Algebraic Groups I. Properties of orthogonal groups

1. Basic definitions

Let V be a vector bundle of constant rank $n \geq 1$ over a scheme S, and let $q : V \to L$ be a quadratic form valued in a line bundle L, so we get a symmetric bilinear form $B_q : V \times V \to L$ defined by

$$B_q(x, y) = q(x + y) - q(x) - q(y).$$

Assume q is fiberwise non-zero over S, so $(q = 0) \subset \mathbf{P}(V^*)$ is an S-flat hypersurface with fibers of dimension $n - 2$ (understood to be empty when $n = 1$). By HW2, Exercise 4 (and trivial considerations when $n = 1$), this is smooth precisely when for each $s \in S$ one of the following holds: (i) B_{q_s} is non-degenerate and either $\mathrm{char}(k(s)) \neq 2$ or $\mathrm{char}(k(s)) = 2$ with n even, (ii) the defect δ_{q_s} is 1 and $\mathrm{char}(k(s)) = 2$ with n odd and $q_s|_{V_s^\perp} \neq 0$. (Likewise, $\delta_{q_s} \equiv \dim V_s$ when $\mathrm{char}(k(s)) = 2$.) In such cases we say (V, q) is *non-degenerate*; (ii) is the "defect-1" case at s. (In [SGA7, XII, §1], such (V, q) are called *ordinary*.)

Clearly the functor

$$S' \rightsquigarrow \{g \in \mathrm{GL}(V_{S'}) \,|\, q_{S'}(gx) = q_{S'}(x) \text{ for all } x \in V_{S'}\}$$

on S-schemes is represented by a finitely presented closed S-subgroup $\mathrm{O}(q)$ of $\mathrm{GL}(V)$, even without the non-degeneracy condition on q. We call it the *orthogonal group* of (V, q). Define the *naive special orthogonal group* to be

$$\mathrm{SO}'(q) := \ker(\det : \mathrm{O}(q) \to \mathbf{G}_{\mathrm{m}}).$$

We say "naive" because this is the wrong notion in the non-degenerate case when n is even and 2 is not a unit on S. The *special orthogonal group* $\mathrm{SO}(q)$ will be defined shortly in a characteristic-free way, using input from the theory of Clifford algebras when n is even. (The distinction between even and odd n when defining $\mathrm{SO}(q)$ is natural, because it will turn out that $\mathrm{O}(q)/\mathrm{SO}(q)$ is equal to μ_2 for odd n but $(\mathbf{Z}/2\mathbf{Z})_S$ for even n. In the uninteresting case $n = 1$ we have $\mathrm{O}(q) = \mu_2$ and $\mathrm{SO}(q) = 1$.)

Definition 1.1. Let $S = \mathrm{Spec}\, \mathbf{Z}$. The *National Bureau of Standards* (or *standard split*) quadratic form q_n on $V = \mathbf{Z}^n$ is as follows, depending on the parity of $n \geq 1$:

$$q_{2m} = \sum_{i=1}^{m} x_{2i-1}x_{2i}, \quad q_{2m+1} = x_0^2 + \sum_{i=1}^{m} x_{2i-1}x_{2i}$$

(so $q_1 = x_0^2$). We define $\mathrm{O}_n = \mathrm{O}(q_n)$ and $\mathrm{SO}'_n = \mathrm{SO}'(q_n)$.

It is elementary to check that $(\mathbf{Z}^n, q_n)$ is non-degenerate.

Remark 1.2. In the study of quadratic forms q over a domain A, such as the ring of integers in a number field or a discrete valuation ring, the phrase "non-degenerate" is often used to mean "non-degenerate over the fraction field". Indeed, non-degeneracy over A in the sense defined above is rather restrictive. In addition to the National Bureau of Standards form q_n, other non-degenerate examples over $\mathbf{Z}$ (in our restrictive sense) are the quadratic spaces arising from even unimodular lattices, such as the E_8 and Leech lattices.

Lemma 1.3. *If (V, q) is a non-degenerate quadratic space of rank $n \geq 1$ over a scheme S then it is isomorphic to $(\mathscr{O}_S^n, q_n)$ fppf-locally on S. If n is odd or 2 is a unit on S then it suffices to use the étale topology rather than the fppf topology.*

Proof. In [SGA7, XII, Prop. 1.2] the smoothness of $(q = 0)$ is used to prove the following variant by a simple induction argument: q is an étale form of q_n when n is even and is an étale form of $ux_0^2 + q_{2m}$ when $n = 2m+1$ is odd, with u a unit on the base. Once the induction is finished, we are done when n is even and we need to extract a square root of u when n is odd. This accounts for the necessity of working fppf-locally for odd n when 2 is not a unit on the base. ■

Lemma 1.3 is very useful for reducing problems with general non-degenerate quadratic spaces to the case of q_n over $\mathbf{Z}$. This will be illustrated numerous times in what follows, and now we illustrate it with Clifford algebras in the relative setting. Consider a non-degenerate (V, q) with rank $n \geq 1$. The *Clifford algebra* $\mathrm{C}(V, q)$ is the quotient of the tensor algebra of V by the relation $x^{\otimes 2} = q(x)$ for local sections x of V. This inherits a natural $\mathbf{Z}/2\mathbf{Z}$-grading from the $\mathbf{Z}$-grading on the tensor algebra, and by considering expansions relative to a local basis of V we see that $\mathrm{C}(V, q)$ is a finitely generated $\mathscr{O}_S$-module. The Clifford algebra is a classical object of study over fields, and we need some properties of it over a general base ring (or scheme) when n is even:

Lemma 1.4. *Assume n is even. The $\mathscr{O}_S$-algebra $\mathrm{C}(V, q)$ and its degree-0 part $\mathrm{C}^+(V, q)$ are vector bundles over S of finite rank. Their quasi-coherent centers are respectively equal to $\mathscr{O}_S$ and a rank-2 finite étale $\mathscr{O}_S$-algebra Z_q.*

Proof. We may work fppf-locally on S, so by Lemma 1.3 we may assume that V admits a basis identifying q with q_n. Since n is even, there are complementary isotropic free subbundles $W, W' \subset V$ (in perfect duality via B_q). This leads to concrete descriptions of $\mathrm{C}(V, q)$ and $\mathrm{C}^+(V, q)$ in [SGA7, XII, 1.4]: $\mathrm{C}(V, q)$ is naturally isomorphic as a $\mathbf{Z}/2\mathbf{Z}$-graded algebra to the endomorphism algebra of the exterior algebra $A = \wedge^\bullet(W)$, where $w \in W$ acts via $w \wedge (\cdot)$ and $w' \in W'$ acts via the contraction operator

$$w_1 \wedge \cdots \wedge w_m \mapsto \sum_{i=1}^{m} (-1)^{i-1} B_q(w_j, w') w_1 \wedge \ldots \widehat{w_i} \cdots \wedge w_m.$$

(It is natural to consider the exterior algebra of W, since $q|_W = 0$ and the Clifford algebra associated to the vanishing quadratic form is the exterior algebra.) The $\mathbf{Z}/2\mathbf{Z}$-grading of the endomorphism algebra of A is defined in terms of the decomposition $A = A_+ \bigoplus A_-$, where A_+ is the "even part" and A_- is the "odd part: an endomorphism of A is *even* when it respects this decomposition and *odd* when it carries A_- into A_+ and vice-versa. We thereby see that $\mathrm{C}^+(V, q)$ is the direct product of the endomorphism algebras of A_+ and A_-. Since the center of a matrix algebra over any ring consists of the scalars, we are done. ■

The action of $\mathrm{O}(q)$ on $\mathrm{C}(V, q)$ preserves the grading and hence induces an action on $\mathrm{C}^+(V, q)$, so we obtain an action of $\mathrm{O}(q)$ on the finite étale center Z_q of $\mathrm{C}^+(V, q)$. The automorphism scheme $\underline{\mathrm{Aut}}_{Z_q/\mathscr{O}_S}$ is uniquely isomorphic to $(\mathbf{Z}/2\mathbf{Z})_S$ since Z_q is finite étale of rank 2 over $\mathscr{O}_S$. Thus, for even n we get a homomorphism

$$D_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_S$$

that is compatible with isomorphisms in (V, q), and its formation commutes with any base change on S. This is called the *Dickson morphism* because D_{q_n} underlies the definition of the *Dickson invariant* (see Remark 2.6).

