Algebraic Groups I. Affine quotients

Let G be a smooth group of finite type over a field k, and H a normal k-subgroup scheme. We have seen that G/H exists as a smooth quasi-projective k-scheme, and that the normality of H provides a unique k-group structure on G/H compatible with the k-group structure on G. If the normality hypothesis is dropped, the coset space G/H is generally not affine even when G is affine. Indeed, G/H is the orbit under a suitable action of G on a projective space, and such orbits can sometimes even be the entire projective space (e.g., $G = \mathrm{GL}(V)$ with its natural action on $\mathbf{P}(V)$).

In this handout, we prove the important fact that when H is a normal k-subgroup scheme in a smooth affine G then G/H is actually affine. (Just as the existence of G/H is valid without smoothness hypotheses on G, the same holds for our affineness claim when H is normal in G. As with the existence of quotients in such extra generality, we refer the interested reader to SGA3 for such generalizations.)

To prove that G/H is affine, it is harmless to first make an extension of the ground field (since we know that the formation of G/H commutes with such extensions), so we may and do assume that k is algebraically closed. Also, the k-homomorphism $G^0 \to G/H$ between smooth k-groups has image with finite index, so it has image $(G/H)^0$. Thus, $(G/H)^0 = G^0/(G^0 \cap H)$. Since G/H is a disjoint union of finitely many copies of $(G/H)^0$, to prove that G/H is affine we may replace G with G^0 so that G is connected.

Let V be a finite-dimensional representation for G such that H is the scheme-theoretic stabilizer of a line L in V (i.e., $H = N_G(L)$). In particular, the H-action on L corresponds to a k-homomorphism $\chi : H \to \mathrm{GL}(L) = \mathbf{G}_m$. Consider the span W of the lines $g.L$ in V for $g \in G(k)$. Clearly W is G-stable in V, and since $gHg^{-1} \subseteq H$ inside of G for each $g \in G(k)$ we see that $g.L$ is also H-stable with action by the twisted character $\chi^g : H \to \mathbf{G}_m$ defined by $h \mapsto \chi(g^{-1}hg)$. Replacing V with W without loss of generality, we can assume that V is spanned by H-stable lines $L = L_1, \dots, L_n$. Thus, $V = \oplus V_i$ for H-stable subspaces V_i on which H acts through pairwise distinct characters $\chi_i : H \to \mathbf{G}_m$, with $\chi_1 = \chi$ and with each χ_i having the form $\chi_i = \chi^g$ for some $g \in G(k)$.

Lemma 0.1. *Necessarily $V = V_1$, which is to say that H acts on V through χ.*

Proof. Since H is normal in G, the condition on $g \in G(R)$ that $\chi_R^g = \chi_R$ as R-homomorphisms $H_R \to (\mathbf{G}_m)_R$ makes sense for any k-algebra R and clearly cuts out a closed k-subgroup G' scheme in G. To prove that this coincides with G, it suffices to prove $G'(k)$ has finite index in $G(k)$, as then the closed k-subscheme G'_{red} has finitely many $G(k)$-translates which cover G, so G' has underlying space that is also open in G, forcing $G' = G$ since G is smooth and *connected*.

The finite index property for $G'(k)$ in $G(k)$ will hold provided that $\chi^g \in \{\chi_i\}$ for each $g \in G(k)$. The normality of H in G implies that the H-action on $V = \oplus V_i$ is given by the pairwise distinct characters $\{\chi_i^g\}$. Hence, it suffices to prove that if $\psi : H \to \mathbf{G}_m$ is any k-homomorphism such that V contains a ψ-eigenline for H then $\psi \in \{\chi_i\}$. Such a line has a nonzero image under projection into some V_i, so the result is clear (as $V_i \neq 0$). ■

Recall that $H = N_G(L)$. For any k-algebra R, if $g \in G(R)$ acts on V_R through scaling by an element of $R^\times$ then clearly $g \in N_{G(R)}(L_R) = H(R)$. Conversely, by the Lemma, if $g \in H(R)$ then g acts on V_R through $R^\times$-scaling. Hence, H is the scheme-theoretic kernel of the k-homomorphism $f : G \to \mathrm{PGL}(V)$. Letting $\overline{G} \subseteq \mathrm{PGL}(V)$ be the smooth closed image of f, the natural map $G \to \overline{G}$ is a surjective homomorphism between smooth k-groups of finite type and the kernel is H, so $G/H = \overline{G}$.

We conclude that G/H is closed in $\mathrm{PGL}(V)$. But $\mathrm{PGL}(V)$ is *affine*, so G/H is also affine.