping $\Phi \to \Phi^\vee$, and satisfying some combinatorial properties.

Theorem 4.3.1. *Over an algebraically closed field k, the torus T together with the above data will turn out determine G up to isomorphism. This is a* huge *theorem; it will be the topic of the final section of the course.*

4.3.1 Construction of Φ

Fix a connected reductive G as above. Consider the set of all **unipotent 1-parameter subgroups** of G; i.e., closed subgroups $u : \mathbf{G}_a \hookrightarrow G$ that are *normalized by* T. In view of the fact that the only automorphisms of $\mathbf{G}_a$ over an algebraically closed field are unit scalings on the coordinate, it follows that there exists a k-homomorphism $\chi : T \to \mathbf{G}_m$ satisfying $t \cdot u(x) \cdot t^{-1} = u(\chi(t)x)$ at the level of functors. The set of these u turns out to be finite, and the set of χ's which arise in this fashion are the roots Φ of T in G.

Example 4.3.2. Let $G = \mathrm{SL}_n$ and T the standard split torus. For $i \neq j$ there is a unipotent 1-parameter group u_{ij} given by $x \mapsto \begin{pmatrix} 1 & & \star \\ & \ddots & \\ \star & & 1 \end{pmatrix}$ where the off-diagonal entry in the ij-spot is x and the other off-diagonal entries are zero. For $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$, one computes $t \cdot u_{ij}(x) \cdot t^{-1} = u_{ij}(t_i t_j^{-1} x)$, so the roots of $T \subset G$ are of the form $\chi_{ij} : t \mapsto t_i t_j^{-1}$ for $1 \leq i \neq j \leq n$.

Example 4.3.3. Let $G = \mathrm{Sp}_{2n} = \mathrm{Sp}(W)$, let T be the standard diagonal split torus in $\mathrm{GL}(U)$, which we identify with its image in G via the map φ. The roots of $T \subset \mathrm{GL}(U) \hookrightarrow G$ are

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto \begin{cases} t_i t_j^{-1} & i \neq j, 1 \leq i, j \leq n, \text{ or} \\ t_i t_j & \text{and } 1 \leq i, j \leq n \text{ (including } i = j). \end{cases}$$

Thus they are χ_{ij} for $i \neq j$ as above, plus new roots χ'_{ij} for all pairs (i, j).

One can check that in both of these examples, multiplication into G from a direct product of T against the images of the roots in a suitable order is an isomorphism onto an open subset of the group G. This is the so-called "open cell', and it exists in general (not just in the above examples). The structure of this open cell underlies why it is reasonable to believe that G can be reconstructed completely from T and the root system. We will make this more concrete at the beginning of the next lecture.

[2]This notion will be defined later.

5 January 13

5.1 Mapping $\mathbf{G}_a$ and $\mathbf{G}_m$ into a reductive group

As a consequence of the theory of maximal tori, roots and coroots, there is an abundance of $\mathbf{G}_a$'s and $\mathbf{G}_m$'s sitting inside a so-called "split connected reductive" group such as SL_n or Sp_{2n}. To illustrate the principles involves, if we consider the standard upper and lower triangular unipotent 1-parameter subgroups in SL_2 then we get a map

$$\begin{array}{rcl} \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_a & \to & \mathrm{SL}_2 \\ (u, t, u') & \mapsto & \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ u' & 1 \end{smallmatrix}\right) \end{array}$$

This is easily check to be an open immersion. More generally, we'll have an open immersion of the form

$$(\prod_i \mathbf{G}_a) \times (\prod_j \mathbf{G}_m) \times (\prod_k \mathbf{G}_a) \to G = \mathrm{SL}_n, \mathrm{Sp}_{2n}, \ldots.$$

The torus T in the middle is maximal in G; the $\mathbf{G}_a$'s are normalized by T; and the maps are determined by the corresponding "root data".

5.2 Classification of 1-dimensional connected linear algebraic groups

Now we come back to a proper development of the subject, rather than a discussion of things to come much later. The goal for this lecture is to prove the following theorem. By convention, "linear algebraic k-group" will mean "smooth affine k-group".

Theorem 5.2.1. *If k is algebraically closed, the only 1-dimensional connected linear algebraic k-groups are $\mathbf{G}_m$ and $\mathbf{G}_a$.*

Corollary 5.2.2. *For any field k, if G is a 1-dimensional connected linear algebraic k-group, then the following hold.*

(i) G is commutative.

*(ii) There exists a finite extension k'/k such that $G_{k'}$ is isomorphic to either $\mathbf{G}_a$ or $\mathbf{G}_m$. The first case is known as the **additive** or **unipotent** case; the second is called the **multiplicative case**.*

Remark 5.2.3. These two cases are in fact different. *Exercise*: Show that

$$\mathrm{Hom}_k(\mathbf{G}_a, \mathbf{G}_m) = \mathrm{Hom}_k(\mathbf{G}_m, \mathbf{G}_a) = 1$$

over a field k, although non-trivial homomorphisms can exists over an artin local ring k. (Such non-trivial homomorphisms cannot be isomorphisms, however.)

Proof of Corollary 5.2.2 *from Theorem* 5.2.1. (i) This is trivial, since by Homework 2, it suffices to prove that $G(\overline{k})$ is commutative.

(ii) Note that $\overline{k} = \varinjlim k'$, the direct limit of $k' \subset \overline{k}$ which are finite over k. Now in general, if X and Y are any finite type k-schemes (for example G and $\mathbf{G}_a$ or $\mathbf{G}_m$), then a $\overline{k}$-map $f : X_{\overline{k}} \to Y_{\overline{k}}$ arises from a k'-map $f' : X_{k'} \to Y_{k'}$ for some k' in this direct system of intermediate fields. In the affine case this is more or less trivial: chase where the generators and relations in a finite presentation of $\Gamma(Y_{\overline{k}})$ go in $\Gamma(X_{\overline{k}})$, and observe that only finitely many elements of $\overline{k}$ are involved in specifying the map f. So just take the subfield generated by these elements. Since the affine case is all we need here, and will be most important for us in this course, we won't prove the general case. But it's proved in [EGA, IV_3, §8] in vast generality, and note that the affine case does not tautologically imply the general case (i.e., some thought is required). □

Refinements of Corollary 5.2.2(ii)

In the additive case, we can actually choose the finite extension k'/k to be purely inseparable. By the direct limit argument from the proof of Corollary 5.2.2, this is equivalent to the claim that over a perfect field k, a 1-dimensional connected linear algebraic k-group which is additive over $\overline{k}$, is actually k-isomorphic to $\mathbf{G}_a$ (since the perfect closure of a field k is the direct limit of the purely inseparable finite extensions of k). In the multiplicative case, we likewise claim that we can choose the finite extension k'/k to be separable. Again by the direct limit argument, this is equivalent to the claim that if k is separably closed then a 1-dimensional connected linear algebraic k-group which is $\mathbf{G}_m$ over $\overline{k}$ is actually isomorphic to $\mathbf{G}_m$ over k (recall that the separable closure of k is a direct limit of finite separable extensions).

The equivalent formulations of these facts – i.e., the claims over perfect and separably closed fields – will follow in the perfect/additive case from the proof of Theorem 5.2.1 and in the separably-closed/multiplicative case from Homework 2.

5.3 Examples of non-split 1-dimensional linear algebraic k-groups

Example 5.3.1 (Non-split multiplicative group). Let k be a field, and k'/k a quadratic Galois extension splitting an irreducible separable polynomial $f(t) = t^2 + at + b$. (If $\mathrm{char}(k) \neq 2$ we can assume $a = 0$, but why would one want to do that?) Consider the curve $G \subset \mathbf{A}^2_k$ defined by $x^2 + axy + by^2 = 1$. Secretly this is the norm-1 subgroup of the Weil restriction $R_{k'/k}\mathbf{G}_m$, which is "$k'^{\times}$ regarded as a k-group" (although that's a sloppy, imprecise way of defining the group). This has a natural k-group structure, defined in analogy to $\mathbf{S}^1$ over $\mathbf{R}$; cf. Example 1.4.4.

We have $G_{k'} \simeq \mathbf{G}_m$. But G is not k-isomorphic to $\mathbf{G}_m$ even as mere k-schemes. One way to see this is that the unique regular compactification $\{x^2 + axy + by^2 = z^2\} \subset \mathbf{P}^2_k$ has only 1 point complementary to G and its residue field is k', while the unique regular compactification $\mathbf{P}^1_k$ of $\mathbf{G}_m$ has two points complementary to $\mathbf{G}_m$ and they are both k-rational.

Example 5.3.2 (Non-split additive group). Let k be an imperfect field of characteristic $p > 0$ and choose $a \in k - k^p$. Let $G \subset \mathbf{A}^2$ be the curve defined by $y^p = x + ax^p$. This equation is easily seen to be additive, so G is a subgroup. For $k' = k(\alpha)$, where $\alpha^p = a$, the curve $G_{k'}$ is the same as $x = (y - \alpha x)^p$, on which the coordinate $y - \alpha x$ gives a k'-group isomorphism $G_{k'} \simeq \mathbf{G}_a$. Over k itself, the compactification of G in $\mathbf{P}^2$ is $y^p = xz^{p-1} + ax^p$. This is actually regular, as we'll see in a moment. At infinity, $y^p = ax^p$ is the only point, with residue field k'. Since this is not k-rational, as long as we know it is regular then we know that $G \not\simeq \mathbf{G}_a$ as k-schemes, since the unique point complementary to $\mathbf{G}_a$ in its regular compactification $\mathbf{P}^1_k$ is k-rational.

To see the asserted regularity, note that when $x = 1$ the equation $y^p = xz^{p-1} + ax^p$ becomes $y^p = a + z^{p-1}$. This actually defines a Dedekind ring (exercise), which proves regularity.

5.4 Start of proof of the classification theorem

Given a connected 1-dimensional linear algebraic k-group G (recall that $k = \overline{k}$), choose an open immersion $G \hookrightarrow X$ where X is a smooth projective curve over k. Then $X - G$ is a finite nonempty set of k-points.

Claim 5.4.1. $X \simeq \mathbf{P}^1_k$.

Proof. Since $k = \overline{k}$, it suffices to prove that the genus g of X vanishes. The finite set $G(k) - \{e\}$ acts on G via fixed-point-free automorphisms. Since the smooth compactification process is functorial, these all extend to automorphisms of X. But it is a general fact that any smooth projective (connected) curve C over an algebraically closed field with genus $g \geq 2$ has finite automorphism group, so $g \leq 1$.

[Here is one method is to show that C has finite automorphism group: we'll show that the locally finite type automorphism scheme of C is both étale and finite type. The étaleness follows because the automorphism scheme of C is a subfunctor of the automorphism scheme of its Jacobian since $g \geq 2$, and the automorphism scheme of an abelian variety is étale via consideration of torsion. To prove the finite type property, the formation of graphs of automorphisms defines a map from the automorphism scheme to the Hilbert scheme of $C \times C$, and one shows this is a locally closed immersion by the same argument that already occurs in the construction of Hom-schemes as subschemes of Hilbert schemes via consideration of graphs. Although the

Hilbert scheme is just locally of finite type, the locus with a given Hilbert polynomial relative to a specified ample line bundle is contained in a finite type part. The graphs of automorphisms of C all have the same Hilbert polynomial on $C \times C$ relative to a fixed ample line bundle arising from one on C and hence these graphs lie in a finite-type part of the Hilbert scheme.]

Next observe that $G(k) \curvearrowright X$ preserving the finite set $X - G$. Hence some finite index subgroup of $G(k)$ must fix $X - G$ pointwise.

Suppose for contradiction that $g = 1$. Choose any point $O \in X - G$. Then (X, O) is an elliptic curve E. But then E has an infinite group of automorphisms, which contradicts the theory of elliptic curves [Sil, Thm. III.10.1]. □

So we know $X = \mathbf{P}^1$. We have an infinite group Γ acting on $\mathbf{P}^1$ fixing a finite set of points $\mathbf{P}^1 - G$. If an automorphism of $\mathbf{P}^1$ fixes 3 points, it is trivial. Thus $\mathbf{P}^1 - G$ consists of either 1 or 2 points. Therefore we obtain, after a change of coordinates, an isomorphism of pointed curves $(G, e) \simeq (\mathbf{A}^1, 0)$ or $(\mathbf{G}_m, 1)$.

Now we are almost done. It suffices to check that $\mathbf{G}_a$ and $\mathbf{G}_m$ each admit a unique group structure.

6 January 20 (Substitute lecture by A. Venkatesh)

6.1 End of proof of classification of 1-dimensional connected linear algebraic groups

To finish the proof of Theorem 5.2.1, we just need to show the following.

Theorem 6.1.1. *The only k-group structures on $\mathbf{A}^1$ and $\mathbf{G}_m$ are the usual ones, up to a curve automorphism moving the identity point. In particular, for the usual identity point the group structure is uniquely determined.*

Proof. We can assume $k = \overline{k}$.

First let's do the case of $\mathbf{A}^1$. Since $\mathrm{Aut}(\mathbf{A}^1)$ acts transitively on $\mathbf{A}^1(k)$, we can assume that the identity is $0 \in \mathbf{A}^1(k)$. Say we have a group law on $\mathbf{A}^1$. For $x \in \mathbf{A}^1(k)$, translation by x is a k-automorphism t_x of $\mathbf{A}^1$. If $x \neq 0$, there are no fixed k-points (this is a fact about groups). Now we know what $\mathrm{Aut}(\mathbf{A}^1)$ is: it consists of affine transformations $y \mapsto ay + b$. To be fixed-point free on k-points, it must be of the form $y \mapsto y + c$. Since it takes 0 to x, $c = x$. So the group is the usual $\mathbf{G}_a$.

The argument for $\mathbf{G}_m$ is much the same. In this case, automorphisms of $\mathbf{G}_m$ extend to automorphisms of $\mathbf{P}^1$ which either fix or swap zero and infinity. That is, they are of the form $y \mapsto y\alpha$ or $y \mapsto \alpha/y$. Since $k = \overline{k}$, $\sqrt{\alpha} \in k$, so $y \mapsto \alpha/y$ has fixed points. Hence t_x is $y \mapsto y\alpha$ for some α, and if we pin down the identity in the group law to be $1 \in \mathbf{G}_m(k)$ (as we may) then we must have $\alpha = x$. □

So know we know that over a field k, while there can be many isomorphism classes of 1-dimensional smooth connected linear algebraic groups, they all become isomorphic to $\mathbf{G}_m$ or $\mathbf{G}_a$ over $\overline{k}$. We say that they are **forms** of $\mathbf{G}_m$ or $\mathbf{G}_a$.

Example 6.1.2. If q is a non-degenerate quadratic form on a 2-dimensional vector space V over a field k (say with $\mathrm{char}(k) \neq 2$ for now, since we haven't defined non-degeneracy more generally yet), then $\mathrm{SO}(q)$ is a form of $\mathbf{G}_m$. (This is also valid in characteristic 2 for the right notion of "non-degenerate".)

Example 6.1.3. $\mathbf{G}_a$ can actually have forms U over an imperfect field k which have no nontrivial k-points; i.e. $U(k) = \{0\}$. For an example, let $k = k_0(t)$ for k_0 a field of characteristic p. Let $q = p^r > 2$. Then $U \subset \mathbf{A}^2$ defined by $y^q = x - tx^p$ is such a form.

6.2 Smoothness criteria

Theorem 6.2.1. *Let k be a field. Let A be a complete local Noetherian k-algebra with maximal ideal $\mathfrak{m}_A$ and residue field $A/\mathfrak{m}_A = k$. Then the following are equivalent:*

(i) A *is regular; i.e.* $\dim_k \mathfrak{m}_A/\mathfrak{m}_A^2 = \dim A$.

(ii) $A \simeq k[\![x_1, \ldots, x_d]\!]$ *for some* d.

(iii) A satisfies Grothendieck's "infinitesimal criterion for smoothness": for an Artin local k-algebra R and an ideal $J \lhd R$ satisfying $J^2 = 0$, any k-homomorphism $A \to R/J$ lifts to a k-map $A \to R$.

Before giving the proof, let's see a non-example.

Example 6.2.2. Let $X = \operatorname{Spec} k[x, y]/xy$ be the axes in $\mathbf{A}^2$. Let $A = \widehat{\mathcal{O}}_{X,0}$ be the completed local ring at the origin. Let $R = k[\epsilon]/(\epsilon^3)$ and $R/J = k[\epsilon]/(\epsilon^2)$. Then $x \mapsto \epsilon, y \mapsto \epsilon : A \to R/J$ does not lift to a k-map $A \to R$.

Proof of Theorem 6.2.1*.* We'll prove the equivalences one by one.

(ii)⇒(i) This is easy. One just needs to check that $\dim k[\![x_1, \dots, x_n]\!] = n$, which follows from the characterization of the dimension of a ring via its Hilbert polynomial; i.e. $\dim_k R/\mathfrak{m}^\ell$ is eventually polynomial of degree $\dim R$ in ℓ.

(i)⇒(ii) Pick generators $\overline{x}_1, \dots, \overline{x}_d$ for $\mathfrak{m}_A/\mathfrak{m}_A^2$ as a k-vector space. Lifting them gives a map $\varphi : R = k[\![t_1, \dots, t_d]\!] \to A$ sending $t_i \mapsto x_i$. The claim is that φ is an isomorphism. For surjectivity, we argue by "successive approximation". Given $a \in A$ we can find $r \in R$ so that $\varphi(r) \in a + \mathfrak{m}_A^2$. Write $\varphi(r) = a + \sum m_i m_i'$ where $m_i, m_i' \in \mathfrak{m}_A$. Now we can find $r_i, r_i' \in \mathfrak{m}_R$ so that $\varphi(r_i) \in m_i + \mathfrak{m}_A^2$. Thus $\varphi(r - \sum r_i r_i') \in a + \mathfrak{m}_A^3$. Continue in this manner. We can pass to the limit since R is complete. So there exists r with $\varphi(r) = a$.

For injectivity, suppose otherwise. Then there is a nontrivial power series relation over k among the x_i's. That is, choose a nonzero $f \in \ker(\varphi)$. Then A is a quotient of $R/(f)$. But $\dim R/(f) = \dim R - 1 = d - 1 < \dim A$, which is a contradiction.

(ii)⇒(iii) If $A = k[\![x_1, \dots, x_n]\!]$ then (iii) is easy to verify. This is because $\operatorname{Hom}_{\mathrm{Alg}/k}(A, R)$ for an Artin local k-algebra R is just a product of n copies of $\mathfrak{m}_R$, one for each generator of A. The lifting property thus holds trivially, since $\mathfrak{m}_R$ surjects onto $\mathfrak{m}_{R/J}$. So given $A \to R/J$, wherever you're sending the generators of A in $\mathfrak{m}_{R/J}$, lift those elements arbitrarily to $\mathfrak{m}_R$ to get the desired map $A \to R$.

(iii)⇒(ii) Choose a k-basis $\overline{x}_1, \dots, \overline{x}_d$ for $\mathfrak{m}_A/\mathfrak{m}_A^2$. Lift $\overline{x}_i$ to $x_i \in \mathfrak{m}_A$. We get a map $\varphi : R = k[\![t_1, \dots, t_d]\!] \to A$ as before. By construction φ induces an isomorphism $\overline{\varphi} : R/\mathfrak{m}_R^2 \xrightarrow{\sim} A/\mathfrak{m}_A^2$. So we have a map $\theta^{(2)} : A \to A/\mathfrak{m}_A^2 \xrightarrow{\overline{\varphi}^{-1}} R/\mathfrak{m}_R^2$. Apply the lifting property for $J = \mathfrak{m}_R^2/\mathfrak{m}_R^3 \lhd R/\mathfrak{m}_R^3$. So we can lift $\theta^{(2)}$ to $\theta^{(3)} : A \to R/\mathfrak{m}_R^3$. We can then continue in this manner getting $\theta^{(i)} : A \to R/\mathfrak{m}_R^i$ for all i. By completeness of R, there is thus a map $\theta : A \to R$. The claim is that θ is an isomorphism.

For surjectivity, argue by successive approximation as in (i)⇒(ii). This works because θ induces the isomorphism $\theta_2 = \overline{\varphi}^{-1} : A/\mathfrak{m}_A^2 \xrightarrow{\sim} R/\mathfrak{m}_R^2$, which lets the inductive proof get started.

For injectivity, one could argue by dimension as in (i)⇒(ii), but here is a rephrasing of the same idea. It suffices to show that each $\theta_i : A/\mathfrak{m}_A^i \mathrm{to} R/\mathfrak{m}_R^i$ is injective, since $\ker\theta$ must "show up" at some finite level if it is nonzero (Krull intersection theorem). We know θ_i is surjective for each i, since θ is. Now count k-dimensions. Observe that $\mathfrak{m}_A^{i-1}/\mathfrak{m}_A^i$ is spanned by monomials of degree $n - 1$ in $x_1, \dots, x_d$. They go to $t_1, \dots, t_d$, which are linearly independent in $\mathfrak{m}_R^{i-1}/\mathfrak{m}_R^i$. Hence $\dim_k \mathfrak{m}_A^{i-1}/\mathfrak{m}_A^i \le \dim_k \mathfrak{m}_R^{i-1}/\mathfrak{m}_R^i$ for each i. Thus $\dim_k A/\mathfrak{m}_A^i \le \dim_k R/\mathfrak{m}_R^i$ for each i, since we have the corresponding inequality on the dimensions of the subquotients of the obvious filtrations. So by linear algebra, the surjectivity of θ_i implies that equality holds, and that θ_i is an isomorphism, and in particular is injective.

□

Remark 6.2.3. We did not use the full strength of the infinitesimal lifting hypothesis. It is enough to assume that hypothesis holds when R is a finite local k-algebra, since that's all we used. We can also assume $J = (\epsilon)$ for some $\epsilon \in R$ satisfying $\epsilon\mathfrak{m}_R = 0$, since we can factor the general case $R \to R/J$ into a composition of maps of the form $R \to R/\epsilon \to R/(\epsilon, \epsilon') \to \cdots \to R/J$.

Example 6.2.4. As an example of the usefulness of the infinitesimal criterion, we will rederive the Jacobian criterion for an affine hypersurface. Choose a nonzero $f \in k[x_1, \ldots, x_n]$ and suppose $f(0, \ldots, 0) = 0$. Note that $X = V(f)$ is smooth at 0 if and only if $\widehat{\mathcal{O}}_{X,0}$ is regular.

Claim. A sufficient condition for such regularity to hold is that $(\partial x_1 f, \ldots, \partial x_n f)(0) \neq \vec{0}$.

We will show this by verifying the infinitesimal criterion. A map $\widehat{\mathcal{O}}_{X,0} \to R/J$, or the corresponding map $\varphi : A = k[x_1, \ldots, x_n]/(f) \to R/J$, is specified by choosing $t_i = \varphi(x_i) \in \mathfrak{m}_R + J$ satisfying $f(t_1, \ldots, t_n) = 0 \bmod J$. The claim is that we can lift this to a map $\widetilde{\varphi} : A \to R$ under the Jacobian non-vanishing hypothesis. In other words, we need to find $\widetilde{t}_i \in R$ lifting the $t_i \in R/J$ and satisfying $f(\widetilde{t}_1, \ldots, \widetilde{t}_n) = 0$. First choose arbitrary lifts $\widetilde{t}_i'$ of t_i. So $f(\widetilde{t}_1', \ldots, \widetilde{t}_n') \in J$. Set $\widetilde{t}_i = \widetilde{t}_i' + j_i$. We're trying to solve $f(\widetilde{t}_1, \ldots, \widetilde{t}_n) = 0$ for j_i. The point is that by Taylor expanding, $f(\widetilde{t}_1, \ldots, \widetilde{t}_n) = f(\widetilde{t}_1', \ldots, \widetilde{t}_n') + \sum \partial_{x_i} f(0) j_i + O(J^2)$. The quadratic term vanishes since $J^2 = 0$. So we just need show that the map $J^n \to J$ defined by $\vec{j} \mapsto \sum \partial_{x_i} f(0) j_i$ is surjective, since then we can take $\vec{j}$ to be something mapping to $-f(\widetilde{t}_1', \ldots, \widetilde{t}_n') \in J$. But in fact it is surjective since $\partial_{x_i} f(0) \neq 0$ for some i.

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7.1 More on forms

Definition 7.1.1. Let k be a field and X a finite type k-scheme. A k-*form* of X is a k-scheme X' of finite type such that $X'_{\overline{k}} \simeq X_{\overline{k}}$. Equivalently (but not tautologically so), the condition is $X'_K \simeq X_K$ for some algebraically closed extension field K/k (so we can consider $\mathbf{C}/\mathbf{Q}$ for example). Equivalently (but not tautologically so), the condition is that $X'_K \simeq X_K$ for some finite extension K/k.

This definition has an evident analogue for k-schemes with extra structure, like k-groups.

We saw already that every 1-dimensional connected linear algebraic k-group G is a form of $\mathbf{G}_a$ or $\mathbf{G}_m$. In fact, we also saw refinements of this. On Homework 2 it is shown that if G is additive then in fact it splits (becomes isomorphic to $\mathbf{G}_a$) over a finite, purely inseparable extension k'/k. In the multiplicative case we saw that G splits over a finite separable extension.

The additive case is ill-behaved in the sense that there can be forms of $\mathbf{G}_a$ with only a single rational point. The multiplicative case turns out to be much better. We'll see later that when k is infinite then when G is multiplicative, $G(k)$ is Zariski-dense. Really we'll show something stronger (over an arbitrary, even finite, field): if T is a torus then T is **unirational**. (Recall that a variety X over a field k is unirational if it admits a dominant rational k-map from an open subscheme of an affine space over k. This is equivalent to there being a k-embedding of the function field $k(X)$ into a rational function field $k(t_1, \ldots, t_n)$.)

7.2 A loose end concerning smoothness

One thing not quite addressed by Theorem 6.2.1 is the sensitivity of the notion of smoothness to ground field extension. Here is a lemma to clear this up.

Lemma 7.2.1. *Let A be a local Noetherian k-algebra with residue field k. Let k'/k be a field extension. Let $\mathfrak{m}' = \ker(A \otimes_k k' \twoheadrightarrow k')$. Assume $A' = (A_{k'})_{\mathfrak{m}'}$ is Noetherian. Then A is regular if and only if A' is regular.*

Example 7.2.2. A ring which does *not* satisfy the hypotheses of Lemma 7.2.1 is $A = \mathbf{Q}[\![x]\!]$ and $k' = \mathbf{C}$; i.e., $\mathbf{Q}[\![x]\!] \otimes_{\mathbf{Q}} \mathbf{C}$ has non-noetherian local ring at the "origin". This localization injects into $\mathbf{C}[\![x]\!]$, but it is far from the whole thing. You only get power series whose coefficients all live in a finite dimensional $\mathbf{Q}$-subspace of $\mathbf{C}$.

Example 7.2.3. A ring which does satisfy the hypotheses of lemma 7.2.1 is $A = \mathcal{O}_{X,x} = B_{\mathfrak{m}}$ where $x \in X(k)$ is a rational point of a finite type k-scheme X, and x sits inside an open affine $\operatorname{Spec} B \subset X$. **Exercise.** Show that $A' = (A_{k'})_{\mathfrak{m}'} = \mathcal{O}_{X_{k'}, x'}$ where $x' = x \in X(k) \hookrightarrow X_{k'}(k')$.

An example of this situation is $A = \mathcal{O}_{G,e}$ and $A' = \mathcal{O}_{G_{k'}, e'}$ for an algebraic k-group G.

Proof of Lemma 7.2.1. $A \to A'$ is a local map of local Noetherian rings, and it is flat since $A \to A_{k'}$ is, as it is a localization. So by the dimension formula (e.g. [CRT, §15]) we have $\dim A' = \dim A + \dim(A'/\mathfrak{m}A')$.

But $\mathfrak{m}A' = \mathfrak{m}'$ so $\dim(A'/\mathfrak{m}A') = \dim(k') = 0$. Check that $k' \otimes_k \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathfrak{m}'/\mathfrak{m}'^2$ canonically. Hence by comparing the definitions of regularity, the lemma follows. □

Corollary 7.2.4. *Let* G *be an algebraic* k*-group. Then the following are equivalent.*

(i) G *is smooth.*

(ii) $G_{\overline{k}}$ *is regular.*

(iii) $\mathcal{O}_{G_{\overline{k}}, e_{\overline{k}}}$ *is regular.*

(iv) $\mathcal{O}_{G,e}$ *is regular.*

(v) $\widehat{\mathcal{O}}_{G,e} = k[\![x_1, \dots, x_n]\!]$.

(vi) $\widehat{\mathcal{O}}_{G,e}$ *satisfies the infinitesimal lifting criterion.*

Proof. (i) is equivalent to (ii) by definition. (ii) is equivalent to (iii) by translating by $G_{\overline{k}}(\overline{k})$. (iii) is equivalent to (iv) by the lemma. (vi) is equivalent to (v) and (vi) by Theorem 6.2.1. □

7.3 How to apply Grothendieck's smoothness criterion

Recall the infinitesimal lifting criterion for the smoothness of a complete local Noetherian k-algebra A with residue field k, from Theorem 6.2.1: it says that for all local Artin k-algebras $(R, \mathfrak{m})$ and all square zero ideals $J \lhd \mathfrak{m}$, and local k-map $f : A \to R/J$ can be lifted along $\pi : R \to R/J$ to a map $\widetilde{f} : A \to R$ so that $f = \pi\widetilde{f}$.

How should this criterion be interpreted? Loosely speaking, $A \simeq k[\![x_1, \dots, x_N]\!]/(g_1, \dots, g_m)$ "for free". The power series g_i can be evaluated at *nilpotent* elements of R, even though in general it does not make sense to evaluate a power series at an arbitrary element of a ring. So the map f above is nothing more than a solution to $\{g_i = 0\}$ in $\mathfrak{m}/J$, and $\widetilde{f}$ is a lift of it to a solution in $\mathfrak{m}$. Thus, the criterion is that A is a power series ring over k if and only if there are no obstruction to such lifting problems. Also bear in mind Remark 6.2.3, which says that the class of lifting problems we need to consider is actually quite narrow.

The following special case will be the most important one for us.