Proposition 1.5. *Assume n is even. The map D_q is surjective.*

Flatness properties for D_q require more work; see Proposition 2.3

Proof. We may pass to geometric fibers over S, where the assertion is that the center Z_q of $\mathrm{C}^+(V,q)$ is not centralized by the action of $\mathrm{O}(q)$ on $\mathrm{C}(V,q)$. This is a classical fact from the theory of Clifford algebras, and we now recall the proof.

Let (V,q) be an even-dimensional nonzero quadratic space (V,q) over a field k. Naturally V is a subspace of $\mathrm{C}(V,q)$ and the conjugation action on V by the Clifford k-group

$$\mathrm{Cliff}(V,q) := \{u \in \mathrm{C}(V,q)^\times \mid uVu^{-1} = V\}$$

defines a homomorphism from $\mathrm{Cliff}(V,q)$ onto $\mathrm{O}(q)$ [Bou, IX, §9.5, Thm. 4(a)]. Thus, the nontriviality of D_q amounts to the assertion that the rank-2 center Z_q of $\mathrm{C}^+(V,q)$ is not centralized by the units in $\mathrm{Cliff}(V,q)$. Viewing V as an affine space, the Zariski-open non-vanishing locus of q on V is Zariski-dense in V and lies in $\mathrm{Cliff}(V,q)$ (such $v \in V - \{0\}$ act on V via orthogonal reflection $x \mapsto -x + (B_q(x,v)/q(v))v$ through v [Bou, IX, §9.5, Thm. 4(b)]), so this part of V generates $\mathrm{C}(V,q)$ as an algebra. Hence, if Z_q were centralized by the group $\mathrm{Cliff}(V,q)$ then it would be central in the algebra $\mathrm{C}(V,q)$, an absurdity since $\mathrm{C}(V,q)$ has scalar center. ■

We can finally define special orthogonal groups, depending on the parity of n.

Definition 1.6. Let (V,q) be a non-degenerate quadratic space of rank $n \geq 1$ over a scheme S. The *special orthogonal group* $\mathrm{SO}(q)$ is $\mathrm{SO}'(q) = \ker(\det|_{\mathrm{O}(q)})$ when n is odd and $\ker D_q$ when n is even. For any $n \geq 1$, $\mathrm{SO}_n := \mathrm{SO}(q_n)$.

By definition, $\mathrm{SO}(q)$ is a closed subgroup of $\mathrm{O}(q)$, and it is also an open subscheme of $\mathrm{O}(q)$ when n is even. (In contrast, SO_{2m+1} is not an open subscheme of O_{2m+1} over $\mathbf{Z}$ because we will prove that $\mathrm{O}(q) = \mathrm{SO}(q) \times \mu_2$ for odd n via the central $\mu_2 \subset \mathrm{GL}(V)$, and over $\mathrm{Spec}\,\mathbf{Z}$ the identity section of μ_2 is not an open immersion.)

The group $\mathrm{SO}'(q)$ is not of any real interest when n is even and 2 is not a unit on the base (and we will show that it coincides with $\mathrm{SO}(q)$ in all other cases). The only reason we are considering $\mathrm{SO}'(q)$ in general is because it is the first thing that comes to mind when trying to generalize the theory over $\mathbf{Z}[1/2]$ to work over $\mathbf{Z}$. We will see that SO'_{2m} is not $\mathbf{Z}_{(2)}$-flat. (Example: Consider $m = 1$ and $S = \mathrm{Spec}\,\mathbf{Z}_{(2)}$. We have $\mathrm{O}_2 = \mathbf{G}_\mathrm{m} \times (\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{SO}_2 = \mathbf{G}_\mathrm{m}$, whereas SO'_2 is the reduced closed subscheme of O_2 obtained by removing the open non-identity component in the generic fiber.)

From now on, we only consider non-degenerate (V,q). To orient ourselves, it is useful to record the main properties we shall prove for the "good" groups associated to such (V,q):

Theorem 1.7. *The group* $\mathrm{SO}(q)$ *is smooth of relative dimension* $n(n-1)/2$ *with connected fibers. Its functorial center is trivial for odd* n *and equals the central* $\mu_2 \subset \mathrm{O}(q)$ *for even* n.

(1) *Assume* n *is even. The Dickson morphism* D_q *is a smooth surjection identifying* $(\mathbf{Z}/2\mathbf{Z})_S$ *with* $\mathrm{O}(q)/\mathrm{SO}(q)$. *In particular,* $\mathrm{O}(q)$ *is smooth with* $\#\pi_0(\mathrm{O}(q)_s) = 2$ *for all* $s \in S$.
(2) *Assume* n *is odd. Multiplication against the central* $\mu_2 \subset \mathrm{O}(q)$ *defines an isomorphism* $\mu_2 \times \mathrm{SO}(q) \simeq \mathrm{O}(q)$. *In particular,* $\mathrm{O}(q)$ *is smooth over* $S[1/2]$ *and flat over* S, *and* $\mathrm{O}(q)/\mathrm{SO}(q) = \mu_2$ *(so* $\mathrm{O}(q)_s$ *is connected if* $\mathrm{char}(k(s)) = 2$*).*

We will also determine the functorial centers of these groups when $n \geq 3$: $\mathrm{O}(q)$ has functorial center represented by the central μ_2, whereas the centers of $\mathrm{SO}(q)$ and $\mathrm{SO}'(q)$ coincide and equal μ_2 (resp. 1) when n is even (resp. odd). Since the case of odd rank requires special care in residue characteristic 2, we shall first analyze even rank, where we can use characteristic-free arguments.

2. Even rank

We will prove the smoothness of the closed subscheme $\mathrm{O}(q) \subset \mathrm{GL}(V)$ when n is even by comparing its fibral dimension to the number of local equations that cut it out inside the smooth scheme $\mathrm{GL}(V)$. (A proof can be given by using the infinitesimal criterion, but equation-counting is simpler in this case.)

One subtlety is that since we do not yet know S-flatness for $\mathrm{O}(q)$ (which will be obtained from smoothness results for $\mathrm{SO}(q)$), we do not know that there is a plentiful supply of sections after fppf-local base change on S. In particular, it is not evident whether smoothness near the identity section is sufficient to deduce smoothness of the entire group (as that intuition over a field is based on the ability to do translations after a ground field extension, which is always faithfully flat).

As a concrete counterexample, consider the reduced closed complement G of the open non-identity point in the generic fiber of the constant group $(\mathbf{Z}/2\mathbf{Z})_R$ over a discrete valuation ring R. This R-group is a disjoint union of the identity section and an additional rational point in the special fiber, so it is affine and also R-smooth near the identity section with constant fibers but is not R-flat. To circumvent this problem, we will use a "global" criterion in terms of equation-counting:

Lemma 2.1. *Let R be a ring and G a smooth affine R-group. Let $G' \hookrightarrow G$ be a closed subgroup scheme whose defining ideal admits c global generators $f_1, \dots, f_c$ and whose fibers G'_s satisfy $\dim \mathrm{Tan}_{e(s)}(G'_s) = \dim \mathrm{Tan}_{e(s)}(G_s) - c$. Then G' is R-smooth.*

The special case $G = \mathrm{GL}(V)$ is [DG, II, §5, 2.7].

Proof. Each fiber G'_s has codimension at most c in G_s, so

$$\dim G'_s \geq \dim G_s - c = \dim \mathrm{Tan}_{e(s)}(G_s) - c = \dim \mathrm{Tan}_{e(s)}(G'_s).$$

Thus, the $k(s)$-group G'_s is smooth for all $s \in S$ (due to the homogeneity of the geometric fiber $G'_{\overline{s}}$). Since an open subset of G' that contains all closed points of all fibers must be the entire space, by openness of the smooth locus it suffices to prove that G' is smooth at each point g' that is closed in its fiber G'_s. The $k(s)$-smoothness of G'_s at the closed point g' implies that the local ring $\mathscr{O}_{G'_s,g'} = \mathscr{O}_{G_s,g'}/((f_1)_s, \dots, (f_c)_s)$ is regular with dimension $\dim G'_s = \dim G_s - c = \dim \mathscr{O}_{G_s,g'} - c$. In other words, in the regular local ring $\mathscr{O}_{G_s,g'}$ the sequence $\{(f_j)_s\}$ in the maximal ideal is part of a regular system of parameters. Since G_s and G'_s are $k(s)$-smooth, by computing at a point over g' on geometric fibers over s we see that the elements

$$\mathrm{d}(f_j)(g') \in \Omega^1_{G'/S,g'}/\mathfrak{m}_{g'} = \Omega^1_{G'_s/k(s),g'} \otimes_{\mathscr{O}_{G'_s,g'}} k(g')$$

are $k(g')$-linearly independent. Hence, by the Jacobian criterion for smoothness of a closed subscheme of a smooth scheme [BLR, 2.2/7], G' is R-smooth. ∎

Lemma 2.2. *Inside* $\mathrm{End}(V) = \mathrm{Tan}_e(\mathrm{GL}(V))$, *we have*

$$\mathrm{Tan}_e(\mathrm{O}(q)) = \{T \in \mathrm{End}(V) \mid B_q(v, Tw) \text{ is alternating}\}.$$

This lemma makes no hypothesis on the parity of n.

Proof. We may assume $S = \mathrm{Spec}\, k$ for a ring k. In terms of dual numbers, $\mathrm{Tan}_e(\mathrm{O}(q))$ is the space of linear endomorphisms T of V such that $1 + \epsilon T$ preserves $q_{k[\epsilon]}$ on $V_{k[\epsilon]}$. For any $x \in V_{k[\epsilon]}$ with reduction $x_0 \in V$, clearly $\epsilon T(x) = \epsilon T(x_0)$, so

$$q_{k[\epsilon]}(x + \epsilon T(x_0)) = q_{k[\epsilon]}(x) + \epsilon B_q(x_0, T(x_0))$$

since $\epsilon^2 = 0$. Thus, the necessary and sufficient condition on T is that the bilinear form $B_q(v, Tw)$ vanishes on the diagonal, which is to say that it is alternating. ∎

Proposition 2.3. *If n is even then $\mathrm{O}(q)$ is smooth of relative dimension $n(n-1)/2$. In particular, the open and closed subgroup $\mathrm{SO}(q)$ is smooth and the surjective Dickson morphism $D_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_S$ is smooth, identifying $(\mathbf{Z}/2\mathbf{Z})_S$ with $\mathrm{O}(q)/\mathrm{SO}(q)$.*

Proof. Once smoothness of $\mathrm{O}(q)$ is proved, the Dickson morphism must be smooth since it is visibly smooth over fibers over S. The other assertions are then clear as well. The smoothness of $\mathrm{O}(q)$ is fppf-local on the base, so by Lemma 1.3 it suffices to treat $q = q_n$ over $\mathbf{Z}$ (or over any affine base). By a permutation of the variables, we may equivalently assume $q = \sum_{i=0}^{m} x_i x_{i+m}$ where $2m = n$. To prove the smoothness for this q, we will use the criterion in Lemma 2.1.

We express $n \times n$ matrices in the block form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, B, C, D are $m \times m$ matrices, and we likewise express $(x_1, \dots, x_n)$ as a pair (x, y) where x and y are ordered m-tuples. Thus, our quadratic form is $q(x, y) = \vec{y}^{\mathrm{t}} \vec{x}$ where $\vec{y}$ and $\vec{x}$ are "column vectors" (i.e., $m \times 1$ matrices). For any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ we have $M(x, y) = (Ax + By, Cx + Dy)$, so

$$q(M(x, y)) = \vec{x}^{\mathrm{t}} C^{\mathrm{t}} A \vec{x} + \vec{y}^{\mathrm{t}} D^{\mathrm{t}} B \vec{y} + \vec{y}^{\mathrm{t}} (D^{\mathrm{t}} A + B^{\mathrm{t}} C) \vec{x}.$$

Hence, $M \in \mathrm{O}(q)$ if and only if $D^{\mathrm{t}} A + B^{\mathrm{t}} C = 1_m$ and the matrices $C^{\mathrm{t}} A$ and $D^{\mathrm{t}} B$ are alternating (in the sense that the associated bilinear forms in m variables that they define are alternating; i.e., vanish on pairs (x, x)).

The alternating condition on an $m \times m$ matrix amounts to $m + m(m-1)/2 = m(m+1)/2$ equations in the matrix entries, so the alternating conditions on $C^{\mathrm{t}} A$ and $D^{\mathrm{t}} B$ amount to $m(m+1)$ equations in the matrix entries of A, B, C, D. The condition $D^{\mathrm{t}} A + B^{\mathrm{t}} C = 1_m$ amounts to m^2 such equations, so the closed subscheme $\mathrm{O}(q) \subset \mathrm{GL}_n$ is defined by an ideal generated by $m^2 + m(m+1) = m(2m+1)$ elements. Thus, by Lemma 2.1, to prove $\mathrm{O}(q)$ is smooth we just need to check that over an algebraically closed field k, $\mathrm{Tan}_e(\mathrm{O}(q))$ has codimension $m(2m+1)$ in $\mathfrak{gl}_{2m}(k)$. By Lemma 2.2, an element $M \in \mathfrak{gl}_{2m}(k)$ lies in $\mathrm{Tan}_e(\mathrm{O}(q))$ if and only if the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} C & D \\ A & B \end{smallmatrix}\right)$ is alternating. This says that B and C are alternating and $D = -A^{\mathrm{t}}$, which again amounts to $m(2m+1)$ (linear) equations on $\mathfrak{gl}_{2m}(k)$. We also conclude that the relative dimension of $\mathrm{O}(q)$ is $(2m)^2 - m(2m+1) = 2m^2 - m = n(n-1)/2$. ■

Corollary 2.4. *If n is even then $\mathrm{SO}(q) \to S$ has connected fibers.*

Proof. We proceed by 2-step induction on n, and we can assume $S = \operatorname{Spec} k$ for an algebraically closed field k. Without loss of generality, $q = q_n$. In view of the surjectivity of the Dickson morphism, it is equivalent to show that $\mathrm{O}(q)$ has exactly two connected components. Since $q_2 = xy$, clearly $\mathrm{O}_2 = \mathbf{G}_{\mathrm{m}} \coprod \mathbf{G}_{\mathrm{m}} \iota$ for $\iota = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now assume $n \geq 4$ and that the result is known for $n - 2$.

Since q is not a square (as $n > 1$), it is straightforward to check that the smooth affine hypersurface $H = \{q = 1\}$ is irreducible. The point in $H(k)$ correspond to embeddings of quadratic spaces $(k, x^2) \hookrightarrow (V, q)$. By Witt's extension theorem [Chev, I.4.1], if (W, Q) is a finite-dimensional quadratic space over a field K and B_Q is non-degenerate (so $\dim W$ is even when $\mathrm{char}(K) = 2$) then $\mathrm{O}(Q)(K)$ acts transitively on the set of embeddings of a fixed (possibly degenerate) quadratic space into (W, Q). Hence, $\mathrm{O}(q)(k)$ acts transitively on $H(k)$, so the orbit map $\mathrm{O}(q) \to H$ through e_2 is surjective with stabilizer $G' := \mathrm{Stab}_{e_2}(\mathrm{O}(q))$ that preserves the orthogonal complement $V' := e_2^{\perp} = \sum_{i \geq 2} k e_i$. Since H is connected, it follows that $\mathrm{O}(q)^0$ acts transitively on H too, so $\#\pi_0(\mathrm{O}(q)) \leq \#\pi_0(G')$. Hence, it suffices to show that G' has 2 connected components.

The action map $G' \to \mathrm{GL}(V')$ has kernel consisting of automorphisms of V that fix $\{e_2, \dots, e_n\}$ and preserve q. Such automorphisms must preserve the orthogonal complement $ke_1 + ke_2$ of the span of $\{e_3, \dots, e_n\}$, so it is an elementary calculation that the scheme-theoretic kernel of $G' \to \mathrm{GL}(V')$ is trivial. Let $W = (ke_1 + ke_2)^{\perp} = \sum_{i \geq 3} k e_i$. For any $w \in W$ we have $q(ce_2 + w) = c^2 q(e_2) + B_q(ce_2, w) + q(w) = q(w)$, so relative to the ordered basis $\{e_2, \dots, e_n\}$ of V' the map $G' \to \mathrm{GL}(V')$

is an isomorphism onto the subgroup of $(n-1)\times(n-1)$ matrices whose left column is $(1,0,\ldots,0)$, top row has arbitrary entries beyond the initial 1, and lower right $(n-2)\times(n-2)$ block is $\mathrm{O}(q|_W)$. In other words, G' is an extension of $\mathrm{O}(q|_W)$ by $\mathbf{G}_{\mathrm{a}}^{n-2}$. By induction we know that $\mathrm{O}(q|_W)$ has exactly two connected components, so the same holds for G'. ■

Corollary 2.5. *For even n, the determinant map* $\det:\mathrm{O}(q)\to\mathbf{G}_{\mathrm{m}}$ *factors through μ_2 and kills* $\mathrm{SO}(q)$*. The resulting inclusion* $\mathrm{SO}(q)\subset\mathrm{SO}'(q)$ *is an equality over $S[1/2]$, and* $\mathrm{SO}'(q)\hookrightarrow\mathrm{O}(q)$ *is an equality on fibers at points in characteristic* 2.