Example 7.3.1. Let X be a finite type k-scheme, $x \in X(k)$ a rational point, and $A = \widehat{\mathcal{O}}_{X,x}$. Consider R as in the criterion. Since R is artinian, and in particular complete, by the universal property of completion, to give a local k-map $A \to R$ is the same as to give a local k-map $\mathcal{O}_{X,x} \to R$. By the universal property of the local ring $\mathcal{O}_{X,x}$, to give a local k-map $\mathcal{O}_{X,x} \to R$, or equivalently a map $\operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{X,x}$, is the same as to give a map $\operatorname{Spec} R \to X$ sending the closed point to x.

Thus we actually have a fairly "global" interpretation of the infinitesimal criterion in this situation, or at least one that can be viewed in terms of the functor of points of X: namely, we require that $X(R) \to X(R/J)$ surjects onto the points of $X(R/J)$ that lift $x \in X(k)$.

If $(X, x) = (G, e)$ for a k-group G, it follows that G is smooth if and only if $G(R) \to G(R/J)$ maps $\ker(G(R) \to G(k))$ surjectively onto $\ker(G(R/J) \to G(k))$.

7.4 How to show algebraic k-groups are smooth

Now we apply the method of Example 7.3.1 to show how one might prove that a k-group G is smooth and compute its tangent space, and hence (by smoothness) its dimension. It should be remarked that proving connectedness is an entirely separate issue!

The idea follows Example 7.3.1 closely. Take a k-finite Artin local ring R, and $J \lhd \mathfrak{m}$ a square zero ideal. Sometimes it will be computationally convenient to restrict to **small extensions** $R \twoheadrightarrow R/\epsilon$, i.e. we take $J = (\epsilon)$ with $\epsilon\mathfrak{m} = 0$. It might also be helpful to assume $R/\mathfrak{m} = k$. Then we will show that $G(R) \to G(R/J)$ is surjective, at least when we restrict to points lifting $e \in G(k)$. Next we compute $T_e(G)$, which as a set is $\mathfrak{g} = \ker(G(k[\epsilon]) \to G(k))$. However, that kernel will turn out to have a natural vector space structure, with addition coming from the group structure on G (whether or not G is commutative!) and a k-action from scaling ϵ. On Homework 4 it is shown that $T_e(G) = \mathfrak{g}$ as vector spaces, not just sets.

7.4.1 Smoothness of $\mathrm{GL}_n, \mathrm{SL}_n$ in coordinates, via matrices

In fact $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/J)$ is surjective, which we can see as follows. Take $\overline{M} \in \mathrm{GL}_n(R/J)$, and lift it entry-wise to $M \in \mathrm{Mat}_n(R)$. Is M invertible? Yes, because $\det(M) \equiv \det(\overline{M}) \in (R/J)^\times \bmod J$. So $\det M = r + j$ and there exists r' so that $(r+j)r' = 1 + j'$ for $j' \in J$. Hence some multiple of $\det M$ is in $1 + \mathfrak{m} \subset R^\times$. So $\det M \in R^\times$.

Now $T_1(\mathrm{GL}_n) = \ker(\mathrm{GL}_n(k[\epsilon]) \to \mathrm{GL}_n(k)) = \{\mathbf{1} + \epsilon M \mid M \in \mathrm{Mat}_n(k)\} \simeq \mathrm{Mat}_n(k)$ as an abelian group, since $(\mathbf{1}+\epsilon M)(\mathbf{1}+\epsilon M') = \mathbf{1} + \epsilon(M + M') + O(\epsilon^2)$ and $\epsilon^2 = 0$; in fact the vector space structure is correct, as Homework 4 will address.

Also $\mathrm{SL}_n(R) \to \mathrm{SL}_n(R/J)$ is surjective; lift $\overline{M} \in \mathrm{SL}_n(R/J)$ to $M \in \mathrm{GL}_n(R)$. Suppose $\det M = a \in 1 + J \subset R^\times$. Adjust M to $\begin{pmatrix} a^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} M$, which still lifts $\overline{M}$ and has determinant 1.

Finally we compute $T_1(\mathrm{SL}_n) = \{\mathbf{1} + \epsilon M \mid M \in \mathrm{Mat}_n(k), \det(\mathbf{1} + \epsilon M) = 1\}$. Since $\det(\mathbf{1} + \epsilon M) = 1 + \epsilon \operatorname{tr}(M)$, this shows that $\mathfrak{sl}_n = \mathrm{Mat}_n(k)^{\mathrm{tr}=0}$.

7.4.2 Smoothness of $\mathrm{GL}(V), \mathrm{SL}(V)$ without using coordinates (much)

This "functorial" method will lead to an intrinsic description of $\mathfrak{gl}(V), \mathfrak{sl}(V)$, which is useful because it is obviously functorial in V. When we have representations floating around, this functoriality will be a convenient way to avoid having to think too much.

For $\mathrm{GL}(V)$, we ask whether $\mathrm{Aut}_R(V_R) \to \mathrm{Aut}_{R/J}(V_{R/J})$ is surjective. If $\overline{T} : V_{R/J} \xrightarrow{\sim} V_{R/J}$, observe that since these are free modules we can lift $\overline{T}$ to an R-endomorphism $T : V_R \to V_R$. It's an automorphism because its determinant is a unit mod J, and so it is itself a unit.

For $\mathrm{SL}(V)$ we do similarly, and exhibit an explicit element of $\mathrm{Aut}_R(V_R)$ with any determinant in $1 + J$ which is congruent to $\mathbf{1} \bmod J$. Just use the same matrix as in the previous example.

To do the tangent spaces, we compute $T_1(\mathrm{GL}(V)) = \ker(\mathrm{Aut}_{k[\epsilon]}(V_{k[\epsilon]}) \to \mathrm{Aut}_k(V))$. Since

$$\mathrm{Aut}_{k[\epsilon]}(V_{k[\epsilon]}) \subset \mathrm{End}_{k[\epsilon]}(V_{k[\epsilon]}) = k[\epsilon] \underset{k}{\otimes} \mathrm{End}_k V,$$

we have a natural ambient vector space. Observe that the kernel

$$\{\mathbf{1} + \epsilon T \mid T \in \mathrm{End}_k(V)\} = \mathbf{1} + \epsilon k[\epsilon] \underset{k}{\otimes} \mathrm{End}_k(V)$$

is naturally a k-affine linear subspace of $k[\epsilon] \underset{k}{\otimes} \mathrm{End}_k(V)$, so by subtracting off $\mathbf{1}$ it gets a canonical vector space structure itself. Hence we can identify $\mathfrak{gl}(V) = \epsilon \,\mathrm{End}_k(V)$. Similarly, since $\det(\mathbf{1} + \epsilon T) = 1 + \epsilon \operatorname{tr} T$ [which can be proved in a coordinate-free manner using exterior algebra, should one wish to do so] we obtain $T_1(\mathrm{SL}(V))$ in similar fashion to be $\mathfrak{sl}(V) = \epsilon \,\mathrm{End}_k^0(V)$.

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Our next goal will be to establish smoothness for other classical groups, the series C_n, B_n and D_n corresponding to symplectic groups $\mathrm{Sp}(V, B)$ and orthogonal groups $\mathrm{O}(q)$. Actually, we'll do Sp and defer the orthogonal case to the handout "Properties of orthogonal groups" (where we systematically incorporate characteristic 2). Before doing so, let us consider connectedness.

8.1 How to Show Connectedness

Example 8.1.1. $\mathrm{O}(q)$ is actually always disconnected, scheme-theoretically. A prototypical example is $q = xy$ on $V = k^2$. Then $\mathrm{O}(q) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2 \mid (ax + by)(cx + dy) = xy\}$. These conditions are equivalent to $ac = bd = 0$ and $ad + bc = 1$. Over any local ring one can track units and show that $a = d = 0, bc = 1$ or $b = c = 0, ad = 1$. In generally these equations will define $\mathrm{O}(q)$, Zariski-locally on $\operatorname{Spec} R$ over any ring R. Thus $\mathrm{O}(q) = \mathrm{SO}(q) \sqcup \mathrm{SO}(q) \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, i.e. $\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\} \sqcup \{\left(\begin{smallmatrix} 0 & b \\ b^{-1} & 0 \end{smallmatrix}\right)\}$.

The technique for proving a finite type k-group scheme G to be connected is the following.

1. Find a smooth, geometrically connected k-scheme X (usually not affine!) with a G-action $\alpha : G \times X \to X$.
2. Ensure the action is transitive on $\overline{k}$-points, i.e. $\alpha_{x_0} : G_{\overline{k}} \to X_{\overline{k}}$ should be surjective for $x_0 \in X(\overline{k})$.
3. Prove the stabilizer $\mathrm{Stab}_{G_{\overline{k}}}(x_0)$ is connected.

Proposition 8.1.2. *Given 1-3 above, G is connected (and this holds if and only if $G_{\overline{k}}$ is connected).*[3]

Proof. Rename $k = \overline{k}$, and replace G with G_{red} so that it is smooth. Now G is regular and equidimensional of some dimension d (by translating a neighborhood of the origin!). The surjection $G \twoheadrightarrow X$ has equidimensional fibers all of the same dimension, since they are conjugates of $\mathrm{Stab}_G(x_0)$. The base X is regular, hence irreducible of some pure dimension d'. Therefore all the fibers are of expected dimension $d - d'$. (This holds on a Zariski-open subset of X, by a classical fact; since all the fibers are conjugate - hence isomorphic - it holds everywhere.)

Now we invoke the Miracle Flatness Theorem [Mat, 23.1], which entails that $G \twoheadrightarrow X$ is flat, and hence open. By an easy topological argument, the existence of an open continuous surjection from a topological space Y onto a connected base with connected fibers entails that Y is connected. □

Remark 8.1.3. If G acts on X and $x \in X(k)$ then $\mathrm{Stab}_G(x)$ is the scheme-theoretical fiber of α_x over x. Since it represents the stabilizer-subgroup functor of G, it follows that it is a subgroup scheme of G. But it might not be smooth! (In characteristic p, that is; everything is OK in characteristic 0, by a theorem of Cartier which says *all* k-group schemes locally of finite type are smooth when $\mathrm{char}(k) = 0$.)

8.2 Connectedness of SL_n

Fix an n-dimensional vector space V and let $G = \mathrm{SL}(V) \simeq \mathrm{SL}_n$.

8.2.1 Method 1 - Flag varieties

For V a k-vector space of dimension $n > 0$, let X be the functor on k-algebras given by

$$R \mapsto X(R) = \{0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = V_R \mid \\ F_i = \text{locally free } R\text{-submodule of } V_R \text{ of finite rank}, F_i/F_{i-1} = \text{locally free of rank } 1\}.$$

The following is standard (so we omit the proof):

Proposition 8.2.1. *X is representable by a smooth geometrically connected projective variety, covered by open sets isomorphic to affine spaces having non-empty overlaps.* □

Let $\mathrm{GL}(V)$ act on X in the obvious way. We can restrict this to an action $\alpha : \mathrm{SL}(V) \times X \to X$.

Claim 8.2.2 (Exercise)**.** *The map α is transitive on $\overline{k}$-points.* □

Compute the stabilizer Stab in $\mathrm{SL}(k^n)$ of the standard flag in k^n explicitly: we find as schemes

$$\mathrm{Stab} = \left\{ \begin{pmatrix} a_1 & * & * & * & \cdots & * \\ 0 & a_2 & * & * & \cdots & * \\ 0 & 0 & a_3 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1} & * \\ 0 & 0 & \cdots & 0 & 0 & a_n \end{pmatrix} : \prod a_i = 1 \right\}$$

This is visibly isomorphic to $\mathbf{G}_m^{n-1} \times \mathbf{A}^{\binom{n}{2}}$ as a variety, and is thus irreducible and geometrically connected by inspection. So the method of the previous section will go through.

[3]By Homework 1, Problem 3(i), a connected k-group G of finite type is automatically geometrically connected.

8.2.2 Method 2 - Projective Space

More naively, if you cannot guess what the right homogeneous space is, one can try to use a more obvious action of the linear group one is trying to prove connected, namely the action on $\mathbf{P}^{n-1}$ if $G \subset \mathrm{GL}_n$.

So give the geometrically connected, etc., variety $\mathbf{P}^{n-1}$ the standard (transitive) action of SL_n. The stabilizer in this case is

$$\mathrm{Stab}_{\mathrm{SL}_n}([1:0:\cdots:0]) = \left\{ \begin{pmatrix} a & * & \cdots & * \\ 0 & x & \cdots & x \\ \vdots & \vdots & [g] & \vdots \\ 0 & x & \cdots & x \end{pmatrix} \mid a = \det g^{-1} \right\}.$$

Here the top row is arbitrary after the first entry; the x's denote that the bottom right $(n-1)\times(n-1)$ block is given by $g \in \mathrm{GL}_{n-1}$. Hence $\mathrm{Stab}_{\mathrm{SL}_n}([1:0:\cdots:0]) = \mathrm{GL}_{n-1} \times \mathbf{A}^{n-1}$ as a variety. Again we get lucky and can see that it is connected, so we're done.

8.3 Smoothness of Sp_{2n}

Let $G = \mathrm{Sp}_{2n} \simeq \mathrm{Sp}(V, B)$ where V is an $2n$-dimensional vector space over k and $B : V \times V \to k$ is a symplectic form. We will prove that G is smooth via the infinitesimal criterion. We will also use Remark 6.2.3 to reduce our work to the case where R is a local finite k-algebra with residue field k, $0 \neq \epsilon \in \mathfrak{m}_R$ satisfies $\epsilon\mathfrak{m}_R = 0$, and $J = \epsilon R$; we must prove $G(R) \twoheadrightarrow G(R/J)$.

Choose $\overline{T} \in \mathrm{Aut}_{R/J}(V_{R/J})$ preserving $B_{R/J}$. We want to lift it to an element of $\mathrm{Aut}_R(V_R)$ preserving V_R. In other words, $\overline{T}$ satisfies $B_{R/J}(\overline{T}v, \overline{T}v') = B_{R/J}(v, v')$ for all $v, v' \in V_{R/J}$. But note that by R/J-bilinearity, this condition is completely determined by the fact that it holds for all $v, v' \in V \subset V_{R/J}$. In other words, the right side of the equation is "constant" in $k \subset R/J$.

Start by choosing any R-linear automorphism T of V_R lifting $\overline{T}$. We want to know whether $B_R(Tv, Tv') = B_R(v, v') \in R$, for all $v, v' \in V \subset V_R$. Probably this is not the case. But since T lifts $\overline{T}$, we know, at least, that $B_R(Tv, Tv') = B_R(v, v') \bmod J$. We shall contemplate the automorphism $T + S$ for $S \in \mathrm{Hom}_R(V_R, V \otimes_k J)$. These are precisely the lifts of $\overline{T}$. A choice of $S \in \mathrm{Hom}_R(V_R, V \otimes_k J)$ is equivalent to a choice of $\epsilon S_0 \in \mathrm{Hom}_k(V, V \otimes_k J)$ where $S_0 \in \mathrm{End}_k(V)$. Note that $S_1 \in \mathrm{Hom}_k(V, V \otimes_k J)$ is automatically of the form ϵS_0 since $\dim_k J/\mathfrak{m}_R = \dim_k(\epsilon R/\mathfrak{m}_R) = 1$ since $\epsilon\mathfrak{m}_R = 0$; so we're really using the fact that small extensions are small to make our life easier.

Now we calculate

$$B_R((T + \epsilon S_0)v, (T + \epsilon S_0)v') = B_R(Tv, Tv') + \epsilon(B_R(Tv, S_0 v') + B_R(S_0 v, Tv')).$$

Since $\epsilon\mathfrak{m}_R = 0$, the coefficient of ϵ, namely $B_R(Tv, S_0 v') + B_R(S_0 v, Tv')$, only matters mod $\mathfrak{m}_R$. Let $T_0 = \overline{T} \bmod \mathfrak{m}_R$. The ϵ-coefficient is $B(T_0 v, S_0 v') + B(S_0 v, T_0 v') \in k$. Thus the condition on S_0 for $T + \epsilon S_0$ to be our desired lift is that

$$\epsilon(B(T_0 v, S_0 v') + B(S_0 v, T_0 v')) = B(v, v') - B(Tv, Tv') =: \epsilon h(v, v') \in J = \epsilon R.$$

Here $h : V \times V \to k$ is defined uniquely by the condition that $B(v, v') - B(Tv, Tv') = \epsilon h(v, v')$. By inspection it is alternating and bilinear.

So we have reduced ourselves to a problem in linear algebra: is the map

$$\mathrm{End}(V) \to (\wedge^2 V)^*$$

defined by $S_0 \mapsto [(v, v') \mapsto B(T_0, S_0 v') + B(S_0 v, T_0 v')]$ surjective, so that it must hit h?

If $\dim_k V = 2n$ then $\dim_k \mathrm{End}(V) = 4n^2$ and $\dim_k(\wedge^2 V)^* = \binom{2n}{2} = \frac{2n(2n-1)}{2} = n(2n-1)$.) So we want the dimension of the kernel of the map above to be $n(2n+1)$.

What is the kernel? It is

$$\{S_0 \in \mathrm{End}(V) \mid B(T_0 v, S_0 v') = -B(S_0 v, T_0 v') = B(T_0 v', S_0 v) \text{ for all } v, v' \in V\}.$$

In other words, it is
$$\{S_0 \in \mathrm{End}(V) \mid [(v, v') \mapsto B(T_0 v, S_0 v')] \in (\mathrm{Sym}^2 V)^*\}.$$
Since $T_0 \in \mathrm{Sp}(V, B)$ is invertible, we can set $S_1 = T_0^{-1} S_0$; then the space above is the same as
$$\{S_1 \in \mathrm{End}(V) \mid B((\cdot), S_1(\cdot)) \in (\mathrm{Sym}^2 V)^*\}.$$
But $S_1 \leftrightarrow B((\cdot), S_1(\cdot))$ gives an identification of $\mathrm{End}(V)$ with $V^* \otimes V^* = (V \otimes V)^*$, since B is non-degenerate. Hence the space above is precisely $(\mathrm{Sym}^2 V)^* \subset V^* \otimes V^*$ which has the correct dimension. QED

Finally, we can compute the Lie algebra $\mathfrak{sp}_{2n}$ and determine $\dim \mathrm{Sp}_{2n}$. As usual, we regard $T_{\mathbf{1}}(\mathrm{Sp}(V, B)) \subset T_{\mathbf{1}}(\mathrm{GL}(V)) = \mathfrak{gl}(V) = \mathrm{End}(V)$. Then
$$\mathfrak{sp}_{2n} = \{\mathbf{1} + \epsilon T \in \mathrm{Aut}_{k[\epsilon]}(V_{k[\epsilon]}) \mid T \in \mathrm{End}(V), B((\mathbf{1} + \epsilon T)v, (\mathbf{1} + \epsilon T)v') = B(v, v') \text{ for all } v, v' \in V \subset V_{k[\epsilon]}\}.$$
We compute that the condition on T is that $B(Tv, v') + B(v, Tv') = 0$; i.e. $B(T(\cdot), (\cdot)) \in (\wedge^2 V)^*$, since B is alternating. Hence $\mathfrak{sp}_{2n} = \mathfrak{sp}(B, V) = (\wedge^2 V)^*$ identified as a subspace of $\mathrm{End}(V)$ via B. Thus its dimension is
$$\dim \mathrm{Sp}_{2n} = \frac{(2n)(2n+1)}{2} = n(2n+1).$$
Explicitly, $\mathfrak{sp}_{2n}$ consists of $2n \times 2n$ block matrices $\left(\begin{smallmatrix} a & b \\ c & -a^t \end{smallmatrix}\right)$ such that $b = b^t, c = c^t$.

January 27

Sp_{2n} is connected

The goal for today is to prove the following result.

Theorem 8.3.1. *Let V be a vector space of dimension $2n \geq 0$ equipped with a nondegenerate alternating bilinear form B. Then the symplectic group $\mathrm{Sp}(V, B)$ is connected.*

The proof is by induction following the general strategy outlined earlier; the inductive step comes down to an exercise in (fairly delicate) linear algebra. If anything, it is an indication that **connectedness results for algebraic groups should be treated with respect**!

Proof. We induct on n. The case $n = 0$ is OK (alternatively, $n = 1$ is SL_2, which we know is connected). Let $G = \mathrm{Sp}(V, B)$ act on $\mathbf{P}^{2n-1}$ in the obvious way. After renaming $\overline{k}$ as k, what we need to check is that (i) the action is transitive, and (ii) one stabilizer is connected.

For transitivity, choose two lines $ke, ke' \subset V$. We seek a symplectic automorphism g such that $g(ke) = ke'$. There are two cases.

Case 1: $B(e, e') \neq 0$. After rescaling e' we can take $B(e, e') = 1$. So $H = ke \oplus ke'$ is a **hyperbolic plane**; i.e. the restriction $B|_H$ is nondegenerate, or equivalently $B = H \overset{\perp}{\oplus} H^\perp$. So we can just take g to be the automorphism of V given by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on H and the identity on $H^\perp$. This swaps the lines and is easily seen to be symplectic.

Case 2: $B(e, e') = 0$. Since e and e' are (without loss of generality) linearly independent and B is nondegenerate, there exists a pair of hyperbolic planes H, H' containing e and e' respectively, which are orthogonal to one another; i.e. $V = W \overset{\perp}{\oplus} H \overset{\perp}{\oplus} H'$. Take g to be the automorphism of V which swaps H and H', taking e and e' to one another, and fixes W. Thus transitivity is established.

For the connectedness of the stabilizer, fix a line $L \subset V$. We want $G_L = \mathrm{Stab}_G(L)$ to be connected. Since B is alternating, we have $L \subset L^\perp$, and $L^\perp$ is a hyperplane in V. Let $\overline{V} = L^\perp / L$, which is a vector space of dimension $2(n-1)$. The form $B|_{L^\perp}$ descends (check!) to an alternating (check!) nondegenerate (check!) form on $\overline{V}$. Hence we get a k-group homomorphism $\xi : G_L \to \mathrm{Sp}(\overline{V}, \overline{B})$ sending $g \mapsto \overline{g} = g|_{L^\perp} \bmod L$.

Remark 8.3.2. If $\dim V = 2$ then $G_L = \{\left(\begin{smallmatrix} a & * \\ 0 & a^{-2} \end{smallmatrix}\right)\} \simeq \mathbf{G}_m \times \mathbf{A}^1$ is connected.

By induction $\mathrm{Sp}(\overline{V}, \overline{B})$ is connected. If we knew that ξ were surjective with connected kernel then connectedness of G_L follows, completing the proof. Thus it remains only to show the following.

Claim 8.3.3. ξ *is surjective with connected kernel.*

For surjectivity, choose $L = ke \subset V$, and e' such that $B(e, e') = 1$ (by nondegeneracy) so that $H = ke \oplus ke'$ is hyperbolic and $V = H \overset{\perp}{\oplus} H^\perp$. Since $L^\perp \subset H^\perp$ surjects onto $\overline{V}$ with kernel $L \subset H$, we get an injection $H^\perp \to \overline{V}$, which by dimension considerations is a symplectic isomorphism. So take any $\overline{g} \in \mathrm{Sp}(\overline{V}, \overline{B})$ and view it in $\mathrm{Sp}(H^\perp, B|_{H^\perp})$; extend it by the identity on H to get $g \in G_L$ which induced $\overline{g}$, i.e. such that $\xi(g) = \overline{g}$.

To describe the kernel $\ker\xi$, and in particular show it is connected, we will view its elements as $g = \mathbf{1} + T$ for $T \in \mathrm{End}(V)$ and figure out the conditions on T which encode the defining properties (i) g is an automorphism, (ii) g fixes L, (iii) $g \in \ker\xi$, and (iv) g preserves B. Seeing that the result is connected appears, *a priori*, nontrivial, since while (i)–(-iii) are Zariski-open or linear conditions, (iv) is a quadratic condition. But in fact, it will work out quite nicely. As motivation, consider a special case.

Example 8.3.4. Let $G = \mathrm{Sp}(k^2, B) = \mathrm{SL}_2$ and let $L = ke = L^\perp$. Then G_L is the Borel subgroup, which is of the form $\{\mathbf{1} + \left(\begin{smallmatrix} \lambda & b \\ 0 & \frac{-\lambda}{1+\lambda} \end{smallmatrix}\right) \mid \lambda \neq -1\}$, which is quite tractable.

If we decompose $V = L^\perp \oplus ke'$, then $T : V \to V$ must satisfy some conditions: We have $T_0 = T|_L : L \to L$ is a scalar λ different from -1 (since g is an automorphism fixing L). We have $S = T|_{L^\perp} : L^\perp \to L$ since g must induce the identity on $\overline{V}$. Hence, since g is an automorphism, we must certainly have $T(e') = ce' + \ell^\perp \notin L^\perp$; i.e. $c \neq 0$.

This much ensures that $g = \mathbf{1} + T$ where $T = T_0 \oplus S$ is an automorphism preserving $L^\perp$ and L and inducing the identity on $\overline{V}$. But what is the condition that g preserves B?

Using the condition that $B(e, e') \neq 0$, the condition $B(ge, ge') = B(e, e')$ says exactly that $\lambda c + \lambda + c = 0$, or $(1 + \lambda)(1 + c) = 1$. So necessarily $\lambda \neq 0, -1$ and $c = -\lambda/(1 + \lambda)$; cf. the example above.

Assuming $c = -\lambda/(1 + \lambda)$, one can check that $\varphi_S : L^\perp \to k$ defined by $v \mapsto -cB(v, e) - (1 + c)B(Sc, e)$ kills $L = ke$, and hence induces $\overline{\varphi}_S : \overline{V} \to k$. But we have the nondegenerate form $\overline{B}$ on $\overline{V}$, so all functionals $\overline{\psi} \in \overline{V}^*$, including $\overline{\varphi}_S$, must be of the form $\overline{\psi}(\overline{v}) = \overline{B}(\overline{v}, \overline{x}_S)$ for some $\overline{x}_S \in \overline{V}$.

It then follows (check!) that the full condition that $g = \mathbf{1} + (S \oplus T_0)$ [where now we know T_0 is determined by S, since λ is determined by c!] is encoded by the condition that $\overline{x}_S = \ell^\perp \bmod L$.

Putting all this together, we have a map $\ker\xi \to \mathrm{Hom}(L^\perp, L)$ sending $g \mapsto S$, which is surjective onto $\{S \mid S|_L \text{ acts as } \lambda \neq 0, -1\}$. A point in the fiber over S is given by a choice of $\ell^\perp$ satisfying $\ell^\perp = \overline{x}_S \bmod L$; i.e. the fibers are copies of L.

So the conclusion is that $\ker\xi$ is fibered over a Zariski open subset of an affine space, with connected fibers, and is thus connected, which completes the proof. ☐

9 January 29

9.1 A little more about Sp_{2n}

We proved $G = \mathrm{Sp}(V, B)$ is connected. Here is a useful "matrix description" of the group, using our proof. Recall that we endowed $\mathbf{P}(V)$ with the natural action of $G \subset \mathrm{GL}(V)$, chose a line $L \subset V$ spanned by $e \in V$, and took a complementary line $ke' \subset V$ so that $H = ke \overset{\perp}{\oplus} ke'$ is a hyperbolic plane; i.e. $B(e, e') = 1$. Then $L \subset L^\perp$, $V = L^\perp \overset{\perp}{\oplus} ke' = H \oplus H^\perp \simeq H \oplus L^\perp/L \simeq H \oplus \overline{V}$. The stabilizer G_L had the following structure, where the first two basis vectors are e, e' and the remainder parametrize $\overline{V} = H^\perp = L^\perp/L$:

$$\left\{ \begin{pmatrix} a & * & [& & S \in \mathrm{Hom}(L^\perp/L, L) & &] \\ 0 & a^{-1} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & [] & [& [& [\cdots] &] &] \\ \vdots & \vdots & [& [& \vdots &] &] \\ \vdots & \vec{x}_S \in \overline{V} & [& [& [\overline{g} \in \mathrm{Sp}(\overline{V}, \overline{B})] &] &] \\ \vdots & \vdots & [& [& \vdots &] &] \\ 0 & [] & [& [& [\cdots] &] &] \end{pmatrix} : a = 1 + \lambda, \lambda \neq 0, -1 \right\}$$

This description makes connectedness obvious, since visibly $\mathrm{Sp}_{2n} = (\mathbf{A}^1 - \{0, -1\}) \times \mathbf{A}^1 \times \mathbf{A}^{2n-2} \times \mathrm{Sp}_{2n-2}$ and $\mathrm{Sp}_2 = \mathrm{SL}_2$.

Corollary 9.1.1 (of connectedness, or the matrix description)**.** $\mathrm{Sp}(V, B) \subset \mathrm{SL}(V)$.

Proof. Recall that we know $\mathrm{Sp}(V, B)$ is smooth and connected, For any geometric point $T \in \mathrm{Sp}(V, B)$ we have $(\det T)^2 = 1$ from the equation which says T is symplectic. Hence $\det : \mathrm{Sp}(V, B) \to \mathbf{G}_m$ factors through the finite group scheme $\mu_2 \subset \mathbf{G}_m$ on geometric points (even in characteristic 2). Thus, since $\mathrm{Sp}(V, B)$ is smooth and connected, it follows that det is trivial on $\mathrm{Sp}(V, B)$.

Alternately, use the matrix description and argue by induction, row expanding along the second row. This says that $G_L \subset \mathrm{SL}(V)$. Now you need to do a bit of work to check that this implies all of G has determinant 1, but it shouldn't be so bad. □

9.2 Actions, Centralizers and Normalizers

Now we want to study actions of k-groups on k-schemes.

Definition 9.2.1. An *action* $\alpha : G \times X \to X$ is a map defined on the product of a finite type k-group G and a finite type k-scheme X, such that the induced map on S-points endows $X(S)$ with an action of $G(S)$ functorially in S [or even just in R as S ranges through $\operatorname{Spec} R$ for k-algebras R]. Of course this is equivalent to another definition with diagrams expressing that the identity acts trivially and the action is "associative" in the usual sense.

The key example is when $X = G$ and α is multiplication (left translation). This will be crucial in proving that a linear algebraic k-group actually occurs as a closed subgroup scheme of some GL_n! [4]

Let $\alpha : G \times X \to X$ be a left action, and $W, W' \subset X$ closed subschemes, for example smooth.