Proof. Any point of $\mathrm{O}(q)$ preserves the symmetric bilinear form B_q, and B_q is a perfect pairing (as we may check on fibers, since n is even). Thus, the classical matrix calculation carries over to the relative setting to show that any automorphism of V preserving B_q must have determinant valued in μ_2.

To prove that $\mathrm{SO}(q)$ is killed by the determinant, by working fppf-locally on S and using Lemma 1.3 we may pass to the case $q=q_n$ over $\mathbf{Z}$. The flatness of $\mathrm{SO}(q)$ then reduces to the problem to the generic fiber over $\mathbf{Q}$, so we are over a field not of characteristic 2. Hence, μ_2 is étale, so the *connected* $\mathrm{SO}(q)$ has no nontrivial homomorphism to μ_2.

The determinant is fiberwise nontrivial on O_n over $\mathbf{Z}[1/2]$ since the automorphism of $\mathbf{Z}^n$ that swaps e_1 and e_2 has determinant -1 and preserves q_n. Hence, at points away from characteristic 2 the determinant map factors through a nontrivial homomorphism $\mathrm{O}(q_s)/\mathrm{SO}(q_s)\to\mu_2$. Since $\mathrm{O}(q_s)/\mathrm{SO}(q_s)=\mathbf{Z}/2\mathbf{Z}$, it follows that $\mathrm{SO}(q_s)=\mathrm{SO}'(q_s)$ for such s. In other words, over $S[1/2]$ the closed subscheme $\mathrm{SO}'(q)$ in $\mathrm{O}(q)$ is topologically supported in the open and closed subscheme $\mathrm{SO}(q)$, and that forces the inclusion $\mathrm{SO}(q)\subset\mathrm{SO}'(q)$ to be an equality over $S[1/2]$. At points s of characteristic 2, the smooth group $\mathrm{O}(q_s)$ must be killed by the determinant map into the infinitesimal μ_2, so $\mathrm{SO}'(q_s)=\mathrm{O}(q_s)$ for such s. ■

Remark 2.6. For even n, consider the element $g\in\mathrm{O}_n(\mathbf{Z})$ that swaps x_1 and x_2 while leaving all other x_i's invariant. The section $D_q(g)$ of the constant $\mathbf{Z}$-group $\mathbf{Z}/2\mathbf{Z}$ is equal to 1 mod 2 since it suffices to check this on a single geometric fiber, and at any fiber away from characteristic 2 it is clear (as SO_n coincides with SO'_n over $\mathbf{Z}[1/2]$). Thus, the Dickson morphism $D_q:\mathrm{O}(q)\to(\mathbf{Z}/2\mathbf{Z})_S$ splits as a semidirect product when $q=q_n$.

The induced map $\mathrm{H}^1(S_{\text{ét}},\mathrm{O}_n)\to\mathrm{H}^1(S_{\text{ét}},\mathbf{Z}/2\mathbf{Z})$ assigns to every non-degenerate (V,q) of rank n over S (taken up to isomorphism) a degree-2 finite étale cover of S. This is the *Dickson invariant* of (V,q). If S is a $\mathbf{Z}[1/2]$-scheme (so $(\mathbf{Z}/2\mathbf{Z})_S=\mu_2$) then it recovers the *discriminant* viewed in $\mathbf{G}_{\mathrm{m}}(S)/\mathbf{G}_{\mathrm{m}}(S)^2$. If S is an $\mathbf{F}_2$-scheme then it recovers the *pseudo-discriminant*, also called the *Arf invariant* when $S=\operatorname{Spec}k$ for a field $k/\mathbf{F}_2$. The existence of the section to D_{q_n} implies that every degree-2 finite étale S-scheme arises as the Dickson invariant of some rank-n non-degenerate quadratic space over S.

Corollary 2.7. *For even n, the $\mathbf{Z}$-group SO'_n is reduced and the open and closed subscheme $\mathrm{SO}_n\hookrightarrow\mathrm{SO}'_n$ has complement equal to the non-identity component of $(\mathrm{O}_n)_{\mathbf{F}_2}$. In particular, SO'_n is not $\mathbf{Z}$-flat when n is even.*

Proof. Corollary 2.5 gives the result over $\mathbf{Z}[1/2]$, as well as the topological description of the $\mathbf{F}_2$-fiber. It remains to show that SO'_n is reduced. It is harmless to pass to the quotient by the *smooth* normal subgroup SO_n, so under the identification of $\mathrm{O}_n/\mathrm{SO}_n$ with the constant group $\mathbf{Z}/2\mathbf{Z}$ via the Dickson morphism we see that $G=\mathrm{SO}'_n/\mathrm{SO}_n$ is identified with the kernel of a homomorphism of $\mathbf{Z}$-groups $f:\mathbf{Z}/2\mathbf{Z}\to\mu_2$. The map f is nontrivial since $\mathrm{SO}'_n\neq\mathrm{SO}_n$, and there is only one nontrivial homomorphism from $\mathbf{Z}/2\mathbf{Z}=\operatorname{Spec}\mathbf{Z}[t]/(t^2-t)$ to $\mu_2=\operatorname{Spec}\mathbf{Z}[\zeta]/(\zeta^2-1)$ over $\mathbf{Z}$ (as we see by using flatness to pass to the $\mathbf{Q}$-fiber). This map corresponds to $1\mapsto-1$, or equivalently

$\zeta-1 \mapsto -2t$ on coordinate rings, so the kernel G is equal to $\operatorname{Spec}\mathbf{Z}[t]/(-2t, t^2-t)$. This is precisely the disjoint union of the identity section and a single $\mathbf{F}_2$-point in the $\mathbf{F}_2$-fiber. ■

3. Odd rank

In contrast with the case of even n, our proof of the smoothness of of $\mathrm{SO}(q)$ for odd n has to be different because $\mathrm{O}(q)$ is not smooth over $\mathbf{Z}$. This is already seen in the trivial case $n = 1$, and in general we will show that $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$, so smoothness always fails in characteristic 2. The infinitesimal criterion is ill-suited to this situation, because over a base ring such as $\mathbf{Z}/4\mathbf{Z}$ or $\mathbf{Z}_{(2)}$ in which 2 is neither a unit nor 0 we encounter the problem that the "defect space" $V^{\perp}$ is generally not a subbundle of V. That complicates efforts to verify the infinitesimal criterion when "deforming away from defect-1".

By Lemma 1.3, to prove smoothness of $\mathrm{SO}(q)$ for odd n in general, it suffices to treat the case $q = q_n$ over $\mathbf{Z}$. Our replacement for Lemma 2.1 is a criterion over a Dedekind base that upgrades fibral smoothness to relative smoothness in the presence of a *global* hypothesis of fibral connectedness. First we need to record the useful *fibral isomorphism criterion*:

Lemma 3.1. *Let $h : Y \to Y'$ be a map between finite type schemes over a noetherian scheme S, and assume that Y is S-flat. If h_s is an isomorphism for all $s \in S$ then h is an isomorphism.*

Proof. This is part of [EGA, IV$_4$, 17.9.5], but here is a sketch of an alternative proof. By using that $\Delta_h : Y \to Y \times_{Y'} Y$ satisfies the given hypotheses and is separated, we can reduce to the case when h is separated, and then that Y and Y' are S-separated. For artin local $S = \operatorname{Spec} A$ it is easy to show that h is a closed immersion, and then S-flatness of Y implies that the ideal defining Y in Y' vanishes modulo $\mathfrak{m}_A$, so $Y = Y'$. This settles the result over an artin local base, so in general h is an isomorphism between infinitesimal fibers over S. Hence, h is *flat* (by the so-called "local flatness criterion").

Now we can use a remarkable finiteness criterion of Deligne and Rapoport [DR, II, 1.19]: a quasi-finite separated flat map $f : X \to T$ between noetherian schemes is finite if its fibral rank is Zariski-locally constant on the base (the converse being obvious). Granting this criterion, it follows that h is finite (as its fibral rank is always 1), and being finite flat of degree 1 it must be an isomorphism (check!).