Example 9.2.2. Take $X = G$ and α to be the conjugation action. Given $W = H \subset G = X$ a closed k-subgroup, asking whether H is normal amounts to asking whether it is preserved by the action.

This example motivates the following definition.

Definition 9.2.3. The *functorial centralizer* is

$$\underline{Z}_G(W) : R \mapsto \{g \in G(R) : g : X_R \to X_R \text{ is the identity on } W_R\}.^5$$

The *functorial transporter* is $\underline{\mathrm{Tran}}_G(W, W') : R \mapsto \{g \in G(R) : g(W_R) \subset W'_R\}$. The *functorial normalizer* is $\underline{\mathrm{N}}_G(W) = \underline{\mathrm{Tran}}_G(W, W)$.

Proposition 9.2.4 (Homework 3)**.** *If $W \subset X$ is geometrically reduced, then $\underline{Z}_G(W)$ and $\underline{\mathrm{N}}_G(W)$ are representable by closed k-subgroup schemes $Z_G(W), N_G(W)$ of G.* □

(The idea of the proof is to deduce the general case from the case $k = k_s$ by Galois descent. In the separably closed case, since W is geometrically reduced, W has a dense set of rational points, which allows one to prove representability quite easily.)

Example 9.2.5. Let $X = G$ be a smooth k-group of finite type, and α the conjugation action. Then $Z_G = Z_G(G)$ is the **scheme theoretic center** of G; its R-points are

$$Z_G(R) = \{g \in G(R) \mid g \text{ conjugates trivially on } G_R\} = \{g \in G(R) \mid g \in Z(G(A)) \text{ for all } A \in \mathrm{Alg}/R\}.$$

Example 9.2.6 (Homework 3). $Z_{\mathrm{GL}_n} \simeq \mathrm{GL}_1$ is the diagonal copy of $\mathbf{G}_m$. $Z_{\mathrm{SL}_n} \simeq \mu_n \subset \mathrm{GL}_1$ is the diagonal copy of the scheme of nth roots of unity. $Z_{\mathrm{GL}_n}(T) = T$ where T is the diagonal torus. In particular $Z_{\mathrm{GL}_n}(T)$ is smooth by inspection, which will turn out to be an instance of a more general phenomenon.

Example 9.2.7. When $k = \overline{k}$ is algebraically closed, we have $Z_G(W)(\overline{k}) = \{g \in G(\overline{k}) \mid g \text{ centralizes } W_{\overline{k}} \subset X_{\overline{k}}\}$. But since $W_{\overline{k}}$ is reduced and $\overline{k}$ is algebraically closed, *in this case* the condition is the same as centralizing $W(\overline{k}) \subset X(\overline{k})$!

[4] Briefly, the idea is to have G act on itself by multiplication, endowing it with an action on its own coordinate ring, an infinite dimensional vector space, but with a certain exhaustive filtration by finite-dimensional G-stable subspaces. A sufficiently large one of these subspaces will contain all the generators of the algebra, and G will act faithfully on that subspace, producing the desired closed immersion that is a homomorphism too.

[5] It should be emphasized that the condtion that $g \in G(R)$ be in $\underline{Z}_G(W)(R)$ is **much** stronger than that g act trivially on $W(R) \subset X(R)$!

9.3 Closed orbit lemma

Example 9.3.1. Let $G = \mathbf{G}_m$ act on $X = \mathbf{A}^2$ by $t(x, y) = (tx, t^2y)$. We ask what the G-**orbits** of $X(k)$ are; i.e. what are the set-theoretic images of the orbit maps $G \to X : g \mapsto gx$ for $x \in X(k)$? It is not difficult to check that they consist of the x-axis minus the origin, the y-axis minus the origin, the parabolas $y = \lambda x^2$ for various $\lambda \neq 0$ (each minus the origin), and finally the origin itself. The observations one should make are that

1. All orbits are locally closed, smooth subschemes of X.
2. The only closed orbit is $(0, 0)$, and it is of minimal dimension among the orbits.

These properties turn out to generalize.

Remark 9.3.2. It must be emphasized that the rational points $\alpha_x(G)(k)$ of the set theoretic image of the action map $\alpha_x : G \to X : g \mapsto gx$ for $x \in X(k)$, may be much bigger than the set theoretic image $\alpha_x(G(k))$ of the rational points of G! For example, $GL_1 \to GL_1$ given by squaring is surjective as a map of schemes, so its set theoretic image has lots of rational points. But for most fields, it is not close to being surjective on k-points!

Why do we care about the orbits of k-group actions? One reason is the following.

Example 9.3.3. Let $f : G \to G'$ be a k-homomorphism of smooth k-groups of finite type. Let $G \curvearrowright X = G'$ by left translation through the homomorphism f. The orbit of the identity $e' \in G'(k)$ is precisely the (set theoretic) image of f, as a subset of G'. In this case, at least over $k = \overline{k}$, all the orbits have the same dimension, since they are translates of one another. So if we knew that orbits of minimal dimension are closed (as we will see shortly) then it follows that the set theoretic image of f is a closed subset of G', so it is the underlying space of the schematic image. But the schematic image is geometrically reduced, since G is smooth, and visibly a subgroup at the level of geometric points, so we conclude that the image of f is a smooth closed k-subgroup of G' (since any geometrically reduced k-group of finite type is smooth). See Corollary 9.3.6.

Remark 9.3.4. By the previous example, the image of a k-homomorphism of k-groups is a smooth closed k-subgroup. But there is much danger in conflating the k-points of the image with the image of the k-points – these are usually very different! For example, the projection $\pi : SL_n \twoheadrightarrow PGL_n$ is a degree n finite flat cover; in particular it is surjective. In some sense PGL_n deserves the name "SL_n / μ_n" because any μ_n-invariant map $SL_n \to X$ uniquely factors through π (proof later). But PGL_n is actually more like the sheafification of this quotient. However, traditionally this quotient is called PSL_n, which is "bad" notation because whereas the naive notion $PGL_n(k) := GL_n(k)/k^\times$ gives the points of the expected group scheme, $SL_n(k)/\mu_n(k)$ is usually not the k-points of anything interesting since as functor of the field k it tends to not even satisfy Galois descent. (The distinction is that Hilbert 90 holds for $\mathbf{G}_m$ but not for μ_n.)

Theorem 9.3.5 (Closed orbit lemma). *Let G be a smooth k-group of finite type*[6] *and let $\alpha : G \curvearrowright X$ be an action of G on a finite type k-scheme X. Let $x \in X(k)$ be a rational point, and $\alpha_x : G \to X : g \mapsto gx$ the orbit map. Then the set theoretic image of α_x is locally closed, and with the reduced induced scheme structure it is smooth. Moreover if $k = \overline{k}$ then the orbits of minimal dimension are closed.*

Before we prove this, we record:

Corollary 9.3.6. *Let $f : G \to G'$ be a k-homomorphism of k-groups of finite type, and suppose G is smooth. Then $f(G)$ is closed and smooth.*

Proof of corollary. By the closed orbit lemma, $f(G) \subset \overline{f(G)}$ is an inclusion of an open subset into a closed subscheme [this is a defining property of locally closed subschemes]. Since schematic image for a finite type map commutes with flat base change, the formation of both sides commutes with scalar extension of k (and equality can be tested after such an extension). So we can assume without loss of generality that $k = \overline{k}$, and thus the last part of the closed orbit lemma gives the result. □

[6]Not necessarily connected!

9.4 Start of proof of closed orbit lemma (Theorem 9.3.5)

The first step is to reduce to the case where k is separably closed. Because G is smooth, so reduced, the formation of $\overline{\alpha_x(G)}$ with the reduced induced scheme structure (i.e. the formation of the "schematic" or "scheme theoretic" image) is the same as the formation of the kernel of the map of sheaves $\mathcal{O}_X \to (\alpha_x)_*\mathcal{O}_G$. Since pushforward of quasicoherent sheaves commutes with flat base change, it commutes with extension of scalars. So the formation of the closure of the orbit of $x \in X(k)$ commutes with scalar extension.

The statement that the orbit $\alpha_x(G)$ itself is locally closed in G' is the same as the statement that the subset $\alpha_x(G) \subset \overline{\alpha_x(G)}$ is an open subset. The formation of the set theoretic image always commutes with scalar extension, in the appropriate sense that the set theoretic image $\alpha_x(G_{k'})$ is the preimage in $G'_{k'}$ of $\alpha_x(G) \subset G'$. Thus by Galois descent, it is enough to work over $k = k_s$.

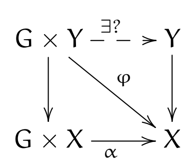
10 February 1

10.1 Conclusion of proof of closed orbit lemma (Theorem 9.3.5)

Above we reduced the proof to the case $k = k_s$. By Homework 2, problem 5(iii), we thus have $G(k)$ dense in G. By Theorem 3.2.1 this means that $G(k)$ is dense in G_K for all field extensions K/k.

The next step is to reduce to the case when the orbit map $\alpha_x : G \to X$ is dominant, so that $X = \overline{\alpha_x(G)}$ with the reduced structure.

Let $Y = \overline{\alpha_x(G)}$ with its reduced structure, i.e. the scheme theoretic image of α_x. Since $G(k) \subset G$ is dense, $Y = \overline{\alpha_x(G(k))}$. Moreover Y is geometrically reduced by Proposition 3.2.4. Finally, since $\alpha_x(G(k))$ is stable under translation by $G(k)$, so is Y; i.e. the group action $G(k) \curvearrowright X$ restricts to a group action $G(k) \curvearrowright Y$. To reduce to the case $X = Y$ we need to show that the action $\alpha : G \times X \to X$ also restricts to an action $G \times Y \to Y$, which is a much stronger statement:



The existence of such a dotted map can be checked by showing that $\varphi^\sharp : \mathcal{O}_X \to \varphi_*\mathcal{O}_{G\times Y}$ kills the quasicoherent ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_X$. To check this factorization we can extend scalars to $\overline{k}$. Since $Y_{\overline{k}}$ is reduced, as is $G_{\overline{k}} \times Y_{\overline{k}}$, it's enough to check on $\overline{k}$-points. The upshot is that it's enough to check that $G(\overline{k})$ acting on $Y(\overline{k}) \subset X(\overline{k})$ stays inside $Y(\overline{k})$; i.e., for all $y \in Y(\overline{k})$ we want $G(\overline{k})y \subset Y(\overline{k})$. In other words, for all $y \in Y(\overline{k})$ we want $\alpha_y : G_{\overline{k}} \to X_{\overline{k}}$ to land in the closed, reduced subscheme $Y_{\overline{k}}$. By topology, it's enough to check that $G(k) \subset G_{\overline{k}}$ (which is a dense subset) lands in $Y_{\overline{k}}$. Since $Y \subset X$ is $G(k)$-stable, so is $Y_{\overline{k}} \subset X_{\overline{k}}$. So we have completed the reduction.

Now we can assume X is geometrically reduced, and $\alpha_x : G \to X$ is dominant (in fact, $G(k)x \subset X(k)$ is dense). We need to prove $\alpha_x(G)$ is open and smooth to conclude the first statement of the closed orbit lemma. Actually, this will give the second statement as well. Because if α_x is not surjective, then its closed G-stable (!) complement has smaller dimension. If $k = \overline{k}$ we can choose a rational point in the complement and look at its orbit, which will thus have smaller dimension (since the complement of a dense open subset of a finite type k-scheme always has strictly smaller dimension). Thus the only way the orbit of x could have been of minimal dimension is if it is closed (since the above reduction steps replaced the original X with a certain *closed* subset).

To prove that the subset $\alpha_x(G) \subset X$ is open, first observe that since $k = k_s$, $\pi : X_{\overline{k}} \to X$ is a homeomorphism, and $\pi^{-1}\alpha_x(G) = (\alpha_x)_{\overline{k}}(G_{\overline{k}}) \subset X_{\overline{k}}$ (exercise). So it's enough to work over $k = \overline{k}$.

Chevalley's constructibility theorem says that $\alpha_x(G)$ is constructible in X. We've rigged it to be dense. A dense constructible set contains a dense open set of the ambient space. (Obvious if you think about constructible sets as finite unions of locally closed subsets.) Now over $k = \overline{k}$, to show a constructible set $\Sigma \subset X$ is open, it's enough to show that $\Sigma(k) \subset X(k)$ is open in the Zariski topology. Since $k = \overline{k}$, $\alpha_x(G)(k) = \alpha_x(G(k)) = G(k)x \subset X(k)$. Since $\alpha_x(G)(k)$ contains $U(k)$ for a dense open $U \subset X$, and since $G(k)x$ is homogeneous, we conclude that $\alpha_x(G)(k) = \bigcup_{g\in G(k)} gU(k)$, and this is visibly open in $X(k)$.

It remains only to prove that the open subset $\alpha_x(G) \subset X$ is smooth with the open subscheme structure. Since $\alpha_x(G) = \overline{\alpha_x(G(k))}$ is geometrically reduced, it is smooth on a dense open. Consider $\alpha_x : G \twoheadrightarrow \alpha_x(G)$. If we base change to $\overline{k}$ we obtain $(\alpha_x)_{\overline{k}} : G_{\overline{k}} \twoheadrightarrow (\alpha_x(G))_{\overline{k}}$, where the target is the image of the orbit map for x on $X_{\overline{k}}$. Hence without loss of generality we can assume $k = \overline{k}$. Now $G(k) \to \alpha_x(G)(k)$ is a surjection, i.e. $\alpha_x(G)(k) = G(k)x$, so $G(k)$ acts transitively on $\alpha_x(G)(k)$, so if there exists one smooth k-point then all k-points are smooth. (Alternately, the $G(k)$-orbits of a smooth dense open cover the space.) This completes the proof of the closed orbit lemma.

Example 10.1.1. Let G be a smooth k-group of finite type, and suppose we have a representation, i.e. a k-homomorphism $G \to \mathrm{GL}(V)$ for a finite dimensional k-vector space V. Then the image is a smooth closed k-subgroup of $\mathrm{GL}(V)$.

10.2 Criterion for a k-homomorphism of k-groups to be a closed immersion

Proposition 10.2.1. *Let $f : G \to G'$ be a homomorphism of finite type k-groups. Suppose G is smooth. Then $T_e \ker f = \ker(T_e f)$ and the following are equivalent:*

(i) $\ker f = 1$ *(trivial as a scheme, not just rationally!).*

(ii) f *is injective on geometric points and* $T_e(f)$ *is injective.*

(iii) f *is a closed immersion.*

We will prove this next time. Note that there are no smoothness hypotheses on $\ker f$.

11 February 3

11.1 Proof of Proposition 10.2.1

Remark 11.1.1. In practice, condition (iii) of Proposition 10.2.1 is the desired conclusion; often (ii) is easiest to check. However, to prove that a smooth linear algebraic k-group is a closed k-subgroup of some GL_n, we will actually verify (i).

First we will prove the claim about the tangent space of the kernel. Let $H = \ker f$. Then we have a commutative diagram

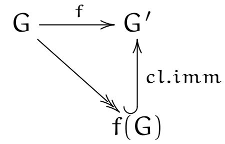
$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & T_e(\ker f) & \longrightarrow & T_e G & \xrightarrow{T_e(f)} & T_e G' \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H(k[\epsilon]) & \longrightarrow & G(k[\epsilon]) & \longrightarrow & G'(k[\epsilon]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H(k) & \longrightarrow & G(k) & \longrightarrow & G'(k)
\end{array}$$

The columns and the bottom two rows are exact. By a diagram chase, the top row is exact. By Homework 4, the group structures on the top rows are really the (additive structures of) the vector space structures of the tangent space. So we are done.

(iii)⇒(ii) is clear.

(ii)⇒(i): As above set $H = \ker f$. By hypothesis $H(\overline{k}) = 1$, so H is artin local; say $H = \operatorname{Spec} A$ for an artin local ring $(A, \mathfrak{m})$ with residue field k. (The closed point is rational because we know H contains the identity!) In particular, the maximal ideal $\mathfrak{m}$ is nilpotent. By the injectivity of $T_e f$, we have $(\mathfrak{m}/\mathfrak{m}^2)^* = T_e(H) = \ker(T_e f) = 0$ by the above. So $\mathfrak{m} = \mathfrak{m}^2$ and hence (by nilpotence) $\mathfrak{m} = 0$, so $H = \operatorname{Spec} k$ is just the identity with no fuzz.

(i)⇒(iii): This is the interesting part. We have a diagram



The image $f(G)$ [set theoretic!] is a closed k-subgroup of G' (smooth, with the reduced structure) by the Closed Orbit Lemma 9.3.5, or more precisely by Corollary 9.3.6. The map $G \twoheadrightarrow f(G)$ has trivial kernel, because functorially we know its kernel is contained in $\ker f$, which is trivial by hypothesis. Hence it is enough to show that $G \twoheadrightarrow f(G)$ is an isomorphism.

So we have a new setup; the proof of the following claim will complete the proof of Proposition 10.2.1.

Lemma 11.1.2. *Let* $f : G \twoheadrightarrow G'$ *be a surjective* k*-homomorphism of smooth, finite type* k*-groups, with* $\ker f = 1$*. Then* f *is an isomorphism.*

Proof. We can assume without loss of generality (Homework 2) that $k = \overline{k}$.

Both G and G' are equidimensional because they are groups, so homogeneous. Say G has pure dimension d, and G' has pure dimension d'. Consider the map $G^0 \to G'^0$. Since G has finitely many connected (= irreducible!) components, it's automatic that $G^0 \twoheadrightarrow G'^0$ is surjective; this follows from the closedness of the image and dimension considerations, since if G^0 didn't fill up G'^0 its image would have strictly smaller dimension than d', and hence the finitely many translates of the image, i.e. the entirety of $f(G)$, could not fill up G'^0, a contradiction since f is surjective.

All the fibers of $G^0 \twoheadrightarrow G'^0$ over k-points are one-point sets, because they are conjugates of $\ker f = 1$. Because fiber dimension behaves correctly over a dense open subset of the base, and because $k = \overline{k}$ we can find a rational point in such a dense open, it follows that $d = d'$.

Because k is algebraically closed, the finiteness of the k-fibers thus entails the finiteness of all fibers; if $x \mapsto y$ we therefore have $[k(x) : k(y)] < \infty$, and hence $\mathrm{trdeg}_k k(x) = \mathrm{trdeg}_k k(y)$ over k. Consequently the generic points of G surject onto the generic points of G', and this restricted set-map gives us the full fibers of f over the generic points.

Claim 11.1.3. *There exists a dense open* $U' \subset G'$ *such that* $f^{-1}U' \twoheadrightarrow U'$ *is finite flat.*

Proof of claim. We use the method of "spreading out" from generic points. Pass to the local ring $\mathcal{O}_{G',\eta'} = k(\eta')$ for a generic point $\eta' \in G'$. Then $\varnothing \neq \coprod_{\eta \in f^{-1}\{\eta'\}} \mathrm{Spec}\, k(\eta) = f^{-1}(\eta') \to \{\eta'\}$ is finite, because is it quasifinite and quasifinite implies finite over a field. (i.e., the fiber $f^{-1}\eta'$ is a localization of $\mathcal{O}_G$, so it is just the reduced product of the function fields of the generic points of G over η'.) Finite implies finite flat, over a field. By general nonsense, viewing $\mathcal{O}_{G',\eta'} = \varinjlim_{\text{affine } U' \ni \eta'} \mathcal{O}_{G'}(U')$, the property of finite flatness thus spreads out to an actual open set containing η'. Doing this for all the generic points η' of G' gives the claim. (Actually all we'll need is any nonempty open set, but we might as well get a dense one.) ☐

Returning to the proof of the lemma, we will now use the group structure to translate the finite flatness of f to the whole of G'.

Since $k = \overline{k}$ we can cover G' by $G'(k)$-translates over U'. Moreover, again because $k = \overline{k}'$, $G'(k) = f(G(k))$ since a surjective map of varieties over an algebraically closed field is surjective on closed points. The property of being finite flat of a given rank is local on the base, and passes through pullback along an automorphism of the base. So these translations by points of $f(G(k))$ give that $f : G \twoheadrightarrow G'$ is actually finite flat of constant rank r. Since we know $f^{-1}(e') = \ker f = \mathrm{Spec}\, k$ by hypothesis (i), in fact the rank $r = 1$. In other words, $f_*\mathcal{O}_G$ is an invertible sheaf on G'.

Since finite morphisms are affine, we are now reduced to the following algebra problem:

Claim 11.1.4. *If* $A \to B$ *is a ring map making* B *an invertible* A*-module, then it is an isomorphism.*

Proof. This can be checked on stalks on $\mathrm{Spec}\, A$. At a prime $\mathfrak{p} \in \mathrm{Spec}\, A$ the map $A_\mathfrak{p} \to B_\mathfrak{p}$ makes $B_\mathfrak{p}$ a free $A_\mathfrak{p}$-module of rank 1. On the other hand, it is a local ring homomorphism, and in particular $1 \mapsto 1$. Since $B_\mathfrak{p} \neq 0$ (it is of rank 1) we must have $1 \neq 0$. Therefore the image of 1 generates the 1-dimensional

$A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$-vector space $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. Thus by Nakayama $A_\mathfrak{p} \to B_\mathfrak{p}$ is surjective, and in fact $B_\mathfrak{p}$ is generated over $A_\mathfrak{p}$ by 1. Since $B_\mathfrak{p}$ is free, this means the map is injective as well. □

The previous claim completes the proof of the lemma. □

This finishes the proof of Proposition 10.2.1.

Remark 11.1.5. (i)⇔(iii) is actually true *sans* smoothness hypotheses; see [SGA3, VI_A, 1.4.2]. In the following when we parenthesize a hypothesis (smooth) we mean that the result is true without that hypothesis, but for our purposes we only need the smooth case, and that it rests upon Proposition 10.2.1 (which we only proved in the smooth case, but is true more generally).

11.2 Embedding linear algebraic groups in GL_n

Does a (smooth) affine k-group G of finite type admit a closed immersion into GL_n as a k-subgroup for some n? The answer is yes. The method will be to study "G acting on the coordinate ring $k[G]$".

We have the following.

Corollary 11.2.1 (of Proposition 10.2.1)**.** *Let G be a (smooth) k-group of finite type. Then the data of closed immersion $\rho : G \hookrightarrow \mathrm{GL}(V)$ which is a k-homomorphism is equivalent to the data of a faithful*[7] *R-linear action of $G(R)$ on V_R, functorial in R.* □

We will find the desired representation (V, ρ) inside $k[G]$.

Remark 11.2.2. "$\mathrm{GL}(W)$" doesn't really make sense if $\dim W = \infty$.

To get around the infinite dimension issue, we make the following definition.

Definition 11.2.3. A *functorial linear representation* of a k-group G on a (possibly infinite dimensional) k-vector space V is an R-linear action of $G(R)$ on V_R, which is functorial in the k-algebra R. That is, it is a map of group functors $\underline{G} \to \underline{\mathrm{Aut}}(V)$. We say a subspace $W \subset V$ of a functorial linear representation V of G, is G-*stable* if the action of $G(R)$ on V_R restricts to a functorial action of $G(R)$ on W_R for all R.

Example 11.2.4. If X is affine and $G \times X \to X$ is a left action, we have a group action of $G(R) \curvearrowright X_R$ for all R, and hence a *right* action $G(R) \curvearrowright \mathcal{O}(X_R) = \mathcal{O}(X) \underset{k}{\otimes} R$. We turn this into a functorial linear representation (i.e. a left action) by setting $g.f = f \circ (\mathrm{act}_{g^{-1}})$.

Example 11.2.5. Let $X = G$ be affine in the previous example, with the action of left multiplication λ. Then $g.f = f \circ \lambda_{g^{-1}}$ for $g \in G(R), f \in R[G]$.

12 February 5

12.1 Actions of affine finite type k-groups on coordinate rings of affine k-schemes

Consider the following setup. Let G be an affine k-group of finite type, and $\alpha : G \times X \to X$ a left action on an affine k-scheme X.

This implies that $V = k[X]$ is a functorial linear representation of G in the sense of Definition 11.2.3; explicitly, $g \in G(R)$ acts on $f \in R[X] = R \underset{k}{\otimes} \mathcal{O}_X(X)$ by $(g.f)(x) = f(g^{-1}x) = f(\alpha(g^{-1}, x))$ for x any point of $X_R(A)$ and A any R-algebra.

Theorem 12.1.1. *The G-stable finite dimensional k-subspaces $W \subset V = k[X]$ form a directed system under inclusion and exhaust V.*

Remark 12.1.2. As usual, it's worth emphasizing that the G-stability of $W \subset V$ is much stronger than saying that W is $G(k)$-stable. It says rather that W_R is $G(R)$-stable in V_R for all k-algebras R.

[7]Meaning $(\ker \rho)(R) = 1$ for all k-algebras R.

12.1.1 Application to embedding a smooth linear algebraic group into GL_n

Take $X = G$ and α left multiplication, so that $f \in R[G], g \in g(R)$, we have $(g.f)(g') = f(g^{-1}g')$ for $g' \in G(R')$, R' an R-algebra.

Choose a finite set of k-algebra generators for $k[G]$. By Theorem 12.1.1 there exists a G-stable finite dimensional subspace $W \subset k[G]$ containing all those generators. Consider the resulting k-homomorphism $\rho : G \to \mathrm{GL}(W)$ arising from the action of $G(R)$ on W_R functorially in the k-algebra R.

Claim 12.1.3. $\ker \rho = 1$*, i.e.* ρ *is injective on* R*-points for all* R.

Proof. We want to prove that $G(R)$ acts on W_R faithfully, i.e. if $g \in G(R)$ acts trivially on W_R then $g = 1$. Observe that $g \curvearrowright R[G] = V_R$ as an R-*algebra* automorphism. But W_R contains a set of R-algebra generators, so $g \curvearrowright R[G]$ trivially. In other words, $f(g') = f(g^{-1}g')$ for all $f \in R[G], g' \in G(R'), R' \in \mathrm{Alg}/R$. Take $g' = 1, R' = R$; this gives $f(g^{-1}) = f(1)$ for all $f \in R[G]$. But this means that g^{-1} corresponds to the map of rings $(g^{-1})^* : R[G] \to R$ given by $f \mapsto f(g^{-1}) = f(1)$; thus g^{-1} coincides with e as maps $\mathrm{Spec}\, R \to G_R$, since they coincide on coordinate rings. [8] Therefore $g^{-1} = e$ so $g = e$. ☐

The previous claim immediately yields the following, via Proposition 10.2.1.

Corollary 12.1.4. *If* G *is a (smooth)*[9] *finite type affine* k*-group, then there exists a* k*-homomorphism* $\rho : G \hookrightarrow \mathrm{GL}(W)$ *that is a closed immersion, with* W *some finite-dimensional* k*-vector space.* ☐

12.2 Proof of Theorem 12.1.1

Observe that if $W_1, \ldots, W_n \subset k[X] = V$ are G-stable subspaces, so is their sum $\sum W_i \subset V$. So it's enough to pick any $w \in V$ and show that there exists a finite-dimensional G-stable subspace $W \subset V$ such that $w \in W$.

The action $\alpha : G \times X \to X$ corresponds to a map $\alpha^* : k[X] \to k[G] \otimes_k k[X]$. Let $\alpha^*(w) = \sum f_i \otimes h_i$; we can arrange so that the set $\{f_i\} \subset k[G]$ is k-linearly independent. Concretely, if R is a k-algebra, R' is an R-algebra, $g \in G(R), x \in X(R')$, then we have

$$(g.w)(x) = w(g^{-1}.x) = w(\alpha(g^{-1}, x)) = (\alpha^* w)(g^{-1}, x) = \sum f_i(g^{-1})h_i(x);$$

by Yoneda's lemma, this implies that $g.w = \sum f_i(g^{-1})h_i \in R[X]$ since they have the same values at all R-algebra valued points x.

By the previous calculation, for example, we have $w = 1.w = \sum f_i(1)h_i \in W$.

Claim 12.2.1. $W = \mathrm{Span}\{h_i\} \subset V$ *is* G*-stable.*

Consider the following diagram.

$$\begin{array}{ccc} k[X] & \xrightarrow{\alpha^*} & k[G] \otimes_k k[X] \\ \uparrow & & \uparrow \\ W & \overset{\exists ?}{\dashrightarrow} & k[G] \otimes_k W \end{array}$$

Suppose the dotted map exists. Then for all $w' \in W$ and $g \in G(R)$, $g.w' = \sum (R-\mathsf{coeff.s}) \cdot w_i$ for $w_i \in W$; so $G(R) \cdot W_R \subset W_R$, which proves G-stability as claimed.[10]

To prove the existence of such a map, we will use the associativity of the left action α:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{1 \times \alpha} & G \times X \\ {\scriptstyle m \times 1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times X & \xrightarrow{\alpha} & X \end{array} \qquad (\dagger)$$

[8] This is where we use that G is affine, crucially!

[9] cf. Remark 11.1.5

[10] The G-stability of W also *implies* the existence of a dotted map, but we won't need this.

We need to compute $\alpha^*(h_i) \in k[G] \otimes_k k[X]$; we hope $\alpha^*(h_i) = \sum_j \varphi_{ij} \otimes h_i$. We will instead work inside $k[G] \otimes_k k[G] \otimes_k k[X]$. We compute:

$$\begin{aligned}
\sum f_i \otimes \alpha^* h_i &= (\mathbf{1} \otimes \alpha^*) \left(\sum f_i \otimes h_i\right) \\
&= (\mathbf{1} \otimes \alpha^*)(\alpha^* w) \\
&\overset{(\dagger)}{=} (m^* \otimes \mathbf{1})(\alpha^* w) \\
&= (m^* \otimes \mathbf{1}) \left(\sum f_i \otimes h_i\right) \\
&= \sum (m^* f_i) \otimes h_i.
\end{aligned}$$

Now $k[G]$ has a k-basis $\{f_j\} \sqcup \{b\}_{b \in B}$, since we chose the $\{f_j\}$ to be linearly independent. Expand $m^* f_i$ with respect to this basis in the first factor:

$$m^*(f_i) = \sum_j f_j \otimes \varphi_{ij} + \sum_{b \in B} b \otimes \varphi_{ib}.$$

Then the computation above gives

$$\sum_i f_i \otimes \alpha^* h_i = \sum_j f_j \otimes \left(\sum_i \varphi_{ij} \otimes h_j\right) + \sum_b b \otimes \left(\sum_i \varphi_{ib} \otimes h_i\right).$$

But $\{f_i\} \sqcup \{b\}$ is a basis. Hence comparing both sides of the equation above says that $\sum_b b \otimes (\sum_i \varphi_{ib} \otimes h_i) = 0$ and $\alpha^* h_i = \sum_j \varphi_{ji} \otimes h_j$ for each i, as desired. This completes the proof of the claim above, and hence of Theorem 12.1.1. □

13 February 8

13.1 Jordan decomposition

For $g \in \mathrm{GL}_n(\overline{k})$ we have a **multiplicative Jordan decomposition**:

$$g \overset{!}{=} g_{ss} g_u \overset{\star}{=} g_u g_{ss}$$

where g_{ss} is semisimple as an operator on $\overline{k}^n$ (which is the same as diagonalizable, since we are over an algebraically closed field) and g_u is unipotent (meaning that $g_u - 1 \in \mathrm{Mat}_n(\overline{k})$ is nilpotent).

Remark 13.1.1. If k is perfect then $T \in \mathrm{End}_k(V)$ [for a finite dimensional k-vector space V] is semisimple if and only if $T_{\overline{k}} \curvearrowright V_{\overline{k}}$ is semisimple.

Remark 13.1.2. We used the identification $\mathrm{GL}_n = \mathrm{Mat}_n^\times$ to define unipotence.

We would like to generalize the Jordan decomposition to any smooth affine k-group, functorially in G.

Remark 13.1.3. Due to the uniqueness of Jordan decomposition over $k = \overline{k}$, by Galois descent we obtain a multiplicative Jordan decomposition in $\mathrm{GL}_n(k)$ for any *perfect* field k; this is also invariant under further extension of the ground field. However a good theory of Jordan decomposition for imperfect fields is basically hopeless, since while one can (in fact) prove that such a decomposition exists and is unique, a semisimple operator on a vector space over an imperfect field may no longer be semisimple after making an inseparable field extension; see the second part of the handout on Jordan decomposition for more on this. Thus in the imperfect case, the Jordan decomposition is not compatible with scalar extension and hence works poorly.

The idea for how to obtain our generalization is to make use of a closed embedding $i : G \hookrightarrow \mathrm{GL}_n$, which we know exists by our previous results. Take $g \in G(\overline{k})$, which gives $i(g) \in \mathrm{GL}_n(\overline{k})$, and hence $i(g)_{ss}, i(g)_u \in \mathrm{GL}_n(\overline{k})$ which are semisimple and unipotent, respectively.

There are two problems:

(i) We must show $i(g)_{ss} = i(g_{ss}), i(g)_u = i(g_u)$ for some $g_{ss}, g_u \in G(\bar{k})$, independent of the choice of embedding i.

(ii) We must establish the functoriality of this construction, and hopefully find an internal characterization which does not reference the GL_n at all.

To avoid the issues of imperfect fields, assume throughout the following that $k = \bar{k}$. (At the end we will make definitions over k by passage to $\bar{k}$.) For $g \in G(k)$, $g \curvearrowright G$ by *right* translation ρ_g; this gives rise to the **right regular representation** $\rho_g : k[G] \xrightarrow{\sim} k[G]$ which sends a function f to $\rho_g f$ where $(\rho_g f)(x) = f(xg) = f \circ \rho_g(x)$ for $x \in G(R)$ for any k-algebra R; this is a left action, as one can easily compute.

Now $k[G] = \varinjlim W$ where the filtered direct limit is over finite-dimensional $G(k)$-stable subspaces W. The action ρ_g restricts to an action $\rho_g|_W \in \mathrm{GL}(W)(k)$ on each piece W. Thus we obtain a multiplicative Jordan decomposition

$$\rho_g|_W = (\rho_g)_{ss,W} \cdot (\rho_g)_{u,W}$$

into commuting semisimple (resp. unipotent) operators on W. To work with this as W varies, we now briefly digress for some elementary generalities.

Let T_{ss}, T_u be a semisimple and a unipotent operator, respectively, on a vector space V. If V' (resp. V'') is a T-invariant subspace (resp. quotient with an induced T-action), then the restriction of the T-action (resp. induced T-action) on V' (resp. V'') is also semisimple or unipotent for $T = T_{ss}$ or $T = T_u$. In a pithy catchphrase, semisimplicity and unipotence are well-behaved with respect to subquotients. Obviously if two operators on V commute then the restricted (resp. induced) operators on V' (resp. V'') also commute.

Since Jordan decomposition is unique, we conclude that $(\rho_g)_{ss,W'} = (\rho_g)_{ss,W}|_{W'}$ and $(\rho_g)_{u,W'} = (\rho_g)_{u,W}|_{W'}$ for any pair of finite-dimensional $G(k)$-stable subspaces $W' \subset W \subset k[G]$. In other words, the construction of $(\rho_g)_{ss,W}$ and $(\rho_g)_{u,W}$ is *compatible with change in* W.

By this compatibility, we can pass to the direct limit and conclude that ρ_g itself factors as a product $\rho_g = (\rho_g)_{ss}(\rho_g)_u$ of commuting operators on $k[G]$ with semisimple (resp. unipotent) restriction to any finite dimensional $G(k)$-stable subspace W.

We now make a temporary definition.

Definition 13.1.4. Say that g is *right semisimple* (resp. *right unipotent*) if $\rho_g = (\rho_g)_{ss}$ (resp. $\rho_g = (\rho_g)_u$).

Remark 13.1.5. Observe that g is right semisimple (resp. right unipotent) if and only if $\rho_g|_W = (\rho_g)_{ss,W}$ (resp. $\rho_g|_W = (\rho_g)_{u,W}$) for a single finite dimensional $G(k)$-stable subspace $W \subset k[G]$ containing algebra generators for $k[G]$ over k. To see this, recall that semisimplicity (resp. unipotence) is inherited by tensor products, direct sums, and quotients thereof for the spaces in question, and we have a surjection $\mu : \mathrm{Sym}^{\cdot} W \twoheadrightarrow k[G]$ given by multiplication, when W is as just specified. Since ρ_g acts by k-algebra automorphisms on $k[G]$, so μ is ρ_g-equivariant, semisimplicity (resp. unipotence) of $\rho_g|_W$ passes to its action on $W^{\oplus r}$, hence to the tensor algebra, hence to its quotient the symmetric algebra, hence to its quotient $k[G]$.

Proposition 13.1.6. *Let* $G = \mathrm{GL}_n$. *Then* $g \in \mathrm{GL}_n(\bar{k})$ *is semisimple (resp. unipotent) if and only if* g *is right semisimple (resp. right unipotent).*

Sketch, cf. Borel, Ch. I, §4.3. We have $k[G] = k[\mathrm{End}(V)][\frac{1}{\det}] = \sum_{n \in \mathbf{Z}} k[\mathrm{End}\, V] \cdot \det^n$ as a "g-module"; that is, $\mathrm{GL}(V) \subset \underline{\mathrm{End}}(V)$ is an open subfunctor and the action of ρ_g is the restriction of the action – also denoted ρ_g – of g on $\underline{\mathrm{End}}(V)$ by the same formula (multiplication of matrices on the right). Inside $k[G]$, $\det^n$ is a ρ_g-eigenvector with eigenvalue $\det(g)^n \in k^\times$. One can show that g is right-semisimple (resp. right-unipotent) if and only if ρ_g acting on $k[\mathrm{End}(V)]$ is semisimple (resp. unipotent) on finite-dimensional $G(k)$-stable subspaces of $k[\mathrm{End}(V)]$. The crucial idea here is the eigenvector property of the $\det^n$'s. See Borel's book for details.

Now $\mathrm{End}(V) = V \otimes_k V^*$. Thus $k[\mathrm{End}(V)] = \mathrm{Sym}^{\cdot}(V \otimes V^*)$. We wish to describe the right regular action ρ_g of g on $\mathrm{End}(V) \xleftarrow{\sim} V \otimes_k V^*$. First, some generalities. If W is a finite dimensional k-vector space and $\mathbf{A}(W)$ is the associated affine space $R \mapsto W_R$, then $A = \mathrm{Spec}\,\mathrm{Sym}\, W^*$; note the duality, which is essential for the variance to be correct. If $W = \mathrm{End}(V)$ the coordinate ring $\mathbf{A}(W)$ is thus $k[\mathrm{End}(V)^*]$; the right-translation action by $\mathrm{GL}(V)$ is given by $(g^*.\varphi)(T) = \varphi(Tg)$ for $g \in \mathrm{GL}(V)(k), \varphi \in \mathrm{End}(V)^*$. If we identify $\mathrm{End}(V)^*$ with $V \otimes V^*$ via $v \otimes \ell \mapsto \varphi_{v\otimes\ell} = [T \mapsto \ell(Tv)]$ then this action is $g^*\varphi_{v\otimes\ell}(T) = \varphi_{v\otimes\ell}(Tg) = \ell(Tgv) = \varphi_{gv\otimes\ell}(T)$;

i.e. $g^*\varphi_{v\otimes\ell} = \varphi_{gv\otimes\ell}$. Hence the compatible GL(V)-action on $V \otimes V^*$ is $g \otimes \mathbf{1}$ (which is a left action, as it should be).

Consequently ρ_g is semisimple (resp. unipotent) on $k[\mathrm{End}(V)]$, if and only if it is such on End(V) if and only if $g \otimes \mathbf{1}$ is such on $V \otimes V^*$, if and only if g is such on V. □

The previous proposition is by way of motivation for the main theorem of today.

Theorem 13.1.7. *Let G be a smooth affine k-group, $k = \overline{k}$, $g \in G(k)$, $j : G \hookrightarrow \mathrm{GL}_n$ a closed immersion which is a k-homomorphism. Consider ρ_g acting on $k[G]$ and the Jordan decomposition $j(g) = j(g)_{ss}j(g)_u \in \mathrm{GL}_n(k)$. Then $j(g)_{ss} = j(g_{ss})$ and $j(g)_u = j(g_u)$ for some $g_{ss}, g_u \in G(k)$. Moreover, $\rho_{g_{ss}} = (\rho_g)_{ss}$ and $\rho_{g_u} = (\rho_g)_u$ as operators on $k[G]$, g_{ss} and g_u are independent of j, and the formation of g_{ss} and g_u is functorial in G.*

In particular, $g = g_{ss}$ (resp. $g = g_u$) if and only g is right semisimple (resp. right unipotent).

Proof. In a handout on Jordan decomposition it is proved (conditional on Theorem 14.1.1 to be discussed next time) that g_{ss} and g_u exist in $G(k)$ giving rise to $j(g)_{ss}$ and $j(g)_u$ respectively. Note that the actions of $\rho_{j(g)}$ and ρ_g on $k[\mathrm{GL}_n]$ and $k[G]$ respectively are compatible with the surjection j^*, so $\rho_{j(g)}$ determines ρ_g. Since the formation of Jordan decomposition is compatible with passage to quotients, and the operators $\rho_{j(g)_{ss}}$ and $\rho_{j(g)_u}$ are the "Jordan components" of $\rho_{j(g)}$ (due to the unique characterization of Jordan decomposition in terms of commuting semisimple and unipotent operators), we conclude that $\rho_{g_{ss}}$ and ρ_{g_u} are the respective semisimple and unipotent Jordan components of ρ_g on $k[G]$. This gives an intrinsic characterization of $\rho_{g_{ss}}$ and ρ_{g_u} in terms of $g \in G(k)$ without reference to j, to we conclude that $\rho_{g_{ss}}$ and ρ_{g_u} are independent of j, so likewise for g_{ss} and g_u (indeed, ρ_h determines h, since as an endomorphism of the variety G it carries e to h).

It remains to consider the issues of functoriality. That is, if $f : G \to G'$ is a k-homomorphism between smooth affine k-groups and $g \in G(k)$ then we claim that $f(g)_{ss} = f(g_{ss})$ and $f(g_u) = f(g)_u$. By factoring f into a surjection onto a smooth closed subgroup, it suffices to separately treat the cases when f is surjective and when f is a closed immersion. The closed immersion case is immediate from the "independence of j" established above. If instead f is surjective then $k[G'] \hookrightarrow k[G]$ via f^*, and this maps $k[G']$ a $G(k)$-stable subspace via the right regular action. More specifically, ρ_g on $k[G]$ restricts to $\rho_{f(g)}$ on the subspace $k[G']$. Since Jordan decomposition passes to subspaces, it follows that the semisimple and unipotent parts of $\rho_{f(g)}$ are respectively obtained by restriction to $k[G']$ of the semisimple and unipotent parts of ρ_g. In other words, $\rho_{f(g)_{ss}} = \rho_{g_{ss}}|_{k[G']} = \rho_{f(g_{ss})}$ and similarly for the unipotent parts. It follows that $f(g_{ss}) = f(g)_{ss}$ and $f(g_u) = f(g)_u$. □

14 February 10

14.1 All subgroups of a smooth finite type k-group are stabilizers of a line

To tie up a loose end in the proof of Theorem 13.1.7, and more specifically to fill in a key step used in the handout referenced in that argument, we need the following result, somewhat remarkable for its extraordinary generality and usefulness.

Theorem 14.1.1. *Let $j : G \hookrightarrow G'$ be a closed k-subgroup scheme of a (smooth) affine k-group G' of finite type. Then there exists a k-linear representation $\pi : G' \to \mathrm{GL}(V)$ and a line $L \subset V$ such that $G = N_{G'}(L)$.*

Proof. Let $I = \ker(j^* : k[G'] \twoheadrightarrow k[G])$. The group G acts on G' by right translation, through j. The induced action on $k[G']$ is compatible (equivariant) with the right translation action on $k[G]$. Hence I is a G-stable subspace of $k[G']$. Now $k[G'] = \varinjlim V$ is the rising union of finite dimensional G'-stable (hence G-stable) subspaces V. Since $k[G']$ is Noetherian, I is finitely generated, so we can choose V once and for all to be a finite dimensional G-stable subspace of $k[G']$ which contains ideal generators for I. Let $W = I \cap V$.

We will show that $G = \mathrm{Stab}_{G'}(W)$ for the induced action of G' on V.

Note that without loss of generality we can assume $I \neq 0$, hence $W \neq 0$, since if $I = 0$ then $G = G'$ and we can take any stupid representation of G' to fulfill the conditions of the theorem.

Consider $\pi : G' \to \mathrm{GL}(V)$.

Claim 14.1.2. $G = N_{G'}(W)$, *i.e. for all* k*-algebras* R, $g' \in G'(R)$ *lies in* $G(R)$ *if and only if* $g'(W_R) \subset W_R$.

Suppose $g'(W_R) \subset W_R$. Since W generates I and $\rho_{g'}$ acts on $k[G'_R]$ by an R-algebra automorphism, we have $\rho_{g'}(I_R) \subset I_R$. Hence $\rho_{g'}$ acting in G'_R preserves the closed subscheme G_R. Applying this to $e' = j(e)$, we see that $\rho_{g'}(j(e)) = g' \in j(G(R)) \subset G'(R)$.

Conversely if $g' \in G(R)$ then the action of $\rho_{g'}$ by right translation on G'_R preserves $G_R \subset G'_R$. Hence the action of $\rho_{g'}$ on $k[G'_R]$ preserves I_R. Since V_R is G'-stable, $\rho_{g'}$ also preserves V_R. Hence it preserves the intersection $V_R \cap I_R$. Since $k \to R$ is flat, $V_R \cap I_R = (V \cap I)_R = W_R$. Hence g' preserves W_R.

Thus the claim holds.

To deduce the theorem from the claim, we need only improve V to be a line. Set $d = \dim W > 0$. Take $\wedge^d \pi : G' \to \mathrm{GL}(\wedge^d V)$. Then (exercise) $N^{\pi}_{G'}(W) = N^{\wedge^d \pi}_{G'}(\wedge^d W)$. But $L = \wedge^d W$ is a line, so we are done. □

Remark 14.1.3. Use the direct sum of $\wedge^d \pi$ and any faithful G'-representation to ensure that G is the stabilizer of a line in a *faithful* G'-representation.

15 February 12

15.1 Unipotent groups

Definition 15.1.1. A smooth affine k-group U is *unipotent* if for all $g \in U(\bar{k})$, $g = g_u$ (where g_u is the unipotent factor in the Jordan decomposition).

Remark 15.1.2. It is possible to define unipotence for possibly non-smooth group schemes over a field k; see [SGA3, XVII, 1.3, 3.5(i)-(v)].

Example 15.1.3. The ur-unipotent group is

$$U_n = \left\{ \begin{pmatrix} 1 & & * & * \\ & 1 & & * \\ 0 & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_n .$$

A special case is U_2, which is easily seen to be isomorphic to $\mathbf{G}_a$.

Example 15.1.4. If U is unipotent, then any smooth closed k-subgroup $U' \subset U$ and any smooth image (quotient) $U \twoheadrightarrow U''$ are unipotent.

This is trivial in the subgroup case, since $U'(\bar{k}) \subset U(\bar{k})$. For quotients, use the functoriality of Jordan decomposition. Thus we see that if $u \in U(\bar{k})$ maps to $u'' \in U''(\bar{k})$, then $u'' = u''_u u''_{ss}$ where u''_u and u''_{ss} are the images of the unipotent and semisimple factors of u. But $u_{ss} = 1$, so $u''_{ss} = 1$, so $u'' = u''_u$.

Example 15.1.5. If $\mathrm{char}(k) = p > 0$ then the constant subgroup $\mathbf{Z}/p\mathbf{Z} \subset \mathbf{G}_a$ given as $\mathrm{Spec}\, k[t]/(t^p - t)$ is unipotent.

Example 15.1.6. A non-example is any k-torus T of positive dimension. For given any nontrivial $t \in T(\bar{k})$, since $T_{\bar{k}}$ is isomorphic to the standard diagonal torus in $\mathrm{GL}_{\dim T}$ and we can compute the Jordan decomposition of $t \in T(\bar{k}) = T_{\bar{k}}(\bar{k})$ with respect to that representation, so we see that $t_u = 1$. Thus in fact tori are *maximally* non-unipotent in some vague sense.

Remark 15.1.7. Later we'll see that for all smooth connected affine k-groups G such that $g = g_{ss}$ for all $g \in G(\bar{k})$, G is in fact a torus. See Homework 5 for the case where G is a priori assumed commutative.

Theorem 15.1.8. *Let* U *be unipotent. Then for any* k*-linear representation* $\rho : U \to \mathrm{GL}_n$*, some* $\mathrm{GL}_n(k)$*-conjugate* ρ' *of* ρ *satisfies* $\rho'(U) \subset U_n \subset \mathrm{GL}_n$.

We will prove this next time. Now we deduce some consequences.

Corollary 15.1.9. *If* U *is unipotent over* k *and* K/k *is any algebraically closed extension field, then* $g = g_u$ *for all* $g \in U(K)$. □

(This can of course also be seen more directly, by considering the characteristic polynomial of g.)

Corollary 15.1.10. *Suppose* U *is a finite unipotent group (hence smooth). If* $\mathrm{char}(k) = 0$ *then* U *is trivial, while if* $\mathrm{char}(k) = p > 0$ *then* $U(\overline{k})$ *is a* p*-group.*

Proof. Without loss of generality $k = \overline{k}$. By the theorem, $U(k)$ is a subset of $U_n(k)$. Now $U_n(k)$ has a composition series by subgroups which have zeroes on several superdiagonal rows, followed by arbitrary entries in a top-right corner triangle. The Jordan-Holder factors are diagonal lines which are easily seen to be isomorphic to products of the additive group k. [11] These Jordan-Holder factors are thus torsion-free in characteristic zero; hence the image of $U(k)$ in each is finite and torsion-free, so trivial. In characteristic p, the Jordan-Holder factors are exponent p. Hence $U(k)$ has a composition series with abelian p-group subquotients, so it is a p-group. $\square$

Corollary 15.1.11. *If* U *is unipotent and* $\mathrm{char}(k) = 0$ *then* U *is connected. If* $\mathrm{char}(k) = p > 0$*, the component group* $U(\overline{k})/U^0(\overline{k})$ *is a* p*-group.*

Proof. Without loss of generality $k = \overline{k}$. Let $\Gamma = U(k)/U^0(k)$, regarded as a finite constant k-group. Then $U \simeq \coprod_{\gamma\in\Gamma} \widetilde{\gamma}U^0$ as a scheme, where $\widetilde{\gamma} \in U$ as any k-point of the γ-component of U. Thus U has an evident k-map, and in fact a k-homomorphism $U \twoheadrightarrow \Gamma$, given by sending $\widetilde{\gamma}U^0$ to γ. This is a morphism of schemes, and even a homomorphism of group-schemes, because it is induced by translation from the constant map $U^0 \to \Gamma^0 = \{e_\Gamma\}$. Hence Γ is unipotent by Example 15.1.4. So we can apply Corollary 15.1.10 to deduce the result. $\square$

The main reason why unipotent groups are important is because they and their representations are easy to analyze, due to the filtration with additive subquotients. Moreover, an analysis of unipotent groups is complementary to an analysis of tori (which we will see also behave extremely well), because of the following miraculous result.

Theorem 15.1.12 (Big Miracle)**.** *Let* G *be a smooth connected affine* k*-group, for an arbitrary field* k*. If* G *is not unipotent, then* G *contains a nontrivial* k*-torus as a closed* k*-subgroup.*

This will be proved much later over $\overline{k}$ (Corollary 22.1.4), and lies quite deep over general fields: see Remark 1.4 in the handout on Grothendieck's theorem on tori for the descent from $\overline{k}$.

15.2 Proof of Theorem 15.1.8 on representations of unipotent groups

Write $\rho : U \to \mathrm{GL}(V)$ for the given representation. If $V = 0$, the claim is trivial; so we can assume $V \neq 0$. We seek a nonzero k-subspace $W \subset V$ such that W is U-fixed. Then we can conjugate ρ so that the image of U looks like

$$\begin{pmatrix} \mathbf{1} & ?_1 \\ 0 & ?_2 \end{pmatrix}$$

by taking the first several basis vectors to be a basis for W. We don't care about $?_1$. But now we can look at the induced U-action on V/W – i.e. at $?_2$ – and induct on dimension. Note that we set up Theorem 15.1.8 for all representations, not just faithful ones. This is good, because probably we lose faithfulness when passing to V/W.

By Homework #5, problem 2, for any linear representation of a smooth finite type k-group G on a vector space V, the functorially G-fixed vectors $\underline{V}^G \subset \underline{V} = \mathbf{A}(V)$ [the affine space corresponding to V, equipped with its natural G-action through ρ] constitute the points of a scheme of the form $\underline{W} = \mathbf{A}(W)$ for a unique linear subspace $W \subset V$, which we denote by V^G. So we just need to prove that $V^G \neq 0$ if G is unipotent and V is nonzero.

From the construction of V^G in Homework 5, or from the universal property of $\underline{V}^G$, we see that $(V^G)_K = V_K^{G_K}$ for any K/k field extension. Thus we can assume without loss of generality that $k = \overline{k}$. Now when $k = \overline{k}$, by a schematic density argument or by the construction in Homework 5, it is easy to see that $V^G = V^{G(k)}$.

So we are reduced to showing that $V^{G(k)} \neq 0$. Now since $V \neq 0$, it contains a nonzero irreducible $G(k)$-subrepresentation. Rename that as V. Hence we can assume without loss of generality that V is irreducible for $G(k)$.

We will now show that, in fact, $V = k$ with the trivial G-action, when G is unipotent. The key fact we need is the following.

[11] Once we do quotients, U itself will have a composition series with Jordan-Holder factors being powers of $\mathbf{G}_a$.

Theorem 15.2.1 (Wedderburn)**.** *If* $k = \overline{k}$ *and* $\Gamma \subset \mathrm{GL}(V)$ *is such that* V *is an irreducible* Γ*-representation, then* $\mathrm{End}_k(V)$ *is generated as a* k*-algebra by* Γ. ☐

Since Γ is a group, in fact this means that $\mathrm{End}_k(V)$ is spanned by Γ.

To apply the theorem, we take $\Gamma = \rho(G(k))$. Choose $g \in G(k)$. Note that by functoriality of Jordan decomposition, $\rho(g)$ is unipotent. [E.g. we can enlarge V to a faithful G-representation and take the Jordan decomposition there, where g is unipotent by assumption; hence g acts unipotently on the subrepresentation V.] So write $\rho(g) = \mathbf{1} + x$ for some $x \in \mathrm{End}_k(V)$, which in fact is nilpotent. Then for all $g' \in G(k)$ we have

$$\mathrm{tr}(x\rho(g')) = \mathrm{tr}((\rho(g) - \mathbf{1})\rho(g')) = \mathrm{tr}(\rho(gg')) - \mathrm{tr}(\rho(g)) = \dim V - \dim V = 0$$

since both $\rho(gg'), \rho(g)$ are unipotent on V. Since $\Gamma = \rho(G(k))$ spans $\mathrm{End}_k(V)$, it follows that $\mathrm{tr}(xy) = 0$ for all $y \in \mathrm{End}_k(V)$. But the trace pairing on $\mathrm{End}_k(V)$ is non-degenerate, so $x = 0$. Hence ρ is trivial, so by irreducibility $V = k$, and the theorem follows. ☐

15.3 Remaining ingredients necessary for structure theory

1. Commutator and derived subgroups "as algebraic groups" (in the smooth case). Note that these algebraic groups will not represent the "expected" thing at the level of all field-valued points, but they will be what one expects on geometric points.

2. Lie algebras.

3. Complete reducibility theorem for linear representations of split tori $\mathbf{G}_m^r$. (This is the analogue of Maschke's theorem for compact Lie groups, which implies complete reducibility for representations of "tori" $\mathbf{T}^r$ [$= (\mathbf{S}^1)^r$] in Lie theory.)

4. Coset spaces G/H for not-necessarily-smooth, not-necessarily-normal, not-necessarily connected closed subgroups $H \subset G$. This will use the closed orbit lemma, and it will allow us to show that, in a suitable sense, we have
$$\mathrm{PGL}_n = \mathrm{GL}_n / \mathbf{G}_m = \mathrm{SL}_n / \mu_n, \mathrm{GL}_2 / B = \mathbf{P}^1, \ldots.$$

16 February 17

16.1 Some motivation: a key calculation on $\mathrm{SL}_2(k)$

Proposition 16.1.1. *If* k *is a field other than* $\mathbf{F}_2$ *or* $\mathbf{F}_3$ *(which have the undesirable property that* $(\mathbf{F}_2^\times)^2 = 1, (\mathbf{F}_3^\times)^2 = 1$*) then* $\mathrm{SL}_2(k)$ *is its own commutator subgroup.*

To prove this, we need the following fact.

Lemma 16.1.2. *[L, Ch. XIII, Lemma 8.1] For any field* k*,* $\mathrm{SL}_2(k)$ *is generated by the upper and lower triangle unipotent subgroups*

$$U^+(k) = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \stackrel{\mathrm{def}}{=} x^+(a) : a \in k\}, \qquad U^-(k) = \{\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \stackrel{\mathrm{def}}{=} x^-(a) : a \in k\}. \quad \square$$

Note that $x^\pm : \mathbf{G}_a \xrightarrow{\sim} U^\pm$ is an isomorphism of algebraic groups.

Proof of Proposition 16.1.1. Let $D = \{\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \stackrel{\mathrm{def}}{=} \lambda(t)\}$ be the diagonal torus in SL_2. (So $\lambda : \mathbf{G}_m \to D$ is an isomorphism.)

The first claim is that D normalizes $U^\pm$, i.e. $D \subset N_{\mathrm{SL}_2}(U^\pm)$ as algebraic groups. This follows from the computation (valid over any k-algebra R)

$$\lambda(t)x^\pm(a)\lambda(t)^{-1} = x^\pm(t^{\pm 2}a).$$

This implies that
$$[\lambda(t), x^\pm(a)] = x^\pm((t^{\pm 2} - 1)a).$$

So as long as there exists $t \in k^\times$ such that $t^{\pm 2} - 1 \in k^\times$, i.e. $t \neq \pm 1$, any $x^\pm(a') \in U^\pm(k)$ is a commutator. So $U^\pm(k) \subset [\mathrm{SL}_2(k), \mathrm{SL}_2(k)]$, which, in light of the lemma, proves the proposition. ☐

Proposition 16.1.1 suggests that as a k-group, SL_2 should be **perfect**, i.e. equal to its own derived subgroup in some suitable sense yet to be defined. In particular $SL_2(K) = [SL_2(K), SL_2(K)]$ for any $K = \overline{K}$. The same should hold for any quotient, such as $SL_2 \twoheadrightarrow PGL_2$.

Remark 16.1.3. Over interesting large fields k which are not algebraically closed, $PGL_2(k)$ is not equal to its own commutator subgroup, because it has nontrivial commutative quotients. For example, $PGL_2(k) \twoheadrightarrow k^\times/(k^\times)^2$ via the determinant. Usually the target is nontrivial.

The principle is that we expect Zariski closed conditions to capture group-theoretic constructions [e.g. commutator subgroups] for K-points when $K = \overline{K}$, but not necessarily for k-rational points when $k \neq \overline{k}$.

The difficulty in establishing that this indeed is the case is "bounding the lengths of words". Roughly, if one knows that all the commutators can be realized as products of bounded length, then one has a hope of defining the commutator subgroup of two subgroups of G as an closed subgroup of G. This is certainly false in general; see Remark 16.2.3 below. To overcome the difficulty, we must lean upon connectedness hypotheses.

16.2 Subgroups generated by connected varieties

The key proposition for dealing with derived subgroups is the following.

Proposition 16.2.1. *Let* G *be a smooth* k*-group of finite type and* $\{f_i : X_i \to G\}$ *a collection of* k*-maps from geometrically integral finite type* k*-schemes* X_i*, such that* $e \in f_i(X_i)$ *for all* i*. Then*

(i) There exists a unique smooth closed k*-subgroup* $H \subset G$ *such that for all algebraically closed extension fields* K *of* k*, the* K*-points*

$$H(K) = \langle f_i(X_i(K)) \rangle_i$$

are the subgroup of G *generated by the images of the* K*-points of the* X_i*. Moreover,* H *is connected.*

(ii) There exists a finite sequence $X_{i_1}, \dots, X_{i_n}$ *(indices not necessarily distinct) and signs* $e_1, \dots, e_n \in \{\pm 1\}$ *such that*

$$X_{i_1} \times \cdots \times X_{i_n} \to H$$

defined by

$$(x_1, \dots, x_n) \mapsto f_{i_1}(x_1)^{e_1} \cdots f_{i_n}(x_n)^{e_n}$$

is surjective.