It remains to prove the Deligne–Rapoport finiteness criterion, for which we may assume the fibral rank is constant. Since a proper quasi-finite map is finite, it suffices to prove that f is proper. By the valuative criterion for properness, this reduces the problem to the case when $T = \operatorname{Spec} R$ for a discrete valuation ring R. By Zariski's Main Theorem, the quasi-finite separated X over T admits an open immersion $j : X \hookrightarrow \overline{X}$ into a finite T-scheme $\overline{X}$. We can replace $\overline{X}$ with the schematic closure of X, so $\overline{X}$ has structure sheaf that is torsion-free over R. Hence, $\overline{X}$ is R-flat, so as a finite flat R-scheme it has constant fiber rank. But its open subscheme X also has constant fiber rank and both X and $\overline{X}$ have the same generic fiber. Thus, their constant fiber ranks coincide, and the equality of closed fiber ranks then forces $X = \overline{X}$, so we are done. ■

Now we can prove the desired smoothness criterion:

Proposition 3.2. *Let S be a Dedekind scheme, and G an S-group of finite type such that all fibers G_s are smooth of the same dimension. Then G contains a unique smooth open subgroup G^0 whose s-fiber is $(G_s)^0$ for all $s \in S$. In particular, G is smooth if its fibers are connected.*

Proof. We may assume S is connected, say with generic point η. The smooth open subgroup $G^0_\eta \subset G_\eta$ then "spreads out" over a dense open $U \subset S$ to a smooth open subgroup of G_U with connected fibers. This solves the problem over U, and to handle the remaining finitely many closed points in $S - U$ we may assume that $S = \operatorname{Spec} R$ for a discrete valuation ring R, say with fraction field K. We may and do remove the closed union of the non-identity components of the special fiber, so G has connected special fiber.

Let $\mathscr{G}$ denote the schematic closure in G of the generic fiber G_K, so $\mathscr{G}_K = G_K$. The R-flat $\mathscr{G} \times \mathscr{G}$ is the schematic closure of its generic fiber $G_K \times G_K$, so it follows that the R-flat $\mathscr{G}$ is an R-subgroup of G. This is a flat closed subscheme of G with constant fiber dimension (by flatness), so the closed immersion $\mathscr{G}_0 \hookrightarrow G_0$ between special fibers must be an isomorphism, as G_0 is smooth and connected and $\dim \mathscr{G}_0 = \dim \mathscr{G}_K = \dim G_K = \dim G_0$. Thus, $\mathscr{G}_0$ is smooth (as is $\mathscr{G}_K = G_K$), so $\mathscr{G}$ is smooth.

The closed immersion $\mathscr{G} \hookrightarrow G$ is an isomorphism on fibers, so by flatness of $\mathscr{G}$ it is an isomorphism, due to Lemma 3.1 below. ∎

Remark 3.3. The final assertion in Proposition 3.2 is valid more generally: if $G \to S$ is a finite type group over any reduced noetherian scheme S and if the fibers G_s are smooth and connected of the same dimension then G is smooth. Indeed, the problem is to verify flatness, and by the "valuative criterion for flatness" over a reduced noetherian base [EGA, IV_3, 11.8.1] it suffices to check this after base change to discrete valuation rings, to which Proposition 3.2 applies. See [SGA3, VI_{B}, 4.4] for a further generalization.

The role of identity components in Proposition 3.2 cannot be dropped. For a quasi-finite example, consider the constant group $(\mathbf{Z}/d\mathbf{Z})_R$ over a discrete valuation ring R with $d > 1$. This contains a reduced closed subgroup $G^{(d)}$ given by the reduced closed complement of the open non-identity points in the generic fiber. The R-group $G^{(d)}$ has étale fibers but is not flat over R (the non-identity points of the special fiber are open). A more interesting example is SO'_{2m} over $\mathbf{Z}_{(2)}$ (which is the pushout of $G^{(2)}$ along the identity section $\operatorname{Spec} \mathbf{Z}_{(2)} \hookrightarrow \mathrm{SO}_{2m}$; see the proof of Corollary 2.7).

To apply Proposition 3.2 to $\mathrm{SO}(q)$ when n is odd, we need to verify three things in the theory over an algebraically closed field: connectedness, smoothness, and dimension depending only on n. We first address the connectedness and dimension aspects by a fibration argument in the spirit of the proof of connectedness for SO_{2m}:

Proposition 3.4. *Let (V, q) be a non-degenerate quadratic space over a field k, with $n = \dim V$ odd. The group $\mathrm{SO}(q)$ is connected with dimension $n(n-1)/2$ and multiplication against the central μ_2 defines an isomorphism $\mu_2 \times \mathrm{SO}(q) \simeq \mathrm{O}(q)$*

Proof. We may assume k is algebraically closed and $q = q_n$. The case $n = 1$ is trivial, so we assume $n \geq 3$. We treat characteristic 2 separately from other characteristics, due to the appearance of the defect space $V^{\perp} = ke_0$ in characteristic 2.

First assume $\mathrm{char}(k) \neq 2$, so the symmetric bilinear form B_q is non-degenerate. Points of $\mathrm{O}(q)$ preserve B_q and hence must have determinant valued in μ_2 (by a classical calculation with matrices). Since n is odd, the restriction of $\det : \mathrm{O}(q) \to \mu_2$ to the central μ_2 is the identity map on μ_2. Thus, $\mu_2 \times \mathrm{SO}(q) = \mathrm{O}(q)$. Hence, $\mathrm{O}(q)$ has at least 2 connected components, and exactly 2 such components if and only if $\mathrm{SO}(q)$ is connected. Since $n > 1$, the hypersurface $H = \{q = 1\}$ is irreducible, and exactly as in the proof of Corollary 2.4 we may apply Witt's extension theorem (valid for odd n since $\mathrm{char}(k) \neq 2$) to deduce that the action of $\mathrm{O}(q)$ on H is transitive. The orthogonal complement V' of e_0 is spanned by $\{e_1, \ldots, e_{2m}\}$ since $\mathrm{char}(k) \neq 2$, and it is preserved by $\mathrm{Stab}_{e_0}(\mathrm{O}(q))$. It is straightforward to check that the action of this stabilizer on V' defines an isomorphism onto $\mathrm{O}(q|_{V'}) \simeq \mathrm{O}_{2m}$. We already know that O_{2m} has 2 connected components and dimension $2m(2m-1)/2 = m(2m-1)$, whereas H is irreducible of dimension $n - 1 = 2m$, so it follows that $\mathrm{O}(q)$ has dimension $2m + m(2m-1) = n(n-1)/2$ and at most 2 connected components (hence exactly 2 such components). This settles the case $\mathrm{char}(k) \neq 2$.

Now assume $\mathrm{char}(k) \neq 2$. The non-vanishing defect space obstructs induction using the action on H, so instead we will use an entirely different procedure that is specific to characteristic 2. When $\mathrm{char}(k) \neq 2$ we passed to the hyperplane $ke_0^{\perp}$ spanned by $e_1, \dots, e_{2m}$ on which B_q restricted to a non-degenerate symmetric bilinear form. In characteristic 2, we will use the $(n-1)$-dimensional quotient $V' := V/V^{\perp} = V/ke_0$ rather than a hyperplane. More specifically, because $V^{\perp}$ is the defect space, B_q factors through a non-degenerate symmetric bilinear form B'_q on V'. This has an extra property: it is *alternating*, since $\mathrm{char}(k) = 2$. Any point of $\mathrm{O}(q)$ preserves the defect space, and the induced automorphism of V' preserves $\overline{B}_q$. This defines a homomorphism

$$h : \mathrm{O}_{2m+1} = \mathrm{O}(q) \to \mathrm{Sp}(V', B'_q) \simeq \mathrm{Sp}_{2m}$$

from our orthogonal group to a symplectic group.

The kernel seen by direct calculation to be $\alpha_2^{n-1} \rtimes \mu_2$ (along the top row of matrices, with μ_2 in the upper left), where μ_2 acts on the Frobenius kernel $\alpha_2 \subset \mathbf{G}_a$ by the usual scaling action. Indeed, rather explicitly, since a point T of $\mathrm{O}(q)$ must restrict to an automorphism of the quadratic space $V^{\perp} = (ke_0, x_0^2)$, it has the block form

$$T = \begin{pmatrix} \zeta & \alpha & \alpha' \\ 0 & A & B \\ 0 & C & D \end{pmatrix}$$

for $m \times m$ matrices A, B, C, D, a point ζ of μ_2, and ordered m-tuples α and α'. Writing a typical ordered $(2m+1)$-tuple as (x_0, x, x') for ordered m-tuples x and x', we see that

$$q(T(x_0, x, x')) = x_0^2 + \langle \alpha, x \rangle^2 + \langle \alpha', x' \rangle^2 + B'_q(Ax + By, Cx + Dy),$$

where $\langle \cdot, \cdot \rangle$ is the standard bilinear form $(w, z) \mapsto \sum w_j z_j$. Setting this equal to $q(x_0, x, x')$ then imposes equations on $\alpha, \alpha', A, B, C, D$ that define the closed subscheme $\mathrm{O}(q) \subset \mathrm{GL}_n$. This not only implies the description of $\ker h$, but also shows that h is surjective.

The smoothness of symplectic group schemes is easily proved by the infinitesimal criterion, and the dimension is likewise easily determined by direct computation of the tangent space. This gives that Sp_{2m} has dimension $m(2m+1) = n(n-1)/2$. Finally, the connectedness of symplectic groups is easily proved by an inductive fibration argument (using lower-dimensional symplectic spaces). Since h is surjective with infinitesimal kernel, we conclude that $\mathrm{O}(q)$ is connected of dimension $n(n-1)/2$.

Since $\mathrm{Sp}_{2m} \subset \mathrm{SL}_{2m}$ as k-groups, the above functorial description of points T of O_{2m+1} shows that $\det T = \zeta \in \mu_2$. Thus, $\det : \mathrm{O}(q) \to \mathbf{G}_{\mathrm{m}}$ factors through μ_2, and once again the oddness of n implies that the central μ_2 in $\mathrm{O}(q)$ thereby splits off as a direct factor. Hence, $\mu_2 \times \mathrm{SO}(q) = \mathrm{O}(q)$. This implies that $\mathrm{SO}(q)$ is connected of dimension $n(n-1)/2$ since μ_2 is infinitesimal in characteristic 2. ■

Now we can pass to the relative case and establish smoothness too:

Proposition 3.5. *If n is odd then $\mathrm{SO}(q) \to S$ is smooth with connected fibers of dimension $n(n-1)/2$, and the multiplication map $\mu_2 \times \mathrm{SO}(q) \to \mathrm{O}(q)$ against the central μ_2 is an isomorphism.*

Proof. By Lemma 1.3, we may assume $q = q_n$ over $S = \mathrm{Spec}\,\mathbf{Z}$. The case $n = 1$ is trivial, so we assume $n = 2m+1$ with $m \geq 1$. Since the fibers are connected of the same dimension $(n(n-1)/2)$, by Proposition 3.4, to prove smoothness we may apply Proposition 3.2 to reduce to proving fibral smoothness. In other words, we wish to show that over a field k, the tangent space $\mathrm{Tan}_e(\mathrm{SO}(q))$ has dimension $n(n-1)/2$. To do this we will treat characteristic 2 separately from other characteristics.

First assume $\mathrm{char}(k) \neq 2$, so the equality $\mu_2 \times \mathrm{SO}(q) = \mathrm{O}(q)$ implies $\mathrm{Tan}_e(\mathrm{SO}(q)) = \mathrm{Tan}_e(\mathrm{O}(q))$. This latter tangent space is identified in Lemma 2.2: it is the space of linear endomorphisms T of V such that $B_q(v, Tw)$ is alternating. But B_q is non-degenerate since $\mathrm{char}(k) \neq 2$, so $T \mapsto B_q(\cdot, T(\cdot))$ identifies $\mathrm{Tan}_e(\mathrm{O}(q))$ with the space $\mathrm{Alt}^2(V)$ of alternating bilinear forms on V. This is the dual of $\wedge^2(V)$, so it has dimension $n(n-1)/2$, as desired.

Now assume $\mathrm{char}(k) = 2$. Let $V' = V/V^{\perp}$, and let B'_q the induced non-degenerate alternating form on V'. Since $\mu_2 \times \mathrm{SO}(q) = \mathrm{O}(q)$ and $\mathrm{Tan}_e(\mu_2)$ is 1-dimensional, it is equivalent to show that $\mathrm{Tan}_e(\mathrm{O}(q))$ has dimension $1 + n(n-1)/2$. We will construct a short exact sequence

$$0 \to \mathrm{Hom}(V, V^{\perp}) \to \mathrm{Tan}_e(\mathrm{O}(q)) \to \mathrm{Alt}^2(V/V^{\perp}) \to 0,$$

from which we will get the desired dimension count

$$n + (n-1)(n-2)/2 = m(2m-1) + (2m+1) = 2m^2 + m + 1 = m(2m+1) + 1 = 1 + n(n-1)/2.$$

To construct the exact sequence, we will compute using dual numbers as in the proof of Lemma 2.2. Using notation as in that calculation, since the alternating property for $B_q(v, Tw)$ implies skew-symmetry and hence symmetry (as $\mathrm{char}(k) = 2$), T must preserve the defect line $V^{\perp}$ (as $B_q(v, Tw) = 0$ for $v \in V^{\perp}$ and any w, and $B_q(v, Tw)$ is symmetric for general $v, w \in V$). Thus, $\mathrm{Tan}_e(\mathrm{O}(q))$ consists of those T which preserve $V^{\perp}$ and whose induced endomorphism T' of $V' = V/V^{\perp}$ makes $B'_q(v', T'w')$ alternating. By non-degeneracy of B'_q, every bilinear form on V' is $B'_q(v', Lw')$ for a unique endomorphism L of V', so the vector space $\mathrm{Tan}_e(\mathrm{O}(q))$ fits into the asserted exact sequence since $\mathrm{Hom}(V, V^{\perp})$ is precisely the ambiguity in T when T' is given.

Smoothness has now been proved in the general relative setting, and it remains to prove that the natural homomorphism $f : \mu_2 \times \mathrm{SO}(q) \to \mathrm{O}(q)$ is an isomorphism. The map f_s between fibers over any $s \in S$ is an isomorphism (Proposition 3.4), and the source of f is S-flat (as $\mathrm{SO}(q)$ is even S-smooth). Thus, f is an isomorphism due to the fibral isomorphism criterion in Lemma 3.1. ■

Corollary 3.6. *Assume n is odd. The map* $\det : \mathrm{O}(q) \to \mathbf{G}_{\mathrm{m}}$ *factors through μ_2, and its kernel is* $\mathrm{SO}(q)$. *In particular,* $\det$ *identifies* $\mathrm{O}(q)/\mathrm{SO}(q)$ *with* μ_2.

Proof. Since $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ via multiplication and the determinant on the central μ_2 is the inclusion $\mu_2 \hookrightarrow \mathbf{G}_{\mathrm{m}}$ (as n is odd), we are done. ■

Remark 3.7. In the proof of Proposition 3.4, over any field k of characteristic 2 we constructed a surjective homomorphism $\mathrm{O}(q) \to \mathrm{Sp}(V', B'_q)$ with infinitesimal geometric kernel $\mu_2 \ltimes \alpha_2^{n-1}$ when n is odd. This kernel meets the kernel $\mathrm{SO}(q)$ of the determinant map on $\mathrm{O}(q)$ in α_2^{n-1}, so by smoothness of $\mathrm{SO}(q)$ we obtain a purely inseparable isogeny $\mathrm{SO}(q) \to \mathrm{Sp}(V', B'_q)$ with kernel that is a form of α_2^{n-1}. This "unipotent isogeny" is a source of many weird phenomena related to algebraic groups in characteristic 2 (e.g., see [CP, A.3]).

For the benefit of those who have some prior awareness of the theory of root systems (perhaps in the context of connected compact Lie groups or semisimple Lie algebras over $\mathbf{C}$), here is the broader significance of the preceding strange isogenies. In the setting of connected semisimple algebraic groups over arbitrary fields to be taken up in the sequel course, special orthogonal groups in $2m+1$ variables are type B_m and symplectic groups in $2m$ variables are type C_m (as in Lie theory over $\mathbf{C}$). These types are distinct for $m \geq 3$ (for $m = 1$ and $m = 2$ the types coincide; see Example 5.2 and Example 5.5 respectively), and the deeper structure theory of semisimple groups via root systems shows that in characteristics distinct from 2 and 3 there are no isogenies between (absolutely simple) connected semisimple groups of different types. However, in characteristic 2 we have just seen that isogenies exist between the distinct types B_m and C_m for all $m \geq 3$. See [SGA3, XXI, 7.5] for further details.