An important example of the setup above is the single map $G \times G \to G$ given by taking commutators. See Example 16.2.4 below. Here are some others.

Example 16.2.2. Let $H, H' \subset G$ be smooth connected closed k-subgroups of a smooth k-group G of finite type. Using the inclusions $H \hookrightarrow G, H' \hookrightarrow G$, Proposition 16.2.1 we obtain a smooth, connected closed k-subgroup $H \cdot H' \subset G$ which deserves to be called "the subgroup generated by H and H′"; it has the correct geometric points: for algebraically closed K/k we have $(H \cdot H')(K) = H(K) \cdot H'(K) \subset G(K)$.

Remark 16.2.3. In Example 16.2.2, H and H′ must be connected. Otherwise the conclusion is false. Take $G = SL_{2/\mathbf{Q}}$ and $H, H' \subset SL_2(\mathbf{Z}) \subset SL_{2/\mathbf{Q}}$ finite (disconnected) subgroups of orders 3 and 4 which generate $SL_2(\mathbf{Z})$. Thus the subgroup of $SL_{2/\mathbf{Q}}(\mathbf{C})$ generated by H and H′ is $SL_2(\mathbf{Z})$ which is not algebraic. (It is an infinite disjoint union of points.)

Example 16.2.4. Assume G is smooth, and for now assume G is connected (but see Example 16.4.4 below). Then taking

$$[\cdot, \cdot] : G \times G \to G$$

in Proposition 16.2.1 yields a smooth connected closed k-subgroup $\mathcal{D}G \subset G$, such that for $K = \overline{K}$ over k we have

$$(\mathcal{D}G)(K) = [G(K), G(K)].$$

Example 16.2.5. Let $G = \mathrm{GL}_n$. We have

$$\mathrm{SL}_n = \ker(\mathrm{GL}_n \stackrel{\det}{\to} \mathbf{G}_m).$$

Now $\mathcal{D}\,\mathrm{GL}_n$ must map trivially to $\mathbf{G}_m$ under det, because everything is smooth, so we can compute on geometric points, and we know that commutators die when mapped to an abelian group. So $\mathcal{D}\,\mathrm{GL}_n$ factors through $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$.

Later we will see that $\mathrm{SL}_n = \mathcal{D}\,\mathrm{GL}_n = \mathcal{D}\,\mathrm{SL}_n$; we'll deduce the outer equality from the case $n = 2$.

Caution: this does not mean that $\mathrm{SL}_n(k) = [\mathrm{SL}_n(k), \mathrm{SL}_n(k)]$ on the level of rational points. The latter is actually true, however, when k is not too small. The proof requires some structure theory for split reductive groups (in terms of which SL_2 plays a central role, akin to the special role of $\mathfrak{sl}_2$ in the theory of complex semisimple Lie algebras).

Example 16.2.6. Let $G \twoheadrightarrow G'$ be a surjective homomorphism of smooth k-groups. Then we get an induced map $\mathcal{D}G \to \mathcal{D}G'$ and this is surjective because it can be checked on the level of geometric points. Consequently PGL_n is perfect as an algebraic group; i.e., $\mathrm{PGL}_n = \mathcal{D}(\mathrm{PGL}_n)$, because of the surjection $\mathrm{SL}_n \twoheadrightarrow \mathrm{PGL}_n$.

16.3 Proof of Proposition 16.2.1

First note that uniqueness in (i) is guaranteed, because H is determined by its geometric points.

Next observe that without loss of generality we can add maps $g_i : X_i \to G$ to our collection, where $g_i(x) = f_i(x)^{-1}$. Now we don't need to mention inverses, and can restrict our attention to taking products of the $f_i(X_i)$'s.

Now for $I = \{i_1, \dots, i_n\}$ a multiset of indices, define

$$m_I : X_I \stackrel{\mathrm{def}}{=} X_{i_1} \times \cdots \times X_{i_n} \stackrel{\prod f_{i_j}}{\to} G \times \cdots \times G \stackrel{\cdot}{\to} G.$$

The set theoretic image $W_I = m_I(X_I)$ is constructible by Chevalley's theorem. By hypothesis, W_I contains the identity e. Therefore W_I contains a dense open U in $\overline{W}_I$, the schematic image of m_I (= the Zariski closure of W_I), which is a geometrically integral closed subscheme of G passing through e. [Note that $\overline{W}_I$ is geometrically integral because $X_I \to \overline{W}_I$ is dominant, so locally $\mathcal{O}_{\overline{W}_I} \subset \mathcal{O}_{X_I}$, and the sections of $(\mathcal{O}_{X_I})_{\overline{k}}$ are domains (as X_I is geometrically integral) so the same is true for $(\mathcal{O}_{\overline{W}_I})_{\overline{k}}$.]

Next choose I such that $\dim \overline{W}_I$ is maximal. We will show that $H \stackrel{\mathrm{def}}{=} \overline{W}_I$ satisfies the conclusions of the proposition.

Claim 16.3.1. *The map $\overline{W}_J \times \overline{W}_{J'} \to \overline{W}_{J \sqcup J'}$ given by multiplication is dominant, where $J \sqcup J'$ denotes concatenation of multisets.*

The proof is similar to that of Proposition 3.2.4(iii), and we omit it.

Now for any J we have on K-points ($K = \overline{K}$) that $\overline{W}_J(K), \overline{W}_I(K) \subset \overline{W}_J(K) \cdot \overline{W}_I(K) \subset \overline{W}_{I \sqcup J}(K)$. But $\overline{W}_I(K) = \overline{W}_{I \sqcup J}$ by the maximality of I; both are irreducible and closed and they have the same dimension, so since one is contained in the other they coincide.

The upshot is that $\overline{W}_I$ is stable under left (and by analogous reasoning, right) multiplication by any $\overline{W}_J$, on K-points. But we get more: since $\overline{W}_I = \overline{W}_{I \sqcup J}$, we have $\overline{W}_J \subset \overline{W}_I$ for all J. Thus $\overline{W}_I$ is stable under left and right multiplication against itself. Moreover $\overline{W}_I =$ "$\overline{W}_I^{-1}$" $= \overline{W}_{I^{\mathrm{opp},-1}} \subset \overline{W}_I$ where $I^{\mathrm{opp},-1}$ denotes the set of indices corresponding to taking the inverse maps g_i of the f_i for $i \in I$, in the opposite order. Thus H is stable under multiplication and inversion, and by construction it is smooth and connected. Therefore H is a smooth connected closed k-subgroup of G, and moreover it contains $f_j(X_j)$ for all j by construction.

Now we have $U \subset W_I \subset H$ and $U \hookrightarrow H$ is open and dense On K-points, $W_I(K)$ comes from $X_i(K)$'s for $i \in I$ multiplied in order. Therefore to show the last bit of (i) as well as (ii), it's enough to show that $U \times U^{-1} \stackrel{\cdot}{\to} H$ is surjective, or equivalently, surjective on K-points.

We can extend scalars to K, rename K as k. Then we are reduced to showing the following lemma.

Lemma 16.3.2. *Let $k = \overline{k}$ and let H be a smooth k-group of finite type. Let $U \subset H$ be a dense open.*[12] *Then $U(k)U(k)^{-1} = H(k)$.*

[12]In practice this density will be automatic if H is connected and U is nonempty.

Proof. Choose $h \in H(k)$. We just need to show that $h \cdot U(k) \cap U(k)$ is nonempty. But $h \cdot U(k) = (hU)(k)$ where hU is the translation of U by h. But hU and U are dense opens in H, so their intersection is nonempty, so since $k = \overline{k}$ it contains a rational point. □

16.4 Improvements on Proposition 16.2.1

Corollary 16.4.1. *Let* H, H' *be smooth closed subgroups of a smooth finite type* k*-group* G*. Suppose* H *(but not necessarily* H'*) is connected. Then there exists a unique smooth closed connected subgroup* $[H, H'] \subset G$ *which on geometric points is the commutator subgroup of* $[H(K), H'(K)]$.

Proof. Uniqueness follows from existence because H is determined by its geometric points. By uniqueness and Galois descent, we can therefore assume without loss of generality that $k = k_s$.

Write $H' = \coprod H'_i$ as a finite disjoint union of connected components, which because $k = k_s$ are geometrically connected. Consequently the $(H'_i)_{\overline{k}}$ are the connected components of $H'_{\overline{k}}$. Hence they are translates of the irreducible variety $(H'_{\overline{k}})^0$, and are thus themselves irreducible. Since the H'_i are smooth, it follows that the H'_i are geometrically integral over k, and hence the rational points $H'_i(k) \neq \varnothing$. So there exist $h'_i \in H'_i(k)$ such that $H'_i = (H')^0 h'_i$. Thus "the disconnectedness of H' is completely explained by a finite set of rational points". In particular

$$H' = \coprod_{h'_i \in H'(k) \text{ (finite)}} (H')^0 h'_i.$$

Form the maps

$$f_i : H \times (H')^0 \to G$$

by $f_i(h, h') = [h, h'h'_i]$. Applying Proposition 16.2.1 to these maps yields the corollary. □

Proposition 16.4.2. *If* $H, H' \subset G$ *are smooth closed* k*-subgroups a smooth finite type* k*-group* G *and* $H \subset N_G H'$ *then there exists a unique smooth closed commutator* k*-subgroup* $[H, H'] \subset G$ *with the expected geometric points.*

Remark 16.4.3. In Proposition 16.4.2, $[H, H']$ is generally not connected if neither H nor H' is.

Example 16.4.4. Take $H = H' = G$ in Proposition 16.4.2. We obtain $\mathcal{D}G$ even when G is disconnected. Consequently (by Noetherianness) any smooth G of finite type has a finite derived series.

Start of proof of Proposition 16.4.2. By Galois descent (as in Corollary 16.4.1) we can assume without loss of generality that $k = k_s$.

Since $H \subset N_G H'$, there is a k-homomorphism $H \ltimes H' \to G$. By Corollary 9.3.6, the image is a smooth closed k-subgroup of G. Since smooth closed subgroups of $H \ltimes H'$ thus map to smooth closed subgroup of G, it is enough to treat the case $G = H \ltimes H'$. In particular we can assume $H' \lhd G$.

Exercise: If $H' \lhd G$ then $(H')^0 \lhd G$.

By Corollary 16.4.1, we have smooth connected closed k-subgroups

$$[H, (H')^0], [H^0, H'] \subset G.$$

Let $L = [H, (H')^0] \cdot [H^0, H'] \subset G$ be the smooth closed k-subgroup they generate, using Proposition 16.2.1 in the guise of Example 16.2.2.

Now consider gLg^{-1} for $g \in G(k)$, and form the smooth connected closed subgroup

$$N = \langle gLg^{-1} \rangle_{g \in G(k)} \subset G$$

generated by all of them, again using Proposition 16.2.1. Since $G = H \ltimes H'$, $N \subset \mathcal{D}G$. Since $G(k) \subset G$ is Zariski-dense (as $k = k_s$) $N \lhd G$.

Now we are almost done; we'll finish the proof next time. □

17 February 19

17.1 Conclusion of proof of Proposition 16.4.2

Last time we reduced to the case $H, H' \subset H \ltimes H' = G$ all smooth k-groups of finite type over $k = k_s$, and we constructed a smooth connected normal closed k-subgroup $N \lhd G$ such that

$$[H, H'^0] \cdot [H^0, H'] \subset N, \qquad N(K) \subset [H(K), H'(K)] \text{ for all } K = \overline{K}/k.$$

Since $k = k_s$ we can choose representatives $h_i \in H(k)$ and $h'_j \in H'(k)$ for the component groups, so that

$$H = \coprod_{\text{finite}} H^0 h_i, \qquad H' = \coprod_{\text{finite}} H'^0 h'_j.$$

Now for words $w \in \langle h_i \rangle \subset H(k)$ and $w' \in \langle h'_j \rangle \subset H'(k)$ we can contemplate the maps

$$H^0 \times H'^0 \to G$$

given by

$$(h, h') \mapsto [wh, w'h'].$$

Since in the quotient group $G(\overline{k})/N(\overline{k})$ the component $H^0(\overline{k})$ centralizes $H'(\overline{k})$ and $H'^0(\overline{k})$ centralizes $H(\overline{k})$ [this is because $[H, H'^0] \cdot [H^0, H'] \subset N$] we compute that

$$[wh, w'h'] \equiv [w, w'] \bmod N(\overline{k})$$

on geometric points.

Lemma 17.1.1. *If the set of commutators* $\{[w, w'] \bmod N(\overline{k})\}$ *as* w, w' *range through words as above, is finite, then we are done.*

We leave the proof as an exercise; the idea is that by taking representative commutators

$$[w_1, w'_1], \dots, [w_n, w'_n] \in G(k)$$

the disjoint union

$$\coprod_{\text{finite}} N[w, w']$$

is a group and is in fact the group $[H, H']$ we seek.

So we are reduced to proving the following claim, which is even more than we need.

Claim 17.1.2. $\{[h, h'] \bmod N(\overline{k})\}_{(h,h') \in (H \times H')(\overline{k})} \subset G(\overline{k})/N(\overline{k})$ *is finite.*

We omit the proof of this because it is pure group theory. It is called the "Lemma of Baer"; see [Bor, end of §I.2]. □

17.2 Solvable groups

Taking $H = H' = G$ in Proposition 16.4.2, we obtain for *any* smooth finite type k-group (possibly disconnected!) G a derived subgroup $\mathcal{D}G$ and hence a derived series

$$G \supset \mathcal{D}G \supset \mathcal{D}^2 G := \mathcal{D}(\mathcal{D}G) \supset \cdots.$$

By Noetherian-ness, the series eventually stabilizes

Lemma 17.2.1. *Let* K/k *be any algebraically closed extension field. Then* $G(K)$ *is a solvable group if and only if* $\mathcal{D}^n G = 1$ *for all* $n \gg 0$.

Remark 17.2.2. Note that the solvability of $G(K)$ is thus independent of K.

Proof. We have $\mathcal{D}^n G = \mathcal{D}^{n+1} G$ for $n \gg 0$. So $(\mathcal{D}^n G)(K) = \mathcal{D}^n(G(K)) \stackrel{?}{=} 1$ if and only if $\mathcal{D}^n G = 1$ because $K = \overline{K}$. In other words, the triviality of $\mathcal{D}^n G$ for large n can be checked on geometric points. $\square$

The last lemma motivates the definition of a solvable group.

Definition 17.2.3. We say that a smooth k-group G is *solvable* if $\mathcal{D}^n G = 1$ for $n \gg 0$.

Example 17.2.4. If G is commutative then $\mathcal{D}G$ is trivial, so G is solvable.

Example 17.2.5. Given maps $G' \hookrightarrow G \twoheadrightarrow G''$, if G is solvable then so are G' and G''. This follows from the analogous fact from group theory, since it can be checked on geometric points. If moreover $G'(\overline{k}) = \ker(G \twoheadrightarrow G'')(\overline{k})$ then G', G'' solvable implies G is solvable, by similar reasoning. In particular G is solvable if and only if G^0 is solvable and $G(\overline{k})/G^0(\overline{k})$ is solvable in the usual group theoretic sense.

Example 17.2.6. If G is unipotent, then G is solvable. This is because by Theorem 15.1.8 we can find an embedding $G \hookrightarrow U_n$. And U_n is solvable because this can be checked on geometric points, and $U_n(\overline{k})$ has an obvious composition series with successive quotients isomorphic to products of additive groups, hence abelian.

Example 17.2.7. If G has a composition series in the sense of algebraic groups, i.e. a chain $1 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$ all smooth closed k-subgroups, and $\mathcal{D}G_i \subset G_{i+1}$ for all i, then G is solvable. Just check on geometric points!

17.3 Structure of smooth connected commutative affine k-groups

Theorem 17.3.1. *Let k be a perfect field and G a smooth connected commutative affine k-group. Then there exists a decomposition $G = M \times U$ where M, U are smooth closed k-subgroups of G, such that U is unipotent and $M(K)$ consists of semisimple elements of $G(K)$ for any algebraically closed extension field K/k. Moreover this decomposition is functorial in G, M^0 is a torus, and the component group $M(\overline{k})/M^0(\overline{k})$ has order not divisible by the characteristic of k.*

Remark 17.3.2. This is false for imperfect k, as the example of the Weil restriction of scalars $\mathrm{R}_{k'/k}(\mathbf{G}_m)$ for an inseparable field extension k'/k shows.

Proof. Uniqueness is immediate because the geometric points of M and U must be given by the Jordan decomposition as the semisimple and unipotent elements of $G(\overline{k})$, respectively, and these determine the groups M and U uniquely.

Thus by Galois descent we may assume without loss of generality that $k = k_s$, and hence since k is perfect that $k = \overline{k}$.

Since G is commutative, gg' is semisimple (resp. unipotent) if $g, g' \in G(k)$ are both semisimple (resp. unipotent). Upon passing to a faithful representation of G, this is because commutating diagonalizable matrices are simultaneously diagonalizable, so their product is diagonalizable; likewise commuting nilpotent matrices have nilpotent product, which implies the claim for unipotent matrices.

Thus we have abstract *subgroups* $G(k)_{\mathrm{ss}}$ and $G(k)_{\mathrm{u}}$ of $G(k)$ consisting of exactly the semisimple (resp. unipotent) elements. Define U to be the Zariski closure of $G(k)_{\mathrm{u}}$, which is a smooth closed k-subgroup by previous results.

Lemma 17.3.3. *U is unipotent, which is equivalent to $U(k) = G(k)_{\mathrm{u}}$.*

To prove the lemma, choose a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ and consider the condition that the characteristic polynomial of $g \in G(k)$ is $(T-1)^{\dim V}$. This is manifestly Zariski closed on $\mathrm{GL}(V)$, and hence on G. So $G(k)_{\mathrm{u}}$ is a Zariski closed locus in $G(k)$, which implies $U(k) = G(k)_{\mathrm{u}}$ as desired.

Next define M to be the Zariski closure of $G(k)_{\mathrm{ss}}$. The analogous lemma is

Lemma 17.3.4. *There exists a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ such that M maps to a torus.*

In particular, by Homework 5, this implies that all elements of $M(k)$ are semisimple in G, so $M(k) = G(k)_{\mathrm{ss}}$.

To prove this lemma, take any faithful representation at all. Then the subgroup $G(k)_{ss} \subset GL(V)$ is a commutative group of diagonalizable matrices (albeit an infinite one) and is thus simultaneously diagonalizable over $GL(V)(k)$. So $G(k)_{ss} \subset T(k)$ for a $GL(V)(k)$-conjugate T of the standard diagonal torus in $GL_{\dim V}$. Since $T \subset GL(V)$ is closed, the Zariski closure M can be computed inside $GL(V)$ to be contained in T.

By the preceding lemma and Homework 5, we conclude that M^0 is a torus and the component group has order prime to the characteristic.

Finally consider the k-homomorphism $M \times U \to G$ given by multiplication. This is a homomorphism precisely because G is commutative.

Claim 17.3.5. *This is an isomorphism*

The map is surjective because this can be checked on geometric points, where one can appeal to Jordan decomposition.

To prove injectivity, notes that the kernel is precisely $M \cap U$. This has no nontrivial geometric points by Jordan decomposition. Hence $M \cap U \subset M^0 \cap U$, so it's enough to prove the latter is trivial (as a scheme!). But that is the intersection of a torus and a unipotent group. Take any faithful representation of G; this can by conjugated so that U lands in U_n by Theorem 15.1.8. So $M^0 \cap U$ is a subgroup of $T \cap U_n$ for a torus $T \subset GL_n$. But by Homework 5, $T \cap U_n$ is trivial (as a scheme). □

17.4 Coset spaces for closed subgroups (that is, quotients)

Definition 17.4.1. Let G be a finite type k-group and $H \subset G$ a closed k-subgroup. A *quotient* G/H is a map $\pi : G \to X$ for a finite type k-scheme X, which is flat, surjective, and invariant under the right translation action of H on G, such that the map

$$G \times H \to G \underset{X}{\times} G$$

given by

$$(g, h) \mapsto (g, gh)$$

is an isomorphism.

Remark 17.4.2. The last condition is equivalent to the condition that on R-points, the fibers of $G(R) \to X(R)$ are precisely the $H(R)$-orbits of the right translation action on $G(R)$.

Remark 17.4.3. If $\pi : G \to X$ is a quotient G/H then $\pi_{k'} : G_{k'} \to X_{k'}$ is easily seen to be a quotient $G_{k'}/H_{k'}$. Moreover that π is a quotient G/H can be *checked* after scalar extension.

Remark 17.4.4. By passing to $pr_1^{-1}(e)$ in the isomorphism $G \times H \simeq G \underset{X}{\times} G$, one can easily check that $\pi^{-1}(\pi(e)) = H$.

Example 17.4.5. A surjective k-homomorphism $\pi : G \twoheadrightarrow G'$ of smooth finite type k-groups is a quotient $G/\ker \pi$. To prove this, note that π is surjective by assumption, flat by the Miracle Flatness Theorem [Mat, 23.1] and that $G \times \ker \pi \simeq G \underset{G'}{\times} G$ via the specified map can be checked group-theoretically on R-points for any k-algebra R.

The definition of quotients leaves open three crucial questions, which we will address in the sequel.

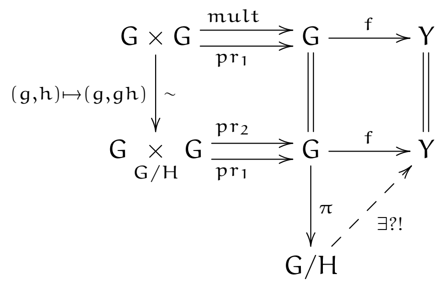
1. Under what circumstances does G/H exist?
2. If $H \lhd G$ is normal, does G/H necessarily have a unique k-group structure making the projection $G \to G/H$ a k-homomorphism?
3. Does the projection $\pi : G \to G/H$ (for any closed subgroup H) satisfy the right universal mapping property, i.e. is it initial among right-H-invariant maps to k-schemes?

We address question 3 in the next lemma, and question 2 in the following corollary.

Lemma 17.4.6. *If* $\pi : G \to G/H$ *is a quotient then it is initial among right-*H*-invariant* k*-maps* $f : G \to Y$ *to* k*-schemes* Y.

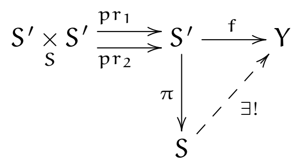
(An immediate consequence is that if the quotient exists, it is unique up to unique isomorphism.)

Proof. Consider the diagram



In both rows, both compositions agree. But by using the isomorphism in the first column from the definition of a quotient, the second row has lost all mention of group theory. Since π is flat and surjective and finite type (hence quasicompact) the existence of the dotted map is thus reduced to the following, which is essentially the content of faithfully flat descent.

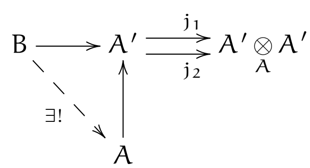
Theorem 17.4.7 (Grothendieck)**.** *Let $S' \xrightarrow{\pi} S$ be a flat surjective quasicompact map. Given a diagram with both compositions in the top row agreeing,*



there exists a unique dotted map making the diagram commute.

For the proof, see [BLR, §6.1]. □

Example 17.4.8. The proof of faithfully flat descent essentially reduces to the affine case, where one must actually do something. The setup is that $A \to A'$ is a faithfully flat algebra and we have a diagram



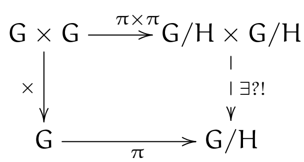
such that both composition in the row agree. Then there exists a unique dotted map making the diagram commute. The real content of this is that $A \hookrightarrow A' \rightrightarrows A' \otimes_A A'$ is exact, which one must prove.

Example 17.4.9. The toy example is the case of S quasicompact and separated, $S' = \coprod U_i'$ a disjoint union of U_i''s giving a finite open affine cover of S. Then $S' \times_S S' = \coprod U_i' \cap U_j'$, and the content of faithfully flat descent in this case is that morphisms glue.

Now we address question 2 above.

Corollary 17.4.10. *If $H \lhd G$ is normal and the quotient G/H exists, then there exists a unique k-group structure on G/H such that $\pi : G \to G/H$ is a k-homomorphism.*

Proof. One can check that $\pi \times \pi : G \times G \to G/H \times G/H$ is a quotient map for $(G \times G)/(H \times H)$; this follows easily form the definition. Therefore it has the universal property of the preceding lemma. But consider the diagram



The induced map, if it exists, is unique and must be the unique multiplication law giving the group structure on G/H compatible with π.

To see that the dotted map exists, we just need to check $\pi \circ (\times) : G \times G \to G/H$ is right G-invariant, which can be checked functorially on R-points. Here this says that $\pi(ghg'h') = \pi(gg')$. This follows from the usual group theoretic computation that $ghg'h' = gg'g'^{-1}hg'h \in gg'H(R)$ because $H(R) \lhd G(R)$ is normal. □

Example 17.4.11. $\mathrm{SL}_n/\mu_n \simeq \mathrm{PGL}_n$, so given a μ_n-invariant map $\mathrm{SL}_n \to Y$, there exists a unique induced map $\mathrm{PGL}_n \to Y$ making the triangle commute. Thus for all intents and purposes, PGL_n is the quotient of SL_n by μ_n, although this is manifestly false on the level of rational points.

18 February 22

18.1 Existence of quotients (of smooth affine groups)

Theorem 18.1.1. *If* G *is a smooth affine* k*-group and* $H \subset G$ *is a closed* k*-subgroup, then* G/H *exists as a smooth quasiprojective* k*-scheme, and* $(G/H)(K) = G(K)/H(K)$ *when* K/k *is algebraically closed.*

Example 18.1.2. If $G = \mathrm{GL}_n$ and H is the subgroup $\left\{\begin{pmatrix} * & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ \vdots & & & & \\ \vdots & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \end{pmatrix}\right\}$ (where anything can go in the $\star$'s), then $G/H \simeq \mathbf{P}^n$. The method of proof will in fact show that if $G \curvearrowright X$, a finite type k-scheme, and $H = \mathrm{Stab}_G(x)$ for a rational point $x \in X(k)$, then G/H sits as a locally closed subscheme of X as the orbit of x, with the reduced structure.

Proof. Using that G is smooth and affine and H is closed, by Theorem 14.1.1 there is a representation $\rho : G \to \mathrm{GL}(V)$ such that $H = N_G(L)$ for a line $L \subset V$. Consider the projective representation $G \curvearrowright \mathbf{P}(V)$, and let $x_0 \in \mathbf{P}(V)$ be the point corresponding to $L \subset V$. Then $H = Z_G(x_0)$ is the scheme theoretic centralizer of x_0 for this action. Let $X \subset \mathbf{P}(V)$ be the locally-closed G-orbit of x_0, with the reduced structure, so that X is smooth. (We are here invoking the closed orbit lemma 9.3.5.)

Now look at $\pi : G \to X$, sending $g \mapsto gx_0$ (the orbit map). This certainly induces an isomorphism $G(K)/H(K) = X(K)$ on geometric points, because on geometric points $X(K)$ is the $G(K)$-orbit of x_0. In particular π is geometrically surjective, so surjective. Moreover G and X are smooth varieties of pure dimension. Since the fibers $\pi^{-1}(x)$ are (geometrically, hence scheme-theoretically) translates of the equidimensional variety H, they are all equidimensional of dimension $\dim H$. Since generically the fiber dimension plus $\dim X$ is equal to $\dim G$, this equation must therefore hold everywhere. So by the Miracle Flatness theorem [Mat, 23.1] π is flat. Finally, by definition π is right H-invariant. So we just need to show $G \times H \to G \underset{X}{\times} G$ sending $(g, h) \mapsto (g, gh)$ is an isomorphism. But on R-points we have

$$G \underset{X}{\times} G(R) = G \underset{\mathbf{P}(V)}{\times} G(R) = \{(g, g') \in G(R)^2 : gx_0 = g'x_0 \in \mathbf{P}V(R)\}$$

$$= \{(g, g') : g^{-1}g' \in Z_G(x_0)(R) = H(R)\} = \{(g, g') : g' = gh, \text{for a unique } h \in H(R)\} = (G \times H)(R).$$

□

Remark 18.1.3. The existence of quotients in more general settings (over a field) is discussed in [SGA3, $\mathrm{VI_A}$]; for even more general settings on must use the theory of algebraic spaces.

Example 18.1.4. The important example of the above theorem is when $H \lhd G$ is normal. In this case G/H is actually affine. For a proof, see the handout on the webpage. The idea is to go back to the construction and rig the representation V so that H acts on all of V by the same character $\chi \in \mathrm{Hom}(H, \mathbf{G}_m) = \mathrm{Hom}(H, \mathrm{GL}(L))$ it acts by on L. Then one finds that $G/H \hookrightarrow \mathrm{PGL}(V)$ is a closed (by the closed orbit lemma) subscheme of $\mathrm{PGL}(V)$, and is thus affine.

Remark 18.1.5. See Homework 7 for a discussion of exact sequences of k-groups, as well as more examples.

18.2 Lie algebras

Let G be a locally finite type k-group, but *not* necessarily a smooth one. Let $\mathfrak{g} = \mathsf{T}_e\mathsf{G}$. For $\nu \in \mathfrak{g} \subset \mathsf{G}(\mathsf{k}[\epsilon])$, consider right multiplication over $\mathsf{k}[\epsilon]$ as a map

$$\mathsf{G}_{\mathsf{k}[\epsilon]} \xrightarrow{\sim} \mathsf{G}_{\mathsf{k}[\epsilon]}$$

$$g \mapsto g\nu.$$

This is the identity on the special fiber, since ν is a tangent vector at the identity. In particular, on the underlying topological space $|\mathsf{G}_{\mathsf{k}[\epsilon]}| = |\mathsf{G}|$ this map is the identity. On structure sheaves, regarded as sheaves on $|\mathsf{G}|$, the map is an isomorphism

$$\mathcal{O}_{\mathsf{G}_{\mathsf{k}[\epsilon]}} = \mathcal{O}_\mathsf{G} \oplus \mathcal{O}_\mathsf{G}\epsilon \simeq \mathcal{O}_\mathsf{G} \oplus \mathcal{O}_\mathsf{G}\epsilon.$$

More particularly, it is a $\mathsf{k}[\epsilon]$-algebra automorphism deforming the identity on $\mathcal{O}_\mathsf{G}$. By an easy computation, it is uniquely of the form

$$f_1 + f_2\epsilon \mapsto f_1 + f_2\epsilon + D_\nu(f_1)\epsilon$$

for $D_\nu \in \mathrm{Der}_\mathsf{k}(\mathcal{O}_\mathsf{G}, \mathcal{O}_\mathsf{G})$.