4. Center

The remaining structural property of the groups $\mathrm{SO}(q)$ and $\mathrm{O}(q)$ (and $\mathrm{SO}'(q)$) that we wish to determine is the functorial center. Since $\mathrm{O}(q)$ is commutative when $n \leq 2$, we will assume $n \geq 3$. For odd n, the central μ_2 in $\mathrm{O}(q)$ has trivial intersection with $\mathrm{SO}'(q)$, and hence with $\mathrm{SO}(q)$, so there is no obvious nontrivial point in the center. If n is even then the central μ_2 is contained in $\mathrm{SO}'(q)$, and we claim that it also lies in $\mathrm{SO}(q)$. In other words, for even n we claim that the Dickson morphism $D_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_S$ kills the central μ_2. It suffices to treat the case of $q = q_n$ over $\mathbf{Z}$, in which case we just need to show that the *only* homomorphism of $\mathbf{Z}$-groups $\mu_2 \to \mathbf{Z}/2\mathbf{Z}$ is the trivial one. By flatness, to prove such triviality it suffices to check after localization to $\mathbf{Z}_{(2)}$. But over the local base $\operatorname{Spec} \mathbf{Z}_{(2)}$ the scheme μ_2 is connected and thus it must be killed by a homomorphism into a constant group.

Proposition 4.1. *Assume $n \geq 3$. The functorial centers of $\mathrm{SO}(q)$ and $\mathrm{SO}'(q)$ coincide. This common center is represented by μ_2 in the central $\mathbf{G}_{\mathrm{m}} \subset \mathrm{GL}(V)$ when n is even, and it is trivial when n is odd.*

Proof. By Lemma 1.3, it suffices to treat the Bureau of Standards form q_n and $S = \operatorname{Spec} k$ for a ring k. We will use a method similar to the treatment of $Z_{\mathrm{Sp}_{2n}}$ in HW4 Exercise 1: we will exhibit a specific torus T that we show to be its own centralizer in $G := \mathrm{SO}'(q)$ (so T its own centralizer in $\mathrm{SO}(q)$) and then we will look for the center inside this T. To write down an explicit such T, we will use the standard form of q.

First suppose $n = 2m$, so relative to a suitable ordered basis $\{e_1, e'_1, \dots, e_m, e'_m\}$ we have $q = \sum_{i=1}^m x_i x'_i$. In this case we identify GL_1^m with a k-subgroup T of $\mathrm{SO}'(q)$ via

$$j : (t_1, \dots, t_m) \to (t_1, 1/t_1, \dots, t_m, 1/t_m).$$

To prove that $Z_G(T) = T$, we consider the closed subgroup $T_j \simeq \mathbf{G}_m$ given by the jth factor of GL_1^m (so $T = \prod T_j$). It is easy to compute that the centralizer of T_j in $\mathrm{GL}(V) = \mathrm{GL}_{2m}$ is a direct product $D_j \times \mathrm{GL}_{2m-2}$ according to the decomposition

$$V = (ke_j \oplus ke'_j) \bigoplus (\bigoplus_{i \neq j} ke_i \oplus ke'_i),$$

where $D_j \subset \mathrm{GL}_2$ is the diagonal torus. Thinking functorially, the centralizer of T in $\mathrm{GL}(V)$ is the (scheme-theoretic) intersection of the centralizers of the T_j's, so this is the diagonal torus D in $\mathrm{GL}(V)$. But the explicit form of q shows that $D \bigcap \mathrm{SO}'(q) = T$.

Now suppose $n = 2m+1$ with $m \geq 1$. Pick a basis $\{e_0, e_1, e'_1, \dots, e_m, e'_m\}$ relative to which

$$q = x_0^2 + \sum_{i=1}^m x_i x'_i.$$

If we define T in the same way (using the span of $e_1, e'_1, \dots, e_m, e'_m$) then the same analysis gives the same result: T is its own scheme-theoretic centralizer in $\mathrm{SO}'(q)$. The point is that there is no difficulty created by e_0 because we are requiring the determinant to be 1. (If we try the same argument with $\mathrm{O}(q)$ then the centralizer of T would be $\mu_2 \times T$.)

With $Z_{\mathrm{SO}'(q)}(T) = T$ proved in general, we are now in position to identify the center of $\mathrm{SO}'(q)$ when $n \geq 3$. First we assume $n \geq 4$ (i.e., $m \geq 2$). In terms of the ordered bases as above, consider the automorphisms obtained by swapping the ordered pairs (e_i, e'_i) and (e'_1, e_1) for $1 < i \leq m$. (Such i exist precisely because $m \geq 2$.) These automorphisms lie in $\mathrm{SO}'(q)$ since the determinant is $(-1) \cdot (-1) = 1$, and a point of $T = \prod S_j$ centralizes it if and only if the components along S_1

and S_i agree. Letting i vary, we conclude that the center is contained in the "scalar" subgroup $\mathrm{GL}_1 \hookrightarrow T$ given by $t_1 = \cdots = t_m$. This obviously holds when $m = 1$ (i.e., $n = 3$) as well.

Letting t denote the common value of the t_j, to constrain it further we consider more points of $\mathrm{SO}'(q)$ against which it should be central. First assume $m \geq 2$. Consider the automorphism f of V which acts on the plane $ke_i \bigoplus ke'_i$ by the matrix $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (and leaves all other basis vectors invariant) for exactly *two* values $i_0, i_1 \in \{1, \dots, m\}$, so $\det f = 1$. Clearly f preserves q, so f lies in $\mathrm{SO}'(q)(k)$. But f-conjugation of t viewed in $\mathrm{SO}'(q)$ (or $\mathrm{GL}(V)$) carries t_i to $1/t_i$ for $i \in \{i_0, i_1\}$. Thus, the centralizing property forces $t \in \mu_2$. This is the central μ_2 in $\mathrm{GL}(V)$, so $Z_{\mathrm{SO}'(q)} = \mu_2$ when $n \geq 4$ is even. If n is odd then the central μ_2 in $\mathrm{GL}(V)$ has trivial intersection with $\mathrm{SO}'(q) = \mathrm{SO}(q)$, so $Z_{\mathrm{SO}'(q)} = 1$ for odd $n \geq 5$.

Next, we give a direct proof that $Z_{\mathrm{SO}_3} = 1$. The action of PGL_2 on $\mathfrak{sl}_2$ via conjugation defines an isomorphism $\mathrm{PGL}_2 \simeq \mathrm{SO}_3$; see the self-contained calculations in Example 5.2. By HW3 Exercise 4(ii) the scheme-theoretic center of PGL_r is trivial for any $r \geq 2$ (and for PGL_2 it can be verified by direct calculation), so SO_3 has trivial center.

We have settled the case of odd $n \geq 3$, and for even $n \geq 4$ we have proved that $\mathrm{SO}'(q)$ has center μ_2 that also lies in $\mathrm{SO}(q)$. It remains to show, assuming $n \geq 4$ is even, that the functorial center of $\mathrm{SO}(q)$ is no larger than this μ_2. We may and do assume $q = q_n$. The torus T constructed above in SO'_n lies in the open and closed subgroup SO_n for topological reasons, and $Z_{\mathrm{SO}_n}(T) = T$ since T has been shown to be its own centralizer in SO'_n. Thus, it suffices to show that the central μ_2 is the kernel of the adjoint action of T on $\mathrm{Lie}(\mathrm{SO}_n) = \mathrm{Lie}(\mathrm{O}_n)$. The determination of the weight space decomposition for T acting on $\mathrm{Lie}(\mathrm{O}_n)$ is classical, from which the kernel is easily seen to be the diagonal μ_2. ■

Corollary 4.2. *For $n \geq 3$, the functorial center of $\mathrm{O}(q)$ is represented by the central μ_2.*

Proof. If n is odd then the identification $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ yields the result since $\mathrm{SO}(q)$ has trivial functorial center for such n. Now suppose that n is even. In this case the open and closed subgroup $\mathrm{SO}(q)$ contains the central μ_2 is its functorial center. To prove that μ_2 is the functorial center of $\mathrm{O}(q)$ we again pass to the case $q = q_n$. It suffices to check that the diagonal torus T in SO_n is its own centralizer in O_n.