Remark 18.2.1. The data of D_ν is equivalent to that of a map $\Omega^1_{\mathsf{G}/\mathsf{k}} \to \mathcal{O}_\mathsf{G}$, and hence in the smooth case (by duality) to a global vector field $\mathcal{O}_\mathsf{G} \to \mathsf{T}_{\mathsf{G}/\mathsf{k}}$. It is easy to see that this vector field is left-G-invariant, relating $\mathfrak{g}$ to the interpretation of Lie algebras in terms of left invariant vector fields on Lie groups in the classical, analytic setting.

As in the remark, D_ν (as a derivation) is left-invariant, in the sense that

$$D_\nu(f \circ \ell_g) = D_\nu f \circ \ell_g \text{ for all } g \in \mathsf{G}(R), f \in \mathcal{O}_{\mathsf{G}_R},$$

as is easy to check by hand.

Lemma 18.2.2. *The map we just produced*

$$\mathfrak{g} \to \{\textit{left-invariant } \mathsf{k}\textit{-algebra derivations } D : \mathcal{O}_\mathsf{G} \to \mathcal{O}_\mathsf{G}\}$$

is a k*-linear isomorphism of vector spaces.*

Proof. See [CGP, A.7.1, A.7.2]. □

Remark 18.2.3. It is this lemma which underlies Cartier's theorem on the smoothness of algebraic groups in characteristic zero.

Definition 18.2.4. Let $\mathrm{Lie}(\mathsf{G}) = (\mathfrak{g}, [\cdot,\cdot]_\mathsf{G})$ where $[\cdot,\cdot]_\mathsf{G}$ is the commutator of (left-invariant) k-derivations $\mathcal{O}_\mathsf{G} \to \mathcal{O}_\mathsf{G}$, pulled back to a Lie bracket on $\mathfrak{g}$ along the isomorphism of the previous lemma.

(Note that one must check that the commutator of left-invariant derivations is a left-invariant derivation, but this is trivial.)

Example 18.2.5. If $\mathsf{G} = \mathrm{GL}(V)$ then $\mathfrak{g} = \mathfrak{gl}(V) = \mathrm{End}(V)$ and $[\cdot,\cdot]_{\mathrm{GL}(V)}$ is the commutator on $\mathrm{End}(V)$; cf. Homework 7.

18.2.1 Functoriality of Lie(·)

Proposition 18.2.6. *The derivative*

$$\mathrm{ad}_\mathsf{G} = \mathsf{T}_e(\mathrm{Ad}_\mathsf{G}) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$$

of the adjoint representation

$$\mathrm{Ad}_\mathsf{G}; \mathsf{G} \to \mathrm{GL}(\mathfrak{g})$$

$$g \mapsto \mathsf{T}_e(c_g), \qquad c_g : x \mapsto gxg^{-1}$$

satisfies

$$\mathrm{ad}_\mathsf{G}(X) = [X, \cdot]_\mathsf{G}.$$

Proof. See [CGP, A.7.5]. Note that Ad_G is well-defined because $\mathfrak{g}_R = T_{e_R}(G_R)$, so $g \mapsto T_e(c_g)$ really gives a functorial representation of G on $\mathfrak{g}$. □

Now given $f : G \to G$ we have the identity

$$f \circ \mathrm{Ad}_{G'} = \mathrm{Ad}_G \circ f$$

because f is a homomorphism and so commutes with conjugation. Differentiating this identity we obtain

$$T_e f \circ \mathrm{ad}_{G'} = \mathrm{ad}_G \circ T_e f.$$

In light of the proposition, this establishes that the Lie bracket, and hence the Lie functor, is functorial in G.

19 February 24

19.1 Linear representations of split tori, over general rings

Let k be any commutative ring, not equal to the zero ring. Let $T \simeq \mathbf{G}_m^r$ be a split torus over k. Set $\Lambda = \mathbf{Z}^r$ We have an injection

$$\Lambda \hookrightarrow \mathrm{Hom}_{k\text{-gp}}(T, \mathbf{G}_m) =: X(T) \tag{$\star$}$$

given by $\lambda = \vec{n} \mapsto [\lambda : (t_1, \dots, t_r) \mapsto \prod t_i^{n_i}]$.

Remark 19.1.1. On Homework 1, it was shown that $\mathrm{End}_k(\mathbf{G}_m) = \mathbf{Z}$ if $\mathrm{Spec}\, k$ is connected; so under this hypothesis ($\star$) is an isomorphism.

The main theorem about linear representations of split tori is the following.

Theorem 19.1.2. *Let V be a k-module, equipped with a functorial k-linear representation of T, i.e. a map $T(R) \to \mathrm{End}_R(V_R)$ natural in R. Then V has a unique decomposition $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$ where the λ* ***weight space*** *V_λ is T-stable and has action via the character λ; in other words, for all $v \in V_R$ we have*

$$v \stackrel{!}{=} \sum_{\text{finite}} v_\lambda \text{ for some } v_\lambda \in (V_\lambda)_R$$

and for all $t \in T(R)$ we have

$$t.v = \sum \lambda(t) v_\lambda \in V_R,$$

where $\lambda(t) \in R^\times$ is given as above via ($\star$).

Remark 19.1.3. If k is local (e.g. a field or a dvr) or a PID [all we need is that projective finite modules are free] then if V is finite free in Theorem 19.1.2, so are all the weight spaces V_λ (and all but finitely many vanish, of course).

Theorem 19.1.2 will be proved next time and is part of a theme: to give a linear representation of a split torus T on V is to give a k-linear Λ-grading on V. That this works over rings is a fundamental discovery by Grothendieck.

Example 19.1.4. If k is a field then V is a *semisimple* T-representation, in the sense that any T-stable subspace W admits a T-stable direct complement W'.

To prove this, use the theorem to fix decompositions $V = \bigoplus V_\lambda, W = \bigoplus W_\lambda$.

Claim 19.1.5. $W_\lambda = W \cap V_\lambda$.

Granting the claim, we can take $W' = \bigoplus W'_\lambda$ where W'_λ is any complement of W_λ in V_λ; since T acts by scalars on V_λ, any such complement is T-stable, so we win.

To prove the claim, observe that by the uniqueness of the weight space decomposition in Theorem 19.1.2, its formation commutes with scalar extension. The formation of intersection also commutes with scalar extension. So we may assume without loss of generality that $k = \overline{k}$. Now $V_\lambda = \{v \in V : t.v = \lambda(t)v \text{ for all } v \in T(k)\}$ because $T(k) \subset T$ is dense. As W_λ has a similar description, the claim follows.

Example 19.1.6. Let T be the diagonal torus in GL_r. Then $X(\mathsf{T}) = \bigoplus \mathbf{Z}_{\lambda_i}$ where $\lambda_i : \mathsf{T} \stackrel{\mathrm{pr}_i}{\to} \mathbf{G}_m$ is the projection onto the ith diagonal entry. Consider $\mathsf{T} \in \mathrm{GL}_n \stackrel{\mathrm{Ad}}{\to} \mathrm{GL}(\mathfrak{gl}_r) = \mathrm{Aut}(\mathrm{Mat}_r(k))$. [Recall that the adjoint representation is given by $\mathrm{Ad}(g)(X) = gXg^{-1}$ because $g(1+\epsilon X)g^{-1} = 1+\epsilon gXg^{-1}$, where we identify vectors in the Lie algebra with deformations of the identity in $\mathrm{GL}_r(k[\epsilon])$.]

We would like to describe explicitly the weight space decomposition of $\mathfrak{gl}_n$ into T-stable lines. One obvious T-stable piece is the diagonal

$$\mathrm{Lie}(\mathsf{T}) = \mathfrak{t} = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\} \subset \mathfrak{gl}_r.$$

Since T conjugates the diagonal trivially, this has trivial T-action, so $\mathfrak{t}$ is an r-dimensional part of the $\lambda = 0$ weight space. As we will see in a moment, it is the whole 0-weight space.

In general, the other weight spaces are spanned by the elementary matrices $e_{ij} \in \mathfrak{gl}_n$. Let

$$t = \mathrm{diag}(t_1, \dots, t_r) \in \mathsf{T}(R).$$

Then we easily compute

$$\mathrm{Ad}(t)(e_{ij}) = te_{ij}t^{-1} = \frac{t_i}{t_j}e_{ij} = \frac{\lambda_i(t)}{\lambda_j(t)}e_{ij} = (\lambda_i - \lambda_j)(t)e_{ij}$$

where the switch between additive and multiplicative notation when thinking about characters is something that just takes getting used to.

So all non-diagonal elementary matrices span nonzero weight spaces, which are pairwise disjoint, and e_{ij} spans the $\lambda_i - \lambda_j$ weight space. All weight spaces with $\lambda \notin \{0\} \cup \{\lambda_i - \lambda_j : i \neq j\}$ vanish. In sum, the nonzero weight spaces of $\mathrm{Ad} = \mathfrak{gl}_r$ are

$$\mathrm{Ad}_0 = \mathfrak{t}, \text{ of dimension } r$$
$$\mathrm{Ad}_{\lambda_i - \lambda_j} = \mathrm{Span}(e_{ij}), \text{ of dimension } 1, \text{ for } i > j$$
$$\mathrm{Ad}_{-(\lambda_i - \lambda_j)} = \mathrm{Span}(e_{ji}), \text{ of dimension } 1, \text{ for } i > j.$$

Observe that all the characters λ corresponding to nonzero weight spaces factor through the **adjoint torus** $\mathsf{T}/Z_{\mathrm{GL}_r} \subset \mathrm{PGL}_r$, so are in $X(\mathsf{T}/Z_{\mathrm{GL}_r}) \subset X(\mathsf{T})$; this is unsurprising since conjugation by the center always acts trivially. The adjoint torus has dimension $r-1$ and corresponds to the hyperplane $\Lambda_0 = \{\vec{n} \in \Lambda : \sum n_i = 0\} \subset \Lambda$.

Caution: it is *not* always the case that the eigencharacters of a maximal split subtorus T of a linear algebraic group G span the character group of the adjoint torus; however, in this example they do. (E.g. for $r = 3$ one can draw these characters inside Λ_0 and see the A_2 root system, which spans Λ_0.)

19.2 Proof of Theorem 19.1.2

The proof will use Yoneda's lemma rather heavily, so watch out.

Work, for the moment, in the generality of any affine k-group G; later we will specialize to $G = \mathsf{T}$. A functorial linear representation of G on a k-module V is the data of a map $\rho(g) \in \mathrm{End}_R(V_R)$ for all $g \in G(R)$, which satisfy certain multiplicativity and identity axioms, and in particular these force all the $\rho(g)$ to be automorphisms; moreover functoriality in R is obviously required. Now $\rho(g) \in \mathrm{Hom}_R(V_R, V_R)$ corresponds uniquely to $[g] \in \mathrm{Hom}_k(V, V_R)$.

By Yoneda's lemma, all of the above data is equivalent to the data of $\rho(g_0) \in \mathrm{End}_{k[G]}(V_{k[G]})$ coming from $g_0 = \mathbf{1}_{k[G]} : k[G] \to k[G]$ viewed as a point of $G(k[G])$, satisfying certain properties corresponding to the multiplicativity and identity axioms. This again is equivalent to the data of $\alpha_\rho = [g_0] : V \to V \otimes_k k[G]$, a k-linear map satisfying certain properties, which we will make specific in the case $G = \mathsf{T}$ in a moment.

Take $G = \mathsf{T} = \mathbf{G}_m^r$. Then $k[G] = k[X_1^{\pm 1}, \dots, X_r^{\pm 1}] = k[\Lambda] = \bigoplus_{\lambda \in \Lambda} ke_\lambda$ is a free k-algebra with basis $\{e_\lambda : \lambda \in \Lambda\}$ where $e_{\vec{n}} = \prod X_i^{n_i}$ when viewed as an element of the Laurent polynomial ring. Thus the map α_ρ coming from our given k-linear representation ρ of T on V is a map

$$\alpha_\rho : V \to V \otimes_k k[\Lambda]$$

$$v \mapsto \sum_{\text{finite}} f_\lambda(v) \otimes e_\lambda$$

and its properties are encoded by properties of the coefficient functions $f_\lambda : V \to V$.

First of all, since α_ρ is k-linear, so are all the f_λ's.

Next, unwinding the proof of Yoneda's lemma, we see that for all $t \in t(R)$ and $v \in V_R$ we have

$$t.v = \sum \lambda(t) f_\lambda(v) \in V_R \qquad (\dagger)$$

(ramping up f_λ to a map $V_R \to V_R$).

The condition that ρ is multiplicative says that for all k-algebras R, and for all $t, t' \in T(R), v \in V_R$, we have

$$\sum_{\lambda \in \Lambda} \lambda(tt') f_\lambda(v) = \sum_{\lambda,\mu \in \Lambda} \lambda(t)\mu(t')(f_\lambda \circ f_\mu)(v).$$

If we take (t, t') to be the universal pair of R-points of T, namely a point $(t, t') \in T(k[T \times T])$, the previous equation yields

$$f_\lambda \circ f_\mu = \begin{cases} 0, & \lambda \neq \mu, \\ f_\lambda, & \lambda = \mu. \end{cases}$$

For details on this, see [CGP, A.8.8]. In other words, $f_\lambda : V \to V$ is a projection onto a subspace V_λ, and the V_λ's are pairwise disjoint.

By the above, we see that T acts by λ on V_λ.

So it remains only to show that $\sum V_\lambda = V$. Using the identity axiom for ρ, and taking $t = 1$ in (†), we find that for all $v \in V$,

$$v = \sum f_\lambda(v)$$

which says $v \in \sum V_\lambda$.

Combining the above, $V = \bigoplus V_\lambda$ is a direct sum of λ-eigenspaces, with projections given by f_λ. ☐

An important corollary (which is not obviously, but is in fact, related to Theorem 19.1.2) is the following.

Corollary 19.2.1 (Homework 8)**.** *If* k *is a field,* Y *is a smooth separated* k*-scheme of finite type, and a (not necessarily split!)* k*-torus* T *acts on* Y*, then the closed subscheme* $Y^T \subset Y$ *given by functorial fixed-points of the* T*-action on* Y *is smooth.* ☐

Example 19.2.2. If T is a subtorus of a smooth k-group G of finite type upon which T acts upon by conjugation, then $Z_G(T) = G^T$ is smooth.

20 February 26

20.1 k-split solvable groups

Recall from Homework 5 that the category of k-tori is equivalent to the category of finite free abelian groups equipped with a continuous discrete action by $\Gamma = \mathrm{Gal}(k_s/k)$. Given an exact sequence of algebraic groups

$$1 \to T' \to T \to T'' \to 1$$

if T' and T are tori then so is T'' because it is smooth connected affine and its geometric points are semisimple. By Homework 7 we get an exact sequence of Γ-lattices

$$0 \leftarrow X(T'') \leftarrow X(T) \leftarrow X(T') \leftarrow 0.$$

If T' and T'' are k-split then $X(T'')$ and $X(T')$ have trivial Γ-action, which forces $X(T)$ to have trivial Γ-action, so T is also split. This fact that splitting is respected by extensions is particular to groups of "multiplicative type". In particular, it has no analogue for unipotent groups.

The splitting behavior of multiplicative and unipotent groups also contrasts in the type of field extensions over which these groups split: as we know, tori split over *separable* field extensions, but as the next lemma shows, to guarantee that a unipotent group splits we should go to the *perfect* closure (rather than the separable closure).

Lemma 20.1.1. *Let $U \neq 1$ be a smooth connected unipotent group over a perfect field k. Then U has a composition series*

$$1 = U_m \lhd U_{m-1} \lhd U_{m-2} \lhd \cdots \lhd U_1 \lhd U_0 = U$$

such that the successive quotients U_i/U_{i+1} are all k-isomorphic to $\mathbf{G}_a$.

Proof. Choose a faithful representation $U \hookrightarrow U'$ into a standard upper triangle unipotent group U'. We can see by hand that U' has such a composition series, say $\{U'_i\}_{i=0}^N$. (First decompose into the successive subgroups, all normal in U', with 0's on the first several superdiagonals. Each subquotient is abelian, isomorphic to a power of $\mathbf{G}_a$. Now just refine these to obtain the desired composition series.) Define $U_i = (U'_i \cap U)^0_{\mathrm{red}}$. This is connected by fiat, geometrically reduced because k is perfect, and thus is a smooth k-subgroup of U.

It is easy to check that $U_{i+1} \lhd U_i$, as normality can be checked on geometric points. Now the subquotient U_i/U_{i+1} is smooth and connected because U_i is such, and has dimension ≤ 1 because $U'_i/U'_{i+1} \simeq \mathbf{G}_a$ does, and it is unipotent because U is. So if U_i/U_{i+1} is nontrivial, it must be geometrically $\mathbf{G}_a$, and hence isomorphic over k to $\mathbf{G}_a$ because k is perfect (see Homework 2). □

Example 20.1.2. The lemma is false if k is not perfect. Take k'/k to be a degree p purely inseparable extension of an imperfect field k of characteristic p. Then $\mathrm{R}_{k'/k}(\mathbf{G}_m)$ is a smooth p-dimensional k-group with a natural subgroup $\mathbf{G}_m$; the quotient $\mathrm{R}_{k'/k}(\mathbf{G}_m)/\mathbf{G}_m$ is a $(p-1)$-dimensional smooth p-torsion group. On Homework 9 it is shown that it is actually unipotent, and contains no k-subgroup isomorphic to $\mathbf{G}_a$!

Definition 20.1.3. A **k-split solvable group** is a solvable smooth connected[13] affine k-group G with a composition series $\{G_i\}$ by k-subgroups so that $G_i/G_{i+1} \simeq \mathbf{G}_a$ or $\mathbf{G}_m$ (over k) for all i.

Example 20.1.4. A k-torus T is k-split in the usual sense if and only if it is so in the sense of Definition 20.1.3. (Use character theory, as remarked at the start of this section.)

Example 20.1.5. If $k = \overline{k}$ then any solvable smooth connected k-group is k-split.

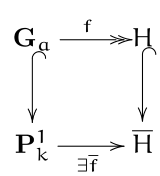
Proof. Using the derived series $\{\mathcal{D}^i G\}$ we can treat each commutative subquotient $\mathcal{D}^i G/\mathcal{D}^{i+1} G$ separately. So without loss of generality G is commutative. Thus, by Theorem 17.3.1, $G \simeq T \times U$ for a torus T and a unipotent group U. So we can treat tori and unipotent groups separately, and this will suffice. Now since $k = \overline{k}$, we have both $k = k_s$ and $k = k_p$. The first guarantees that tori split, and the second (by Lemma 20.1.1) guarantees that unipotent groups split. □

Example 20.1.6. (i) If $G \twoheadrightarrow G''$ is a surjective k-homomorphism of smooth connected affine k-groups and G is k-split solvable, then so is G''.

(ii) If $G' \hookrightarrow G$ is an injective k-homomorphism of smooth connected affine k-groups, k is perfect, and G is k-split solvable, then so is G'.

Proof. (ii) Since k is perfect, the property of being $\mathbf{G}_a$ can be checked over $\overline{k}$. Thus, by using the $(G_i \cap G')^0_{\mathrm{red}}$-trick we just have to observe that if $H \twoheadrightarrow \mathbf{G}_m$ is a k-isogeny and H is smooth and connected then $H \simeq \mathbf{G}_m$ over k. To prove this final k-isomorphism claim, first note that $\dim H = 1$ so $H_{\overline{k}} \simeq \mathbf{G}_a$ or $\mathbf{G}_m$. But $\mathbf{G}_a$ is impossible since there are no nontrivial homomorphisms $\mathbf{G}_a \to \mathbf{G}_m$. So H is a 1-dimensional torus, and $X(H)$ is isogenous to the trivial Galois lattice $\mathbf{Z}$, and is thus itself trivial. So $H \simeq \mathbf{G}_m$.)

(i) Take the image of a splitting solvability series for G; we'd like to show it is a splitting solvability series for G''. This reduces to showing that if $\mathbf{G}_a$ or $\mathbf{G}_m$ maps isogenously onto H then H is accordingly isomorphic to $\mathbf{G}_a$ or $\mathbf{G}_m$ over k. By much the same method as in (i), the $\mathbf{G}_m$ case is fine. For the $\mathbf{G}_a$ case, consider the diagram



[13]Some authors leave out connectedness, and so must change the definition!

where $\overline{H}$ is the regular compactification of the curve H. The map $\overline{f}$ is finite, and a finite extension of Dedekind domains is flat, so $\overline{f}$ is flat. But is $\overline{H}$ actually smooth? That is, does it remain Dedekind after a ground field extension to $\overline{k}$? For this we will apply the following Claim after scalar extension to $\overline{k}$:

Claim 20.1.7. *If $A \hookrightarrow B$ is a finite flat inclusion into a Dedekind domain, the A is Dedekind as well.*

Proof of claim. A is clearly a domain. Its nonzero primes are maximal since this is true for B, and B is A-finite. Thus $\dim A = 1$, so it remains to prove regularity. Since we are in dimension 1, this is equivalent to the invertibility of nonzero ideals as A-modules. Since A is integral, that is equivalent to the flatness of all ideals. Since $A \to B$ is finite flat injective, it is faithfully flat. Hence I is A-flat if and only if I_B is B-flat. But $I_B \to IB$ is an isomorphism since B is A-flat. Hence it's enough to show $IB \lhd B$ is B-flat. But B is Dedekind, so we win. □

Remark 20.1.8. There is a vast generalization: a Noetherian commutative ring with a regular faithfully flat extension is regular. This is shown in [Mat, 23.7(i)] using Serre's homological criterion of regularity.

Let us return to the proof of (i) in the example. Now $\mathbf{P}^1_k$ is geometrically regular, and the "finite flatness" of the surjective $\overline{f}$ is preserved by any ground field extension, so by the Claim we see that $\overline{H}$ is geometrically regular, i.e. smooth. Thus $\overline{H}$ is a smooth projective curve, and by Riemann-Hurwitz considerations it must have genus zero (even though $\overline{f}$ may be highly inseparable, etc.) Moreover $\overline{f}(\infty)$ is a k-rational point of $\overline{H}$, so in particular $\overline{H}(k) \neq \varnothing$. So $\overline{H} \simeq \mathbf{P}^1_k$ and thus $H = \overline{H} - \overline{f}(\infty) \simeq \mathbf{A}^1_k$ as a curve. By our classification of 1-dimensional smooth connected k-groups, we know this implies $H = \mathbf{G}_a$ as k-groups. □

20.2 Borel Fixed Point Theorem

Theorem 20.2.1. *Let X be a proper k-scheme, $X(k) \neq \varnothing$, and suppose X is equipped with an action of a k-split solvable group G. Then $X(k)$ contains a G-fixed point (i.e. the corresponding orbit map $G \to X$ is the constant map.*

Proof. We induct on $d = \dim G$. The case $d = 0$ is fine, so we'll skip that. If $d = 1$ then $G = \mathbf{G}_a$ or $\mathbf{G}_m$. Equipping G with the left-translation action, if we fix a rational point $x_0 \in X(k)$, the orbit map $g \mapsto gx_0 : G \to X$ is G-equivariant. Now $G \hookrightarrow \overline{G} \simeq \mathbf{P}^1_k$ sits inside its regular compactification $\mathbf{P}^1_k$, and by inspect the left translation action on G extends to $\mathbf{P}^1_k$ fixing any added points. Thus by the valuative criterion of properness, the orbit map $G \to X$ extends to a G-equivariant map $\mathbf{P}^1_k \to X$. (To prove G-equivariance, observe that it holds on a dense open and X is proper, so separated, so this is enough.) In particular the image of ∞ (which is G-fixed in $\mathbf{P}^1_k$ for both $G = \mathbf{G}_a$ and $G = \mathbf{G}_m$) is a G-fixed rational point of X.

Next suppose $d > 1$. Then there exists a codimension 1 k-split solvable normal subgroup $G' \lhd G$. By induction there exists $x' \in X(k)$ which is G'-fixed. The corresponding orbit map $G \to X$ induces (by the universal property of quotients) a map $G/G' \to X$ which is equivariant for the obvious G-action on G/G'. But this quotient is again $\mathbf{G}_a$ or $\mathbf{G}_m$, so repeating the argument from the case $d = 1$ we get a G/G'-equivariant, and hence G-equivariant, map $\mathbf{P}^1_k \to X$. The image of ∞ gives us the point we want. □

Corollary 20.2.2 (Lie-Kolchin theorem)**.** *If G is k-split solvable, then any representation $\rho : G \to \mathrm{GL}(V)$ can be conjugated over k into the upper triangular subgroup of $\mathrm{GL}(V)$.*

Proof. Let X be the smooth projective variety of full flags in V. It has a natural G-action via ρ. A rational G-fixed point is precisely a flag in V which is preserved by G acting on V via ρ. So taking a basis for V adapted to this flag proves the corollary. □

Corollary 20.2.3. *If G is k-split solvable then $G = T \ltimes U$ for a k-split torus T and a k-split unipotent group U, although this expression is not unique.*

Proof. We will discuss this next time. (The proof uses Corollary 20.2.2.) □

Corollary 20.2.4. *If G is a smooth connected solvable affine group then $\mathcal{D}G$ is unipotent.*

We will prove this final corollary next time.

21 March 1

21.1 Remark on quotients

See the handout "Quotient formalism" to resolve the following outstanding issue concerning quotients. To really work effectively with quotients, and in particular to make induction arguments go through, e.g. for proving facts about solvable groups, it is essential to know that quotients respect certain basic properties. We also would really like some basic facts the like "second isomorphism theorem" $H'/H'' \xrightarrow{\sim} \overline{H}'/\overline{H''}$ where $\overline{K}$ denotes the reduction of a subgroup $H \subset K \subset G$ to G/H for $H \lhd G$, say.

What is shown on the handout is that

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/H \\ \uparrow & & \uparrow \\ Z = \pi^{-1}\overline{Z} & \xrightarrow[\pi|_Z]{} & \overline{Z} \end{array}$$

sets up a bijection

$$\begin{array}{ccc} \{H\text{-stable closed subschemes } Z \subset G\} & = & \{\text{closed subschemes } \overline{Z} \subset G/H\} \\ \uparrow & & \uparrow \\ \{\text{closed subgroups } H \subset H' \subset G\} & = & \{\text{closed subgroups } \overline{H'} \subset G/H\} \\ \uparrow & & \uparrow \\ \{\text{normal ones}\} & = & \{\text{normal ones}\} \end{array}$$

etc, and everything behaves well with respect to smoothness.

21.2 Lie-Kolchin Corollary 20.2.4

To recapitulate the proof of Corollary 20.2.4, we wish to prove that if G is a solvable smooth connected affine k-group then $\mathcal{D}G$ is unipotent. Without loss of generality we can take $k = \overline{k}$, since unipotence can be detected on geometric points and the formation of derived subgroups is compatible with scalar extension. So G is k-split solvable, so the Lie-Kolchin Theorem 20.2.2 applies. That theorem says we can embed $G \hookrightarrow B_n$, the standard upper triangular subgroup of GL_n, for some n. So we obtain $\mathcal{D}G \hookrightarrow \mathcal{D}(B_n)$. Now $B_n = T \ltimes U$ is the semidirect product of the unipotent upper triangular subgroup $U = U_n \lhd B_n$ and the diagonal torus T. In particular $\mathcal{D}(B_n) \subset U_n$, since T is commutative so the commutators of the T-part die in $\mathcal{D}(B_n)$. But U_n is unipotent, and a subgroup of a unipotent is unipotent (e.g. by functoriality of Jordan decomposition), so the claim follows.

Remark 21.2.1. If G is k-split solvable then one can actually show $\mathcal{D}G$ is k-split solvable. See [Springer, Thm. 14.3.8(i)]. The idea is to prove a more robust (easier to check) criterion for k-split solvability than the definition, along the lines of being dominated as a variety by a product of $\mathbf{G}_a$'s and $\mathbf{G}_m$'s, which is amenable to proving this particular claim.

21.3 Structure theory of solvable groups

What is the general structure of a solvable smooth connected linear algebraic k-group? In general this is very mysterious. But if G is k-split then we will see shortly that $G = T \ltimes U$ is a semidirect product of a k-split torus T and a k-split unipotent group U.

Warning! T is far from unique – e.g. one can conjugate it by any rational point of U. On the other hand we will see that U is intrinsic to G, it is the so-called *unipotent radical*.

Here is the idea: By Corollary 20.2.4, if G is k-split solvable then $\mathcal{D}G$ is unipotent. Consider the projection

$$\pi : G \twoheadrightarrow G/\mathcal{D}G.$$

The quotient $G/\mathcal{D}G$ is k-split commutative (since the image of a split solvable group is split solvable; see Example 20.1.6(i)). By Theorem 17.3.1, it follows that $G/\mathcal{D}G = T_0 \times U_0$ is the product of a split torus T_0 and a split unipotent group U_0. Set $U = \pi^{-1}U_0$; this is unipotent and is the natural candidate for the decomposition $G = T \ltimes U$.

The problem is how to lift T_0 to a torus $T \subset G$. For example, given a short exact sequence

$$1 \to \mathbf{G}_a \to G \to \mathbf{G}_m \to 0$$

is it automatic that $G = \mathbf{G}_m \ltimes \mathbf{G}_a$ using some action $tx = t^n x$ ($t \in \mathbf{Z}$) of $\mathbf{G}_m$ on $\mathbf{G}_a$? It's not entirely clear how we find the $\mathbf{G}_m$ inside G.

We now state the general result (already mentioned last time):

Proposition 21.3.1. *Let G be a k-split solvable group. Then $G = T \ltimes U$ for a k-split torus T and a k-split unipotent group U (which is necessarily $\mathcal{R}_{u,k}(G)$; see the next section).*

We emphasize again that U is intrinsic, but T is *not*.

Proof. The commutative case is already done by Theorem 17.3.1. In the general case, a delicate induction on $\dim G$ is required, using a splitting composition series (i.e. one with consecutive subquotients equal to $\mathbf{G}_a$ or $\mathbf{G}_m$).

There are two problems to deal with. (1) The terms in the composition series need not be normal in the whole group. (2) Smooth connected subgroups of k-split solvable groups are solvable, but need not be k-split.

The crucial input for the induction is Tits's structure theory for (smooth connected) unipotent groups, which is described in [CGP, App. B].

The issue comes down to two crucial cases:

$$1 \to \mathbf{G}_a \to G \to \mathbf{G}_m \to 1,$$

$$1 \to \mathbf{G}_m \to G \to \mathbf{G}_a \to 1.$$

In the first case we must lift $\mathbf{G}_m$ back up to G; in the second we not only need to lift $\mathbf{G}_a$, but to do so essentially uniquely, and after the fact we should find that $G = \mathbf{G}_a \times \mathbf{G}_m$ is commutative.

All of this is worked out in detail in §2–§3 of the handout "Quotient formalism", where the method is to view G in the exact sequences above as a torsor for the subgroup, in the étale topology. Borel's method in his book is to work with the Lie algebras. □

21.4 Unipotent radical

In the following lemma, we will find an intrinsic unipotent normal subgroup in a smooth affine k-group G, which is thus a **characteristic subgroup** of G, in the classical group theoretic sense of being uniquely determined by G and thus invariant under all automorphisms of G, which is a very useful property.

Lemma 21.4.1. *Let G be a smooth affine k-group. There exists a (necessarily unique)*

$$\textit{unipotent normal smooth connected k-subgroup of } G \qquad (\star)$$

(denoted $\mathcal{R}_{u,k}(G)$) containing all subgroups of G satisfying $(\star)$.

Proof. The problem is to show that any two subgroups of G satisfying $(\star)$ are both contained in a third. This shows that there is a unique subgroup of G which is maximal for the conditions $(\star)$. (The assertion here is the uniqueness of the maximal one, since existence follows from connectedness and dimension considerations.)

So let $U, U' \subset G$ satisfy $(\star)$. Then since $U' \lhd G$, the semidirect product $U \ltimes U'$ makes sense. It is unipotent since it sits in a short exact sequence of unipotent groups, and Jordan decomposition is functorial. There is a natural map

$$U \ltimes U' \to G$$

and the image is precisely the subgroup $U \cdot U' \subset G$ generated by U and U'. This is smooth and connected by the general theory of the subgroups generated by smooth connected subvarieties. It is the image of the unipotent group $U \ltimes U'$, and hence unipotent. Finally, it is normal because U and U' are. (This is just a fact from group theory, done on geometric points: two normal subgroups of a big group general a normal subgroup of the big group.) Thus $U \cdot U'$ satisfies ($\star$). ☐

Definition 21.4.2. The subgroup $\mathcal{R}_{u,k}(G) \lhd G$ of Lemma 21.4.1 is called the k-*unipotent radical* of G.

Here are some basic properties of the unipotent radical.

Proposition 21.4.3. *(i) $\mathcal{R}_{u,k}(G)_K \subset \mathcal{R}_{u,K}(G_K)$ for all field extensions K/k, and if $K = k_s$ then equality holds.*

(ii) $\mathcal{R}_{u,k}(G^0) = \mathcal{R}_{u,k}(G)$.

Proof. (i) is elementary; just use Galois descent.

(ii) $\mathcal{R}_{u,k}(G)$ is unipotent, smooth, connected and normal in G. It is thus contained in G^0 and therefore normal in G^0. So certainly $\mathcal{R}_{u,k}(G^0) \supset \mathcal{R}_{u,k}(G)$. For the reverse inclusion it is enough to show that $\mathcal{R}_{u,k}(G^0) \lhd G$. By (i) we can assume without loss of generality that $k = k_s$. Then $G(k) \subset G$ is dense. It is therefore enough to show that conjugation by $G(k)$ preserves $\mathcal{R}_{u,k}(G^0)$. And this is true because such conjugations are honest k-group automorphisms of G^0, which must preserve the characteristic subgroup $\mathcal{R}_{u,k}(G^0)$. ☐

Remark 21.4.4. One might ask why we allow disconnected G in the definition of the unipotent radical, since by Proposition 21.4.3(ii) the component group of G is not detected by $\mathcal{R}_{u,k}(G)$. An example is the centralizer of $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in G(k) = \mathrm{PGL}_2(k)$. We have $Z_{\mathrm{PGL}_2}(g) = D \sqcup D \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ where $D \subset \mathrm{PGL}_2$ is the diagonal torus. This is an interesting (from a group theoretic perspective) yet disconnected group, whose structure we might wish to analyze to study PGL_2.

Example 21.4.5 (Homework 9). If $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n\}$ over any field k then $\mathcal{R}_{u,k}(G) = 1$.

Remark 21.4.6. One must be careful: in [CGP, Ex. 1.1.3] there is given an example over *any* imperfect field of a smooth affine k-group G such that $\mathcal{R}_{u,k}(G) = 1$ but $\mathcal{R}_{u,\overline{k}}(G_{\overline{k}}) \neq 1$!

Proposition 21.4.7. *If K/k is a separable (not necessarily even finitely generated!) extension of fields then $\mathcal{R}_{u,k}(G)_K = \mathcal{R}_{u,K}(G_K)$.*

Proof. Without loss of generality we can take $k = k_s, K = K_s$. From here one must use a "specialization" argument; see [CGP, Prop. 1.1.9(1)]. The idea: if the unipotent radical is bigger after going up to K, then it must become bigger over some finitely generated subextension field K_0/k. Now K_0 is the function field of a smooth k-variety X, and specialization at separable algebraic points is available (after "spreading out" over a dense open subscheme of X). ☐

21.5 Reductive groups

We often write $\mathcal{R}_u$ rather than $\mathcal{R}_{u,k}$ when k is algebraically closed.

Definition 21.5.1. Let G be a smooth affine k-group. We say G is *pseudo-reductive* over k if G is connected and the k-unipotent radical is trivial: $\mathcal{R}_{u,k}(G) = 1$. We say G is *reductive* if G is (not necessarily connected but) the geometric unipotent radical is trivial: $\mathcal{R}_u(G_{\overline{k}}) = 1$.

(Allowing G to be disconnected in the definition of reductive groups over fields is a matter of convention. In contrast, for a good relative theory over rings as in [SGA3] or [C] one must stick to the connected case.)

Remark 21.5.2. If $\mathrm{char}(k) = 0$ it turns out that reductivity (without the connectedness condition) is equivalent to complete reducibility for all finite-dimensional representations; that is the reason for the terminology!

Lemma 21.5.3. *Let G be a smooth affine k-group and $N \lhd G$ a smooth normal closed k-subgroup. Then $\mathcal{R}_u(N_{\overline{k}}) = (N_{\overline{k}} \cap \mathcal{R}_{u,\overline{k}}(G_{\overline{k}}))^0_{\mathrm{red}}$.*

Corollary 21.5.4. *Reductivity passes to smooth normal subgroups.* ☐

Proof of Lemma 21.5.3. Without loss of generality $k = \overline{k}$. Now $(N \cap \mathcal{R}_{u,k}(G))^0_{red}$ is unipotent since it is contained in the unipotent radical. It is connected since we passed to the connected component. It is smooth since we passed to the underlying reduced. Since $k = \overline{k}$ it is actually a group (geometrically connected, geometrically reduced). It is normal in G since the intersection of normal subgroups is normal and passing to $(\cdot)^0_{red}$ preserves normality. Finally it is contained in N, as is clear. So the inclusion $\mathcal{R}_u(N) \supset (N \cap \mathcal{R}_u(G))^0_{red}$ is obvious.

For the reverse inclusion, it suffices to show that $\mathcal{R}_u(N) \lhd N$ is normal in G, not just in N. This is true because $N \lhd G$ is normal and $\mathcal{R}_u(N)$ is a characteristic subgroup of N, so conjugating by rational points (which are dense in G) must preserve it. ☐

Lemma 21.5.5. *Let*

$$1 \to G' \to G \to G'' \to 1$$

be a short exact sequence of smooth affine k*-groups. If* G', G'' *are reductive then so is* G.

Proof. Exercise. The basic idea is that $G \twoheadrightarrow G''$ takes $\mathcal{R}_u(G_{\overline{k}})$ to $\mathcal{R}_u(G''_{\overline{k}})$ because surjections take normal subgroups to normal subgroups and the image of a unipotent thing is unipotent. Thus since G'' is reductive, $\mathcal{R}_u(G_{\overline{k}}) \subset G'_{\overline{k}}$. But it is normal in $G_{\overline{k}}$, hence normal in $G'_{\overline{k}}$, and it is smooth connected unipotent, so it is contained in $\mathcal{R}_u(G'_{\overline{k}}) = 1$ by reductivity of G'. ☐

Example 21.5.6. Take

$$P = \left(\begin{smallmatrix} \mathrm{GL}_2 & \star \\ 0 & \mathrm{GL}_3 \end{smallmatrix}\right) \subset \mathrm{GL}_5\,.$$

Then

$$\mathcal{R}_{u,k}(P) = \left(\begin{smallmatrix} \mathbf{1}_2 & \star \\ 0 & \mathbf{1}_3 \end{smallmatrix}\right) = \ker(P \twoheadrightarrow \mathrm{GL}_2 \times \mathrm{GL}_3).$$

Example 21.5.7. A non-example: note that $U \hookrightarrow \mathrm{GL}_n$ for any unipotent group U. Obviously $\mathcal{R}_{u,k}(U) = U$ but GL_n is reductive. The point is that $\mathcal{R}_{u,k}$ is only functorial for normal inclusions and arbitrary surjections, but not more more general k-homomorphisms.

Example 21.5.8. Let G be smooth connected and affine. Then $G/\mathcal{R}_{u,k}(G)$ is pseudo-reductive (over k), and if k is perfect then $G/\mathcal{R}_{u,k}(G)$ is reductive.

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22.1 Borel subgroups

Theorem 22.1.1. *Let* G *be a smooth connected affine group over* $k = \overline{k}$.

(i) Let R *(for "radical") denote a*

$$\textit{solvable smooth connected } k\textit{-subgroup} \qquad (\dagger)$$

of G*. (Such will be called a* $(\dagger)$*-subgroup of* G *in the sequel.) Then* R *is maximal among* $(\dagger)$*-subgroups of* G*, if and only if* G/R *is proper. If this is the case we call* R *a* ***Borel subgroup*** *of* G.

(ii) All Borel subgroups of G *are* $G(k)$*-conjugate.*

Remark 22.1.2. By Theorem 18.1.1, G/R is automatically quasi-projective for any closed k-subgroup $R \subset G$. So G/R is proper if and only if it is projective.

Proof. First assume the following.

Lemma 22.1.3. *If* R *is a* $(\dagger)$*-subgroup of dimension maximal among the dimensions of all* $(\dagger)$*-subgroups, then* G/R *is proper.*

Note that the hypothesis of the lemma is stronger than the condition that R is maximal among (†)-subgroups; this is because a priori (before proving (ii) of the theorem) we do not know that all maximal (†)-subgroups are of the same dimension.

Granting the lemma, let us deduce the theorem. Consider any (†)-subgroup $R' \subset G$. Then R' is k-split solvable since $k = \overline{k}$ and $R' \curvearrowright G/R$, which is proper by Lemma 22.1.3. So by the Borel fixed point Theorem 20.2.1, R' fixes a coset in $(G/R)(k)$. By Theorem 18.1.1, this coset is of the form gR for some $g \in G(k)$, since $k = \overline{k}$. That R' fixes gR says precisely that $R' \subset gRg^{-1}$. Now gRg^{-1} is the conjugate of a (maximal) (†)-subgroup. So if R' is a *maximal* (†)-subgroup, it must equal gRg^{-1}, and thus both have the same dimension as R and be conjugate to R by $G(k)$. Conversely if G/R' is proper then by the same reasoning (reversing the roles of R and R') we find $R \subset gR'g^{-1}$ and so by maximality of R, $R = gR'g^{-1}$. Thus $\dim R' = \dim R$ is maximal among (†)-subgroups, so in particular R' is a maximal (†)-subgroup.

It remains to do the hardest, but coolest, part: prove Lemma 22.1.3. So fix a (†)-subgroup R of maximal dimension. We need to show G/R is proper. Choose a faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$ such that $R = N_G(L)$ for a line $L \subset v$ under the resulting action $G \curvearrowright V$. Look at the action of R on the flag variety $\mathsf{Flags}_{\mathsf{full}}(V/L)$, which is proper. Since R is k-split solvable, the Borel fixed point Theorem 20.2.1 entails that R fixes a flag in V/L. We can lift this to a maximal flag $\mathcal{F}$ in V starting with L. Let $X = \mathsf{Flags}_{\mathsf{full}}(V)$ and $x \in X(k)$ the point corresponding to $\mathcal{F}$. Observe that $R \subset \mathrm{Stab}^{\rho}_G(x) \subset N_G(L) = R$, where the first inclusion is because R preserves the flag $\mathcal{F}$ and the second is because any g which preserves the flag preserves the first subspace L in it. Hence $R = \mathrm{Stab}_G(x)$.

From our construction of quotients (Theorem 18.1.1) this implies that the orbit map $G \twoheadrightarrow Gx \subset X$ is precisely the quotient G/R. Since X is proper, it is therefore enough to show that $Gx \subset X$ is closed. By the closed orbit lemma 9.3.5, it therefore suffices to show that Gx is an orbit of minimal dimension.

Take $x' = \mathcal{F}' \in X$. Then $Gx = G/\mathrm{Stab}_G(x')$. Set $R' = \mathrm{Stab}_G(x')^0_{\mathrm{red}}$, which is smooth and connected.

In fact, R' is solvable, because with respect to a basis for V adapted to the flag $\mathcal{F}'$, R' is a subgroup via ρ of the upper triangular subgroup of $\mathrm{GL}(V)$, which is solvable because it has an obvious composition series, and a subgroup of a solvable group is solvable, as we've seen earlier.

Thus R' is a (†)-subgroup of G. In particular, $\dim R' \leq \dim R$. So $\dim G/R' \geq \dim G/R$.

But $G/R' = G/\mathrm{Stab}_G(x')^0_{\mathrm{red}} \twoheadrightarrow G/\mathrm{Stab}_G(x')$ has finite fibers: passing to the underlying reduced does not affect the topology and the component group of the finite type k-group $\mathrm{Stab}_G(x')$ is finite. Thus $\dim Gx' = \dim G/\mathrm{Stab}\, G_(x') = \dim G/R' \geq \dim G/R = \dim Gx$. So Gx is an orbit of minimal dimension, so we are done. □

Corollary 22.1.4. *Let $k = \overline{k}$ and G a smooth connected affine k-group. If G is not unipotent, then there exists a nontrivial torus $\mathbf{G}_m \subset G$.*

Proof. Let $B \subset G$ be a Borel subgroup, which exists by Theorem 22.1.1. By the structure of split solvable groups (Proposition 21.3.1) we have $B = T \ltimes U$. (Note that B is split solvable because $k = \overline{k}$!) If $T \neq 1$ then the split torus $T \subset B \subset G$ gives such what we want. So we just need to rule out the case $B = U$ when B is unipotent. Assume for contradiction that B is unipotent. Since G is not unipotent, $B \neq G$. So G/B is proper and of *positive* dimension (since G is *connected*). Choose a representation $\rho : G \to \mathrm{GL}(V)$ such that $B = N_G(L)$ normalizes a line $L \subset V$. So we have a map $B \to \underline{\mathrm{Aut}}(L) \simeq \mathbf{G}_m$. But B is unipotent, so there are no nontrivial such maps. Therefore in fact $B = Z_G(L)$ *centralizes* L. Since L is a line, this is the same as $B = Z_G(v)$ for any nonzero $v \in L$. If we look at the induced orbit map $G/B \simeq Gv \subset V$, we see that it is an isomorphism from a proper variety to a quasi-affine variety. (Since $Gv \subset V$ is locally closed.) Since G/B has positive dimension, this is a contradiction. □

Corollary 22.1.5. *If $g \in G(\overline{k})$ are semisimple then G is a torus.*

Proof. To check whether G is a torus we can extend scalars, so without loss of generality $k = \overline{k}$. Choose a Borel subgroup $B = T \ltimes U \subset G$. Since $B(k)$ consists only of semisimple points, U must be trivial. Hence $B = T$ is commutative.

If we can show $B \subset Z_G$ (i.e., B is central in G) then in particular $B \triangleleft G$ is normal, so G/B is both affine and proper by Theorem 22.1.1 and the handout on quotients. Hence G/B would be finite and connected, so trivial, so $G = B$ is a torus.

To show B is central it suffices to show that conjugation by geometric points of B acts trivially on G, since $k = \overline{k}$ so $B(k) \subset B$ is dense. So pick $b \in B(k)$, and consider $G \to G$ given by $g \mapsto bgb^{-1}$. Since B is commutative, this induces a map $G/B \to G$. But this is a map from a proper connected variety to an affine one, hence constant, hence the trivial map. So the $\mathfrak{b}$-conjugation $G \to G$ is trivial as desired. ☐

Remark 22.1.6. One consequence of Corollary 22.1.5 is that if

$$1 \to T \to G \to T' \to 1$$

is an exact sequence of smooth connected affine k-groups with tori T and T' then G is a torus, since the geometric points of G are forced to be semisimple. But this is definitely overkill. Instead, clearly G is solvable, so by extending scalars to $\overline{k}$ and using the structure theory for split solvable groups, we have $G = T'' \ltimes U$. Now it is clear that $U = 1$ since G has semisimple geometric points.

Remark 22.1.7. The proof of Corollary 22.1.5 shows that if G is not solvable then $B \ntrianglelefteq G$ and B is not commutative. This in fact forces $\dim G \geq 3$: if $\dim G = 1$ then $G = \mathbf{G}_a$ or $\mathbf{G}_m$, which are both commutative; if $\dim G = 2$ then since B is solvable and G is not, $\dim B \leq 1$, and there are no noncommutative such groups.

Definition 22.1.8. A *Borel* k*-subgroup* $B \subset G$ is a solvable smooth connected k-subgroup (i.e. a (†)-subgroup) such that $B_{\overline{k}} \subset G_{\overline{k}}$ is a Borel subgroup in the earlier sense.

This is equivalent to the properness of $G_{\overline{k}}/B_{\overline{k}} = (G/B)_{\overline{k}}$, which by faithfully flat descent is equivalent to the properness of G/B.

Definition 22.1.9. A *parabolic* k-subgroup of G is a smooth k subgroup $P \subset G$ such that G/P is proper.

Thus a Borel k-subgroup is precisely a solvable connected parabolic k-subgroup. Many examples of parabolic subgroups can be found on Homework 9.

Unfortunately, nontrivial parabolics need not exist over general fields. Fortunately we have another way of digging holes into linear algebraic groups: according to the following amazing theorem of Grothendieck, we can *always* find maximal tori.

Theorem 22.1.10 (Grothendieck)**.** *If G is any smooth affine k-group, there exists a k-torus $T \subset G$ such that $T_{\overline{k}} \subset G_{\overline{k}}$ is a maximal torus.*

Proof. This is a huge deal, very non-trivial. We give a proof in the handout "Grothendieck's theorem on tori", based on arguments with finite group schemes (inspired by the proof in [Bor, 18.2(i)] that uses p-Lie-algebra techniques). It is recommended to skip this proof and to read it after the course is over. ☐

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23.1 Properties of parabolics

Let G be a smooth connected affine k-group.

Remark 23.1.1. We saw last time that the parabolicity of a subgroup $P \subset G$ is insensitive to field extension, because properness of the quotient G/P is so insensitive.

Remark 23.1.2. We left connectedness out of the definition of a parabolic k-subgroup, because (it will turn out) we get this for free!

Proposition 23.1.3. *Let $P \subset G$ be a smooth connected closed k-subgroup. Then P is parabolic if and only if $P_{\overline{k}}$ contains a Borel subgroup of $G_{\overline{k}}$.*

Proof. If $P_{\overline{k}}$ contains a Borel B then $G_{\overline{k}}/B$ is proper, and it surjects onto $G_{\overline{k}}/P_{\overline{k}}$, which is separated and finite type (being quasiprojective). Under these conditions, the image of a proper thing is proper, so $P_{\overline{k}}$ and hence P is parabolic.

Conversely suppose P is parabolic. Without loss of generality we can assume $k = \overline{k}$. Thus we can find a Borel $B \subset G$. The split solvable group B acts on the proper quotients G/P, so by the Borel fixed point Theorem 20.2.1 there exists $g \in G(k)$ with $BgP \subset gP$, which says precisely that $g^{-1}Bg \subset P$, so P contains a Borel. ☐

The next thing to consider is connectedness of parabolics.

Example 23.1.4. Connectedness results should be appreciated, as we have stressed before. Suppose $H \subset H' \subset G$ is a containment of closed k subgroups and H is smooth. Then in fact H' is smooth and $H^0 = H'^0$, if and only if $\dim H = \dim H'$, if and only if $\mathfrak{h} \subset \mathfrak{h}'$ is an equality inside $\mathfrak{g}$. So to prove $H = H'$ (assuming H is smooth) from the Lie algebraic fact $\mathfrak{h} = \mathfrak{h}'$, we need to know H' is connected!

Theorem 23.1.5 (Chevalley)**.** *Let $P \subset G$ be a parabolic k-subgroup of a smooth connected affine k-group G. Then P is connected, and $P = N_G(P)$ on geometric points.*

Proof. We sketch the idea, and will address it in full detail without circularity at the start of the next course (so it is taken on faith for this course, or see [Bor, Thm. 11.16] if you are impatient).

The first observation is that the connectedness of P is very closely related to $P = N_G(P)$: it's easy to see that P^0 is automatically parabolic, and $P^0 \lhd P$, so $P \subset N_G(P^0)$; if we knew $N_G(P^0) = P^0$ this gives $P = P^0$ connected.

The ingredients for the proof are the following.

I. Conjugacy of Borels in $G_{\overline{k}}$, which we have already shown.

II. The image of a Borel under a quotient map is a Borel. This will be shown in Theorem 23.2.2 below.

III. The centralizer of a torus is connected. This will be proved next week.

IV. If $\dim G \leq 2$ then G is solvable. This was shown in Remark 22.1.7. ☐

Remark 23.1.6. Granting Chevalley's theorem, $N_G(P)$ is connected because P is connected and $P = N_G(P)$ on geometric points. So by Example 23.1.4 one can prove the scheme theoretic equality $P = N_G(P)$ using the Lie algebras. This is a bit tricky, and will never be needed. (Sketch: Pass to $\overline{k}$, reduce to the case when G is reductive, and use the structure theory of reductive groups.)

Example 23.1.7. Obviously, by Proposition 23.1.3, a solvable smooth connected affine k-group G contains no proper (i.e. $\neq G$) parabolic subgroups. One thing to watch out for is that this can happen for non-solvable G as well, even connected reductive G.

The phenomenon at the end of preceding example is best illustrated and understood via the following remarkable and very important result that we mention just for general awareness but will not use in this course and will prove in the next course (it lies way beyond our present scope):

Proposition 23.1.8. *If $G = \mathcal{D}G$ is reductive and perfect, then G is k-isotropic (i.e., contains $\mathbf{G}_m$ as a k-subgroup) if and only if G contains a proper parabolic k-subgroup.*

For instance, if D is a finite-dimensional central division k-algebra with $\dim D > 1$, then the group norm-1 units SL(D) is such an example. Likewise, if (V, q) is a non-degenerate quadratic space of dimension at least 3 that is anisotropic (meaning q has no nontrivial zeroes in V) then SO(q) is also such an example.

Here is another result that we mention now for general awareness only:

Proposition 23.1.9. *Let G be a connected reductive group. If k is a local field (allowing $\mathbf{R}$) then G is k-anisotropic if and only if $G(k)$ is compact in the analytic topology. If k is a global field then G is k-anisotropic if and only if $G(\mathbf{A}_k)/G(k)$ is compact in its analytic topology.* ☐

(Again, proving this is beyond our scope; the case of local fields will be revisited in the next course. This result underlies the role of parabolic induction in representation theory because it says that the only connected reductive groups whose structure we can't get a handle on using a non-central k-torus are those whose arithmetically interesting associated spaces for representation-theoretic purposes are compact and thus amenable to study in other ways.)

23.2 Conjugacy of maximal tori

The proof of Grothendieck's Theorem 22.1.10 on the existence of geometrically maximal tori rests upon the conjugacy of maximal tori over an algebraically closed field (Theorem 23.2.2 below) plus the following fact, which is also used to prove Theorem 23.2.2.

Proposition 23.2.1. *If $k = \overline{k}$ and G is a solvable smooth connected affine k-group, then all maximal tori in G are conjugate, and the image of a maximal torus under a surjection $G \twoheadrightarrow G'$ is again a maximal torus.*

We will prove this later. First let us deduce the following.

Theorem 23.2.2. *Let* $k = \overline{k}$ *and* G *any smooth connected affine* k*-group. Then*

(i) All maximal tori in G *are* $G(k)$*-conjugate.*

(ii) If $f : G \twoheadrightarrow \overline{G}$ *is a quotient map then the image of a Borel of* G *is a Borel of* $\overline{G}$*, and the image of a maximal torus of* G *is a maximal torus of* $\overline{G}$*.*

Proof. (i) Every torus lies in a Borel, since a torus is in particular solvable smooth and connected, and a Borel is a maximal such. All Borels are conjugate, by Theorem 22.1.1. So we are reduced to the case of showing that two maximal tori of G both contained in a single borel B are conjugate. But in particular they are maximal tori of B. If they are $B(k)$-conjugate then they are $G(k)$-conjugate. So we are reduced to proving (i) for the case when G is solvable. But this follows from Proposition 23.2.1.

(ii) We have a surjection $G/B \twoheadrightarrow \overline{G}/f(B)$. Since G/B is proper, so is $\overline{G}/f(B)$. So $f(B)$ is parabolic. But it is also solvable, since the image of a solvable thing is solvable. It is also smooth and connected (smoothness by earlier results). So $f(B)$ is radical (what we called a (†)-subgroup before) and parabolic, which is equivalent to being a Borel by Theorem 22.1.1.

Now let T be a maximal torus of G. Choose a borel B of G containing T. Then $f(T)$ is a maximal torus of $f(B)$, by Proposition 23.2.1. Any by the above, $f(B)$ is a Borel of $\overline{G}$. So it is enough to show that a maximal torus $\overline{T} = f(T)$ of a borel $\overline{B}$ of $\overline{G}$ is also a maximal torus of $\overline{G}$. But by (i), any maximal torus $\overline{S}$ of $\overline{G}$ can be conjugated into $\overline{B}$. It is then maximal in $\overline{B}$. So by (i) applied to $\overline{B}$ we have $\dim \overline{S} = \dim(\text{some conjugate of } \overline{S}) = \dim \overline{T}$, since maximal tori of $\overline{B}$ must have the same dimension. Hence $\overline{T}$ is of the maximal dimension $\dim \overline{S}$ of tori contained in $\overline{G}$, so it must be maximal in $\overline{G}$. □

24 March 8

24.1 Proof of Proposition 23.2.1

First we assume the conjugacy of maximal tori and deduce that the image of a maximal torus under a quotient map is another maximal torus.

By the conjugacy of maximal tori, all maximal tori in G have the same dimension, say d_G.

Write $G = T \ltimes U$ where $U = \mathcal{R}_u(G)$. We claim that T is maximal. To see this, note that if $T \subset T'$ is another torus, then by group theory (on the functor of points) we have $T' = T \ltimes (T' \cap U)$. But the intersection $T' \cap U$ of a torus and a unipotent is trivial. So $T' = T$ is maximal.

Now write $f : G \twoheadrightarrow \overline{G}$ and $\overline{G} = \overline{T} \ltimes \overline{U}$ where $\overline{T}$ is a maximal torus, and $\overline{U} = \mathcal{R}_u(\overline{G})$ is unipotent. (We can do this since $\overline{G}$ is the image of the solvable group G, so solvable.) Now $\mathcal{R}_u$ is functorial with respect to surjections (because the surjective image of a normal subgroup is normal, which fails for not-necessarily-surjective maps in general). Thus $f(U) \subset \overline{U}$. And since $U \lhd G$, we have $f(U) \lhd \overline{G}$. So we get a map $T = G/U \overset{f}{\twoheadrightarrow} \overline{G}/f(U) = \overline{T} \ltimes (\overline{U}/f(U))$. Since the image of a torus is a torus, we conclude that $\overline{T} \ltimes (\overline{U}/f(U))$ is a torus. Since $\overline{U}$ is unipotent, so is its quotient $\overline{U}/f(U)$, whence $\overline{U}/f(U) = 1$, so $\overline{U} = f(U)$.

So great, we've seen that $f(\mathcal{R}_u(G)) = \mathcal{R}_u(\overline{G})$. Now we have $\overline{G} = f(G) = f(T) \cdot \overline{U}$ is generated by a torus $f(T)$ and a unipotent group $\overline{U}$, which must intersect trivially. Hence $\dim \overline{G} = \dim f(T) + \dim \overline{U}$. So $\dim f(T) = \dim \overline{G} - \dim \overline{U} = \dim \overline{T} = d_{\overline{G}}$. Thus for dimension reasons the torus $f(T)$ must be maximal.

Now we show the conjugacy of maximal tori. Write $G = T \ltimes U$ for a maximal torus T and $U = \mathcal{R}_u(G)$. Let T' be any maximal torus. We want to show that T' is $G(k)$-conjugate to T.

The easy case is when $T \subset Z_G$. For now $T \lhd G$ so $G = T \times U$. Thus the image of T' in G/T (which is unipotent) must be trivial, so $T' \subset T$. Thus by maximality $T' = T$.

If T is not central, but $T' \subset Z_G$ then $T \times T' \to G$ is a homomorphism, so $T \cdot T' \subset G$ is a torus (being the image of the torus $T \times T'$) and contains T'; so by maximality $T' = T \cdot T'$ whence $T \subset T'$ whence $T = T'$ by maximality of T.