Writing $n = 2m$, we may rearrange variables so that $q = \sum_{i=1}^{m} x_i x_{i+m}$. Now the diagonal torus T consists of block matrices $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, and its centralizer in GL_{2m} consists of block matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a' \end{smallmatrix}\right)$ with diagonal $a, a' \in \mathrm{GL}_m$. Membership in the orthogonal group is the condition $aa' = 1$, so indeed T is its own centralizer in O_n. ■

5. Accidental isomorphisms

The study of (special) orthogonal groups provides many examples of *accidental isomorphisms* between low-dimensional members of distinct "infinite families" of algebraic groups. This is analogous to the isomorphisms between small members of distinct "infinite families" of finite groups (such as $\mathbf{Z}/3\mathbf{Z} \simeq \mathrm{PGL}_2(\mathbf{F}_2)$, $S_4 \simeq \mathrm{PGL}_2(\mathbf{F}_3)$, $S_5 \simeq \mathrm{PGL}_2(\mathbf{F}_5)$, and so on). In fact, when such accidental isomorphisms among algebraic groups are applied at the level of rational points of algebraic groups over *finite* fields one obtains many of the accidental isomorphisms among small finite groups.

Just as isomorphisms among small finite groups are due to the limited range of possibilities for finite groups of small size, the accidental isomorphisms between certain low-dimensional algebraic groups are due to a limitation in the possibilities for a "small" case of the root datum that governs the (geometric) isomorphism class of a connected semisimple group.

Example 5.1. Suppose $n = 2$. In this case $\mathrm{O}(q)$ is an étale form of

$$\mathrm{O}_2 = T \coprod T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ for } T = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \right\}.$$

Hence, $\mathrm{O}(q)$ is smooth and $\mathrm{SO}(q)$ is a rank-1 torus, whereas $\mathrm{SO}'(q_s) = \mathrm{O}(q_s)$ is disconnected when $\mathrm{char}(k(s)) = 2$.

Example 5.2. Suppose $n = 3$. In this case $\mathrm{SO}(q)$ is an étale form of SO_3. We claim that $\mathrm{SO}_3 \simeq \mathrm{PGL}_2$. To see this, consider the linear "conjugation" action of $\mathrm{PGL}_2 = \mathrm{GL}_2/\mathbf{G}_{\mathrm{m}}$ on the rank-3 affine space $\mathfrak{sl}_2$.

This action preserves the non-degenerate quadratic form Q on $\mathfrak{sl}_2$ given by the determinant. Explicitly, $Q(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}) = -(x^2 + yz)$ is, up to sign, the Bureau of Standards quadratic form q_3 in 3 variables. Preservation of q_3 is the same as that of $-q_3$, so the sign does not affect the group. We get a homomorphism $\mathrm{PGL}_2 \to \mathrm{O}_3 = \mu_2 \times \mathrm{SO}_3$ over $\mathbf{Z}$ with trivial kernel. By computing on the $\mathbf{Q}$-fibers, the map to the μ_2-factor must be trivial. Thus, the map $\mathrm{PGL}_2 \to \mathrm{O}_3$ factors through SO_3. Since PGL_2 is smooth and fiberwise connected of dimension 3, it follows that the monic map $\mathrm{PGL}_2 \to \mathrm{SO}_3$ is an isomorphism on fibers and hence is an isomorphism (Lemma 3.1).

Example 5.3. Suppose $n = 4$. In this case $\mathrm{SO}(q)$ is not "absolutely simple"; i.e., on geometric fibers it contains nontrivial smooth connected proper normal subgroups. (This is the only $n \geq 3$ for which that happens, and in terms of root systems it corresponds to the equality $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$.) In more concrete terms, we claim that

$$(\mathrm{SL}_2 \times \mathrm{SL}_2)/M \simeq \mathrm{SO}_4$$

with $M = \mu_2$ diagonally embedded in the evident central manner.

To see where this comes from, apply a sign to the third standard coordinate to convert q_4 into $Q = x_1x_2 - x_3x_4$, which we recognize as the determinant of a 2×2 matrix. The group SL_2 acts on the rank-4 space of such matrices in two evident commutating ways, via $(g, g').x = gxg'^{-1}$, and these actions preserve the determinant by the very definition of SL_2. This defines a homomorphism $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}'(Q) \simeq \mathrm{SO}'_4$ whose kernel is easily seen to be M. This map visibly lands in SO_4 since SL_2 is fiberwise connected and $\mathrm{O}_4/\mathrm{SO}_4 = \mathbf{Z}/2\mathbf{Z}$. Hence, we obtained a monomorphism

$$(\mathrm{SL}_2 \times \mathrm{SL}_2)/M \to \mathrm{SO}_4$$

that must be an isomorphism on fibers (as both sides have smooth connected fibers of the same dimension), and therefore an isomorphism.

Remark 5.4. In the preceding example, we can also consider the action by the smooth connected affine group $\mathrm{GL}_2 \times \mathrm{GL}_2$ via $(g, g').x = gxg'^{-1}$. This preserves the determinant provided that $\det g = \det g'$, so if $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ (with fiber product via determinant) then we get a homomorphism $H \to \mathrm{SO}_4$ whose kernel is the diagonal GL_1.

Example 5.5. Suppose $n = 5$. In this case SO_5 is the quotient of Sp_4 by its center μ_2. (This corresponds to the accidental isomorphism of root systems $\mathrm{B}_2 = \mathrm{C}_2$.) To establish the isomorphism, consider a rank-4 symplectic space (V, ω_0) with $\omega_0 \in \wedge^2(V)^* = \wedge^2(V^*)$ the given symplectic form. The rank-6 vector bundle $\wedge^2(V)$ contains a rank-5 subbundle W of sections killed by ω_0, and on $\wedge^2(V)$ there is a natural non-degenerate quadratic form q valued on $L = \det(V)$ defined by $q(\omega, \eta) = \omega \wedge \eta$. The action of $\mathrm{SL}(V)$ clearly preserves q, the restriction $q|_W$ is non-degenerate (by direct calculation), and $\mathrm{Sp}(\omega_0)$ preserves W (due to the definition of W). Thus, the $\mathrm{Sp}(\omega_0)$-action on W defines a homomorphism

$$\mathrm{Sp}(\omega_0) \to \mathrm{O}(q|_W)$$

that must factor through $\mathrm{SO}(q|_W)$ (as we can check by passing to the standard symplectic space (V, ω_0) of rank 4 over $\mathbf{Z}$ and computing over $\mathbf{Q}$), and it kills the center μ_2.

We claim that the resulting map $h : \mathrm{Sp}_4/\mu_2 \to \mathrm{SO}_5$ is an isomorphism. A computation shows that $\ker h$ has trivial intersection with the "diagonal" maximal torus, so $\ker h$ is quasi-finite and hence h is surjective for fibral dimension reasons. This forces h to be fiberwise flat, hence flat [EGA, IV_3, 11.3.10], so h has locally constant fiber rank that we claim is 1. By base change, it suffices to treat the standard symplectic space of rank 4 over $\mathbf{Z}$, for which we can compute the fiber rank over $\mathbf{Q}$. But in characteristic 0, isogenies between connected semisimple groups of adjoint type are necessarily isomorphisms.

Example 5.6. Finally, suppose $n = 6$. In this case SO_6 is the quotient of SL_4 by the subgroup μ_2 in the central μ_4. This corresponds to the accidental isomorphism of root systems $\mathrm{D}_3 = \mathrm{A}_3$, and to explain it we will again use the natural action of $\mathrm{SL}(V)$ on the rank-6 bundle $\wedge^2(V)$ equipped with the non-degenerate quadratic form $q(\omega, \eta) = \omega \wedge \eta$ valued in the line bundle $\det V$. The homomorphism $\mathrm{SL}(V) \to \mathrm{O}(q)$ defined in this way clearly kills the central μ_2, and it factors through $\mathrm{SO}(q)$ (as $\mathrm{O}(q)/\mathrm{SO}(q) = \mathbf{Z}/2\mathbf{Z}$).

To prove that the resulting map $h : \mathrm{SL}(V)/\mu_2 \to \mathrm{SO}(q)$ is an isomorphism, by Lemma 1.3 we may pass to q_6 over $\mathbf{Z}$. As in Example 5.5, we reduce to the isomorphism problem over $\mathbf{Q}$. Isogenies between smooth connected groups in characteristic 0 are always central, and μ_4/μ_2 is not killed by h, so we are done.

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