Thus we can reduce to the case $G' = (Z_G T')^0 \subsetneq G$. Therefore $\dim G' < \dim G$, so $\mathfrak{g} \supsetneq \mathfrak{g}' = \operatorname{Lie}(Z_G T') = \mathfrak{g}^{T'}$, the $\operatorname{Ad}(T')$-invariants of $\mathfrak{g}$, by Homework 7, #4.

Consider the set of **roots** $\Phi(G, T') = \{x \in X(T') : a \neq 1, \mathfrak{g}_a \neq 0\}$ where $\mathfrak{g}_a$ denotes the **a-weight space**. This is some finite set $\{a_1, \dots, a_n\}$. Then $T' - \bigcup \ker a_i \subset T'$ is open, so since $k = \overline{k}$ there is a rational point $s \in (T' - \bigcup \ker a_i)(k)$. This satisfies $a(s) \neq 1$ for all $a \in \Phi(G, T')$. Such an element s is called **regular**. Now we certainly have an inclusion $G' = Z_G(T')^0 \subset Z_G(s)^0$. But by construction $\mathfrak{g}' \subset \mathrm{Lie}(Z_G(s)^0) = \mathrm{Lie}(Z_G(s)) = \mathfrak{g}^{\mathrm{Ad}(s)}$ is an equality. So by Example 23.1.4 we conclude that $Z_G(s)$ is forced to be smooth and $G' = Z_G(s)^0$.

Claim 24.1.1. *Every semisimple element $\gamma \in G(k)$ has a $G(k)$-conjugate contained in $T(k)$.*

Sketch of proof. Use a composition series to induct on $\dim G$, ultimately reducing to the case where $G = \mathbf{G}_m \ltimes \mathbf{G}_a$ (with the standard action of $\mathbf{G}_m$ on $\mathbf{G}_a$) is the "$ax + b$ group" $\{\left(\begin{smallmatrix} t & x \\ 0 & 1 \end{smallmatrix}\right)\}$. Here one can compute by hand. For details of this argument, see the handout "Conjugacy into a maximal torus". ☐

By the claim, choose $g \in G(k)$ such that $gsg^{-1} \in T(g)$, so $s \in g^{-1}Tg$. This is a commutative group so $g^{-1}Tg \subset Z_G(s)^0 = G'$. Hence $g^{-1}Tg$ centralizes the maximal torus T'. As we saw, this forces $g^{-1}Tg \subset T'$. Hence $\dim T \leq \dim T'$. On the other hand $\dim T' \leq \dim G/U = \dim T$ since $T' \cap U = 1$. Therefore for dimension reasons $T' = g^{-1}Tg$. ☐

Corollary 24.1.2. *If G is smooth connected and affine, $T \subset G$ is a maximal k-torus, and K/k is any field extension, then $T_K \subset G_K$ is a maximal K-torus.*

Corollary 24.1.3. *All maximal k-tori in a smooth connected affine group G have the same dimension.* ☐

Proof of Corollary 24.1.2. Let $H = Z_G(T)^0$. The formation of H commutes with scalar extension: $H_K = Z_{G_K}(T_K)^0$. So if T_K were not maximal in G_K, it would not be maximal in H_K. (Since tori are commutative, a bigger torus in G_K would have to centralize T_K.) So renaming H as G we can assume without loss of generality that $T \subset Z_G$. Then we can pass to the quotient G/T to reduce to the case $T = 1$, i.e. assume G has no nontrivial tori and show the same for G_K. (For a nontrivial torus in T' lifts to a subgroup of G which sits in a short exact sequence between T and T', and is thus itself a torus strictly bigger than T.)

Now by Grothendieck's Theorem 22.1.10, $G_{\overline{k}}$ has no nontrivial tori, since if it had one it would descend to a nontrivial torus of G. Hence $G_{\overline{k}}$ is unipotent, by Corollary 22.1.4. So G is unipotent, hence so is G_K, hence G_K contains no nontrivial torus. ☐

24.2 Cartan subgroups

Definition 24.2.1. A *Cartan subgroup* of a smooth connected affine k-group G is one of the form $Z_G(T)$ for a maximal k-torus $T \subset G$.

Remark 24.2.2. From Homework, any Cartan subgroup is smooth and solvable.

Remark 24.2.3. Since $T \subset Z_G(T)$ is central, we have $Z_G(T)_{\overline{k}} = T_{\overline{k}} \times U$ by structure theory for solvable groups.

Remark 24.2.4. For reductive groups G, it will turn out that $Z_G(T) = T$.

In fact, Cartan subgroups are always connected. More generally, we have the following.

Theorem 24.2.5. *Let G be a smooth connected affine k-group, $S \subset G$ a k-torus. Then $Z_G(S)$ is connected.*

Proof. To show connectedness we can assume without loss of generality that $k = \overline{k}$. Since G is connected, we can also assume $S \neq 1$ so $S \simeq \mathbf{G}_m^r$ for some r. Take any decomposition $S = S' \times S''$ into smaller tori. Then $S'' \subset Z_G(S')$ because S is commutative. Hence $Z_G(S) = Z_{Z_G(S')}(S'')$. Thus by induction on $\dim S$ we easily reduce to the case where $S = \mathbf{G}_m$. Then $S \hookrightarrow G$ is given by a cocharacter $\lambda : \mathbf{G}_m \hookrightarrow G$, which happens to be injective, but we don't care. So Lemma 24.3.3 below completes the proof. ☐

24.3 $P_G(\lambda), U_G(\lambda), Z_G(\lambda)$ for a 1-parameter subgroup λ

Now consider any **1-parameter subgroup** $\lambda : \mathbf{G}_m \to G$ for a smooth affine group G. Define an action of $\mathbf{G}_m \curvearrowright G$ by $t.g = \lambda(t)g\lambda(t)^{-1}$.

Example 24.3.1. If $G = \mathrm{GL}_2, \lambda(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ then $t.\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}$.

For any $R \in \mathrm{Alg}/k, g \in G(R)$ say that **the limit** $\lim_{t\to 0} t.g$ **exists** if the orbit map $\mathbf{G}_{m/R} \to G_R$ extends to a map $\mathbf{A}^1_R \to G_R$, and define **the limit** $\lim_{t\to 0} t.g$ to be the image "$t.0$" of $0 \in \mathbf{A}^1_R(R)$ in $G(R)$.

Definition 24.3.2. Let G be a smooth affine k-group and $\lambda : \mathbf{G}_m \to G$ a 1-parameter subgroup. For a k-algebra R, define the functor $P_G(\lambda)$ by

$$P_G(\lambda)(R) = \{g \in G(R) : \lim_{t\to 0} t.g \text{ exists}\}.$$

Define the subfunctor $U_G(\lambda)$ by

$$U_G(\lambda)(R) = \{g \in P_G(\lambda)(R) : \lim_{t\to 0} t.g = 1\}.$$

Define the centralizer $Z_G(\lambda)$ by

$$Z_G(\lambda)(R) = \{g \in G(R) : \mathbf{G}_{m/R} \text{ acts trivially on } g\}..$$

Lemma 24.3.3. *(0)* $P_G(\lambda), U_G(\lambda), Z_G(\lambda)$ *are subgroup functors of* G.

(i) The functors $P_G(\lambda), U_G(\lambda), Z_G(\lambda)$ *are representable by closed* k*-subgroups, and* $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$.

(ii) Multiplication $U_G(\lambda^{-1}) \times P_G(\lambda) \to G$ *is an open immersion.*

(iii) $P_G(\lambda), U_G(\lambda), Z_G(\lambda)$ *are smooth; they are connected if* G *is connected; and* $U_G(\lambda)$ *is unipotent.*

We'll discuss the proof next time.

Example 24.3.4. $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & ty \\ tx & t' + txy \end{pmatrix}$.

Example 24.3.5. If $G = \mathrm{GL}_3$ and $\lambda(t) = \begin{pmatrix} t^3 & & \\ & t^3 & \\ & & t \end{pmatrix}$ one computes

$$Z_G(\lambda) = \left\{ \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \right\}.$$

Using $3 > 1$ one sees

$$P_G(\lambda) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \right\}$$

and

$$U_G(\lambda) = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}, U_G(\lambda^{-1}) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ * & * & 1 \end{pmatrix} \right\}.$$

25 March 10

25.1 Proof of Lemma 24.3.3 on P_G, U_G, Z_G

We leave (0) as an exercise; use that $\mathbf{A}^1$ is a ring functor which lets us multiply the limits, if they exist.

25.1.1 Proof that (i)+(ii)⇒(iii) [except unipotence of $U_G(\lambda)$]

This is pretty easy. Since $U_G(\lambda^{-1}) \times P_G(\lambda)$ is an open subscheme of the smooth scheme G, we see that U_G and P_G are always smooth. Since $P_G = Z_G \times U_G$ we see that Z_G is also smooth. Connectedness is similar.

25.1.2 Proof of (i)

We begin by making some functorial observations.

First, $Z_G(\lambda) = P_G(\lambda) \cap P_G(\lambda^{-1})$.

Proof. The containment $Z_G(\lambda) \subset P_G(\lambda) \cap P_G(\lambda)$ is clear, because for an element which centralizes λ (and hence λ^{-1}), the orbit maps for both λ and λ^{-1} are constant; hence they certainly extend to $\mathbf{A}^1$.

Conversely, if $\mathbf{G}_m \to G_R$ given by $t \mapsto t.g$ for $g \in P_G(\lambda) \cap P_G(\lambda^{-1})$, the induced map $\mathcal{O}(G_R) \to R[t, t^{-1}]$ factors through both $R[t]$ and $R[t^{-1}]$. Hence it lands in their intersection R, which implies $t.g = g$ for all t since $1.g = g$. □

Thus we see that if P_G is representable for all λ, the same is true for Z_G, since we can take scheme theoretic intersections.

Next observe that the map $P_G(\lambda) \to G$ given by $g \mapsto \lim_{t\to 0} t.g$ is a homomorphism of group functors. (Exercise!) By definition its kernel is $U_G(\lambda)$. Thus if $P_G(\lambda)$ is representable, so is $U_G(\lambda)$.

Next note that the above map $P_G(\lambda) \to G$ automatically factors through $Z_G(\lambda)$. This is because for $g \in P_G(\lambda)(R)$ the orbit map $\mathbf{G}_m \to G_R$ extends to a map $\mathbf{A}^1_R \to G_R$ which is equivariant for the standard action of $\mathbf{G}_m$ on $\mathbf{A}^1$ (check!), and the origin is fixed by this action. Moreover $P_G(\lambda) \to Z_G(\lambda)$ is a section of the inclusion $Z_G(\lambda) \hookrightarrow P_G(\lambda)$

Combining the last two observations, we see that $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$ as group functors. Hence (i) reduces to showing the representability of $P_G(\lambda)$. Now we have $\mathbf{G}_m \curvearrowright G$, which gives a functorial linear representation of $\mathbf{G}_m$ on k[G], and hence a weight space decomposition

$$k[G] = \bigoplus_{n \in \mathbf{Z} = X(\mathbf{G}_m)} k[G]_n$$

where $t^*(f) = t^n f$ for $f \in k[G]_n$. One can check that $P_G(\lambda)$ is the zero scheme of the ideal $\langle k[G]_n \rangle_{n<0}$. For details, see [CGP, Lemma 2.1.4].

25.1.3 Proof of (ii) and unipotence of $U_G(\lambda)$

First consider an injective homomorphism (hence closed immersion) $j : G \hookrightarrow G'$; set $\lambda' = j \circ \lambda$ to be the induced 1-parameter subgroup. Check that $G \cap P_{G'}(\lambda') = P_G(\lambda), G \cap U_{G'}(\lambda') = U_G(\lambda), G \cap Z_{G'}(\lambda') = Z_G(\lambda)$.

Consequently multiplication $U_G(\lambda^{-1}) \times P_G(\lambda) \to G$ is monic, if and only if $U_G(\lambda^{-1}) \cap P_G(\lambda) = 1$, if and only $U_{G'}(\lambda'^{-1}) \times P_{G'}(\lambda') \to G'$ is monic, and $U_G(\lambda)$ is unipotent if $U_{G'}(\lambda')$ is unipotent *granting smoothness* of $U_G(\lambda)$.

The"Dynamic approach" handout shows that $G \cap (U_{G'}(\lambda'^{-1}) \times P_{G'}(\lambda')) = U_G(\lambda^{-1}) \times P_G(\lambda)$, assuming $U_{G'}(\lambda'^{-1}) \times P_{G'}(\lambda') \to G'$ is monic. Therefore if we can show that $U_{G'}(\lambda'^{-1}) \times P_{G'}(\lambda') \to G'$ is an open immersion, so is $U_G(\lambda^{-1}) \times P_G(\lambda) \to G$, hence $U_G(\lambda)$ is always smooth, so if we can show that $U_{G'}(\lambda')$ is always unipotent, the same is true for $U_G(\lambda)$.

Putting all this together, we have reduced to proving the claim in a bigger group. Obviously we take $G' = \mathrm{GL}(V)$. Now λ' gives a $\mathbf{G}_m$ action on V, so we have a weight space decomposition $V = \bigoplus V_{e_i}$ for $e_1 > \cdots > e_n$ say, where $t.v = t^e v$ for $v \in V_e$.

Now introduce a partial flag

$$\mathcal{F} = [V_{e_1} \subset V_{e_1} \oplus V_{e_2} \subset \cdots \subset V].$$

On Homework 10 it is shown that $P_{\mathrm{GL}(V)}(\lambda') = \{g : \text{ preserves the flag}\}$ which is a block upper triangle subgroup of GL(V), and hence smooth by inspection. Similarly $U_{\mathrm{GL}(V)}(\lambda') = \{g \in P_{G'}(\lambda') : \mathrm{gr}^\bullet_{\mathcal{F}}(g) = 1\}$ is the standard unipotent subgroup of the block upper triangle subgroup $P_{G'}(\lambda')$, which by inspection is unipotent and smooth, and similarly $U_{G'}(\lambda'^{-1})$ is the corresponding standard unipotent subgroup of the complementary block lower triangle subgroup. Hence $P_{G'}(\lambda') \cap U_{G'}(\lambda'^{-1}) = 1$. This gives monicity. On Homework 10 it is shown that this implies the map is an open immersion as well. □

Next we record another nice fact about P_G, U_G, Z_G.

Proposition 25.1.1. *Consider a diagram*

$$\mathbf{G}_m \xrightarrow{\lambda} G \xrightarrow{f} \overline{G}$$

and assume G *is connected, and set* $\overline{\lambda} = f\lambda$. *Then* $P_G(\lambda)$ *maps onto* $P_{\overline{G}}(\overline{\lambda})$ *and similarly for* U *and* Z.

Proof. We certainly have a commutative diagram

$$\begin{array}{ccc} U_G(\lambda^{-1}) \times P_G(\lambda) & \xhookrightarrow{\text{open}} & G \\ \downarrow \qquad\qquad \downarrow & & \downarrow f \\ U_{\overline{G}}(\overline{\lambda}^{-1}) \times P_{\overline{G}}(\overline{\lambda}) & \xhookrightarrow[\text{open}]{} & \overline{G} \end{array}$$

The problem is to show that the two left maps are surjective. But if either is not, then its image has strictly smaller dimension than the dimension of the guy upstairs. Hence the image of the total map on the left side and then across on the bottom cannot be dense, giving a contradiction (to the openness of the bottom map) since $\overline{G}$ is irreducible.

Since $P = Z \ltimes U$ we get the analogous result for Z too. □

26 March 12

26.1 Classification of split reductive groups of rank 1

Definition 26.1.1. A connected reductive k-group G is called k-*split* if it has a split maximal k-torus.

Example 26.1.2. $\mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{PGL}_n$.

Theorem 26.1.3. *Any split connected reductive group is quasi-split, i.e. has a Borel* k*-subgroup.*

This is a hard theorem, and relies on getting some classification results off the ground, so we cannot invoke it yet. (It will be proved in the next course.)

Definition 26.1.4. The k-*rank* of a connected reductive group G is the dimension of its maximal k-split tori. The *rank* is the $\overline{k}$-rank.

For example, "k-rank zero" means k-anisotropic. The following theorem over general fields will be proved in the next course (whereas we have already established the crucial case of algebraically closed ground fields):

Theorem 26.1.5. *All the maximal* k*-split tori in a connected reductive group are* $G(k)$*-conjugate (so the* k*-rank is well-defined!).*

We already know the rank is well-defined. But the above theorem must be proved before we know the same for the k-rank!

The following theorem, which we will proof via dynamic methods as the punchline to this course (taking for granted Chevalley's theorem on parabolic subgroups that we will prove without circularity near the start of the next course), is the literally the key to getting the classification theory for connected reductive groups off the ground:

Theorem 26.1.6. *Let* G *be a connected reductive* k*-group which is not solvable (i.e. is not a torus). Assume* G *is* k*-split of rank 1, with split maximal* k*-torus* T*. Then* $G \simeq \mathrm{SL}_2$ *or* PGL_2*.*

Proof. We cannot use the (not proved!) theorem that split implies quasi-split, so a priori it is not obvious that G has a Borel k-subgroup. Of course this is no problem over $\overline{k}$. So the plan is to do the proof "in two passes": first show that the result over $\overline{k}$ implies the existence of a Borel k-subgroup, and then prove the result assuming the existence of such a subgroup (an assumption that we know holds over $\overline{k}$!).

So assume the theorem holds over $\overline{k}$. Let $\lambda : \mathbf{G}_m \simeq T \subset G$. Then we have $Z_G(\lambda) = Z_G(T) = T$, since the last equality can be checked over $\overline{k}$ and we know it holds for the maximal torus in $\mathrm{SL}_2, \mathrm{PGL}_2$. Likewise we know $U_G(\lambda^{\pm 1})$ are both 1-dimensional, since geometrically they must be the two $\mathbf{G}_a$'s inside SL_2 or PGL_2. So consider the open subscheme $U(\lambda^{-1}) \times Z(\lambda) \times U(\lambda) \hookrightarrow G$. We have $Z(\lambda) \times U = T \times (\text{unipotent}) = \text{solvable}$, and $\dim U(\lambda^{-1}) = 1$. So for dimension reasons, since G is not solvable, $Z(\lambda) \times U$ is a Borel: there is simply no extra room to make a bigger solvable connected subgroup. Consequently, assuming the theorem holds over $\overline{k}$, there exists a Borel k-subgroup.

So now we can prove the theorem assuming the existence of a Borel k-subgroup B. The idea is as follows. First we will construct a map $G \to \mathrm{PGL}_2 = \underline{\mathrm{Aut}}_{\mathbf{P}^1/k}$ using B. Then we will show the map is a central isogeny, and finally that its kernel must be μ_e for $e|2$. (If $e = 1$ this gives $G = \mathrm{PGL}_2$ and if $e = 2$ this will give $G = \mathrm{SL}_2$.)

First consider the torus action $T \subset G \curvearrowright G/B$. This quotient is proper because B is parabolic, and T is (split) solvable. A refinement of the Borel fixed point theorem [Bor, §13.5]) shows that there exist $\geq 1 + \dim(G/B) \geq 2$ [since $G \neq B$ as G is not solvable] fixed $\overline{k}$-points for the actions $T_{\overline{k}} \curvearrowright (G/B)_{\overline{k}}$. Any such $gB_{\overline{k}}$ for $g \in G(\overline{k})$ gives $T_{\overline{k}} g B_{\overline{k}} \subset g B_{\overline{k}}$ and hence $T_{\overline{k}} \subset g B_{\overline{k}} g^{-1}$, i.e. this conjugate of $B_{\overline{k}}$ is a Borel of $G_{\overline{k}}$ containing $T_{\overline{k}}$. Any such Borel is conjugate to $B_{\overline{k}}$, and the corresponding conjugator g is determined modulo

$N_{G_{\overline{k}}}(B_{\overline{k}}) = B_{\overline{k}}$ [using Chevalley's theorem on parabolic subgroups!]. Thus in fact $T_{\overline{k}}$-fixed $\overline{k}$-points of $(G/B)_{\overline{k}}$ are in *bijection* with the set of Borels of $G_{\overline{k}}$ containing $T_{\overline{k}}$. By Homework 9, #6, the latter is in bijection with the Weyl group $W = (N_G(T)/Z_G(T))(\overline{k})$. So $2 \leq 1 + \dim(G/B) \leq \#\{T_{\overline{k}}\text{-fixed } \overline{k}\text{-points of } (G/B)_{\overline{k}}\} = \#W$. But W injects into $\mathrm{Aut}_{\overline{k}}(T_{\overline{k}}) = \mathrm{Aut}(\mathbf{G}_m) = \pm 1$. So $\#W \leq 2$. Hence we have (by squeezing) $\dim(G/B) = 1$.

So G/B is a smooth projective geometrically connected curve over k with a rational point (the coset of the identity). We claim it is $\mathbf{P}^1$. It suffices to show its genus is zero, which can be checked over $\overline{k}$. But there are only two fixed points, geometrically, and plenty of rational points. If we consider the orbit map $\mathbf{G}_m = T_{\overline{k}} \to (G/B)_{\overline{k}}$ for any *non*-fixed point, the map is dominant. Hence $(G/B)_{\overline{k}}$ is a rational curve, and so has genus zero.

So we have shown $G/B \simeq \mathbf{P}^1_k$ over k. Great! Thus the action $G \curvearrowright G/B$ defines a k-homomorphism $\varphi : G \to \underline{\mathrm{Aut}}_{\mathbf{P}^1/k} = \mathrm{PGL}_2$. Of course the last equality requires serious stuff (the theorem on formal functions,...). Instead we can get the map $G \to \mathrm{PGL}_2$ in a more elementary fashion, since G is a variety: we can spread out from the generic fiber G_η using that $k(\eta)$ is a field and we know the automorphisms of $\mathbf{P}^1_{k(\eta)}$ are $\mathrm{PGL}_2(k(\eta))$; this gives us a map from a dense open of G to PGL_2, and using that $\varphi : G \to \underline{\mathrm{Aut}}_{\mathbf{P}^1/k}$ is a homomorphism we can translate it around to all of G. More precisely, we appeal to the general fact that a rational map between smooth connected k-group varieties that is multiplicative in the evident "rational maps" sense extends uniquely to a k-homomorphism. To prove it we may pass up to k_s and then use translation by rational points to extend the domain of definition, since the k-point translates of any dense open *do* cover the group: exercise!

The next claim is that $\varphi : G \to \mathrm{PGL}_2$ is surjective. We have $g \in \ker\varphi \Rightarrow g$ acts trivially on G/B $\Rightarrow gB \subset B \Rightarrow g \in B$, totally functorially. This gives $\ker\varphi \subset B$. Consequently $(\ker\varphi_{\overline{k}})^0_{\mathrm{red}}$ is solvable. Now we have a map $G_{\overline{k}}/(\ker\varphi_{\overline{k}})^0_{\mathrm{red}} \to \mathrm{PGL}_2$ with finite kernel. If this map is not surjective, then the image (being closed) has dimension $\leq \dim \mathrm{PGL}_2 - 1 = 3 - 1 = 2$ and is thus solvable. But this gives a short exact sequence

$$1 \to (\ker\varphi_{\overline{k}})^0_{\mathrm{red}} \to G_{\overline{k}} \to (\text{something}) \to 1$$

and the outer terms are both solvable. So $G_{\overline{k}}$ is solvable too, a contradiction. Thus φ must be surjective.

Net $N = (\ker\varphi_{\overline{k}})^0_{\mathrm{red}}$. This is smooth and connected, and normal in the reductive group $G_{\overline{k}}$. On Homework 10 it is shown that this implies N is reductive as well. But we already saw N is solvable. Hence N is a torus.

On Homework 6 #3(ii) it was shown that a normal torus in a connected group is central. Therefore N is a central torus. If N is nontrivial, then it is one dimensional (this being the dimension of maximal split tori in the rank 1 group $G_{\overline{k}}$) and hence equal to $T_{\overline{k}}$. [By centrality $N \cdot T_{\overline{k}}$ is a torus, so contains $T_{\overline{k}}$, so is equal to $T_{\overline{k}}$.] Therefore $T_{\overline{k}}$ is central in $G_{\overline{k}}$. So the quotient $G_{\overline{k}}/T_{\overline{k}}$ is a group with no nontrivial maximal torus, and hence is unipotent, hence solvable. The same short exact sequence trick gives a contradiction since G is not solvable. Thus N must be trivial, so $\ker\varphi$ is finite, so $\dim G = 3$ and φ is an isogeny.

Next we want to show that φ is a central isogeny. Inside G we have T, a one dimensional split maximal torus. Hence $\varphi(T)$ is a split maximal torus in PGL_2. On the "dynamic approach..." handout on the website, it is shown that all split maximal tori in PGL_2 are rationally conjugate. So composing φ with an appropriate $\mathrm{PGL}_2(k)$-conjugation, which amounts to changing the isomorphism $G/B \simeq \mathbf{P}^1_k$, we can assume without loss of generality that $\varphi : G \twoheadrightarrow \mathrm{PGL}_2$ maps T surjectively onto the diagonal split torus in PGL_2.

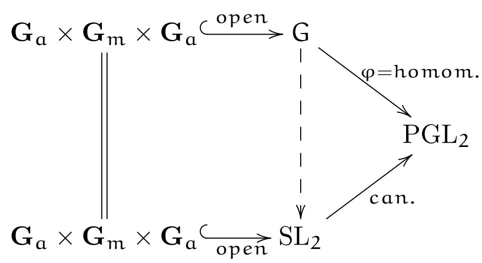
In PGL_2 the maximal torus is its own centralizer. Since φ is finite, for dimension reasons we see that $Z_G(T) = T$ as well. Choose an isomorphism $\lambda : \mathbf{G}_m \simeq T$. This gives $\varphi\lambda : \mathbf{G}_m \to (\mathrm{diag.torus}) \simeq \mathbf{G}_m$, which must be given by a map $t \mapsto t^e$ for some $e \neq 0$ (by surjectivity). Replacing λ by λ^{-1} if necessary, we can assume $e \geq 1$. Now $G \supset U(\lambda^{-1}) \times Z(\lambda) \times U(\lambda)$, and since φ is surjective, $U(\lambda^{\pm 1})$ maps surjectively via φ onto $U_{\mathrm{PGL}_2}(\varphi\lambda^{\pm 1}) = U^{\pm} \simeq \mathbf{G}_a$, where $U^{\pm}$ are the standard upper and lower triangular unipotent subgroups of PGL_2. (That $U_{\mathrm{PGL}_2}(\varphi\lambda^{\pm 1}) = U^{\pm}$ can be seen by direct calculation.) Consequently $\dim U(\lambda^{\pm 1}) = 1$, as $U(\lambda^{\pm 1})$ is smooth unipotent and connected.

The action of $T \curvearrowright U_G(\lambda^{\pm 1})$ lies over the action of the diagonal $\mathbf{G}_m$ in PGL_2 on $U^{\pm}$ (by conjugation). Said action is nontrivial, so since φ is surjective, T acts nontrivially on $U_G(\lambda^{\pm 1})$. By Homework 9,#1 (structure of unipotent groups a la Tits in [CGP, App. B]), this forces $U_G(\lambda^{\pm 1}) \simeq \mathbf{G}_a$ as k-groups. Since $\dim G = 3$, $B^{\pm} := T \ltimes U_G(\lambda^{\pm 1})$ are Borels, for dimension reasons; their intersection is T. Geometrically, it is easy to see, $(\ker\varphi)(\overline{k})$ is contained in the intersection of all Borels. Hence in particular $\ker\varphi \subset B^+ \cap B^- = T = \mathbf{G}_m$. Since φ maps $T \to \mathrm{diag}$ by $t \mapsto t^e$, we can compute directly that $\ker\varphi = \mu_e \lhd G$. Thus we get an action of the connected group G on μ_e, hence a morphism $G \to \underline{\mathrm{Aut}}(\mu) \simeq (\mathbf{Z}/e\mathbf{Z})^\times$ (which is discrete), the last isomorphism being by Homework 10. Thus G acts trivially on μ_e, so $\ker\varphi = \mu_e$ is central, and thus φ is a central isogeny.

It remains to show $e|2$ and to verify this forces $G \simeq \mathrm{SL}_2$ or PGL_2. Go back and choose $n \in N_G(T)(\overline{k})$ representing the unique nontrivial element of the Weyl groups (which we saw has order 2). Conjugation by n must be the unique nontrivial automorphism of $T = \mathbf{G}_m$, and hence be inversion. Since μ_e is central it is fixed pointwise by this automorphism. So inversion acts trivially on μ_e. This indeed forces $e|2$.

If $e = 1$ obviously φ is an isomorphism $G \simeq \mathrm{PGL}_2$.

If $e = 2$, observe that since $\ker\varphi \subset T$ and φ maps $U_G(\lambda^{\pm 1})$ isomorphically onto $U^\pm$, we obtain a diagram



There are actions of $\mathbf{G}_m$ on both copies of $\mathbf{G}_a$ on both the top and the bottom (the identification of $\mathbf{G}_a$'s expressing exactly that $U_G(\lambda^{\pm 1}) \to U^\pm$ are isomorphisms). By design the actions are compatible with the identification given by the vertical $=$, so this diagram commutes. The vertical $=$ induces the indicated rational map $G \to \mathrm{SL}_2$. The finiteness of the kernels of the diagonal maps forces $G \dashrightarrow \mathrm{SL}_2$ to be a homomorphism. By translation, any rational homomorphism is an actual morphism. (This is shown on Homework 10 by Galois descent from the case $k = k_s$.) The theorem follows. □

Why do we care about the theorem? Inside any reductive group G suppose we have a k-split maximal torus T. The torus acts on the Lie algebra $\mathfrak{g}$ giving a weight space decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_a$. Let $T_a = (\ker a)^0_{\mathrm{red}}$ for $a \neq 0$ that arise. Then it turns out that $\mathcal{D}Z_G(T_a)$ is a k-split non-solvable reductive group of rank 1, and thus is SL_2 or PGL_2 by the theorem. So these groups show up all over the place, and are what will allow us to classify reductive groups! The first step towards this is to use this rank-1 classification to build the root datum associated to a split connected reductive group over a general field as part of the structure theory of connected reductive groups in the sequel course.

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