Algebraic Groups I. Conjugacy into a maximal torus

This handout addresses an intermediate step in the general proof of conjugacy of maximal tori in a smooth connected affine group over an algebraically closed field. We wish to prove:

Proposition 0.1. *Let G be a solvable smooth connected group over an algebraically closed field k, and choose a semidirect product expression $G = T \ltimes U$ with T a torus and U unipotent. Then every semisimple $s \in G(k)$ admits a $G(k)$-conjugate contained in T.*

Recall from the handout on "covering by Borel subgroups" (which only required $G(k)$-conjugacy of Borels, and no solvability hypotheses) that every semisimple element must lie in *some* torus. The problem is to relate things to a *specific* torus, and we cannot appeal to conjugacy of maximal tori since the proof of that rests on the above proposition in the solvable case (applied to a Borel subgroup). So to prove the proposition, we need to give a direct argument making essential use of the solvability of G.

The idea of the proof is to induct on dimension with the help of a composition series, but we will use a composition series whose terms are *normal* in G and have as successive quotients not individual $\mathbf{G}_a$'s and $\mathbf{G}_m$'s but rather vector groups and tori of possibly big dimension. Ultimately the problem will be reduced to the 2-dimensional case with T and U each of dimension 1, in which case a direct calculation becomes possible with little difficulty.

As a first step, we reduce to the case when U is commutative. To do this, first note that if $\{U_i\}$ is any characteristic composition series of U (i.e., each U_i is smooth connected and stable under all k-automorphisms of U) then all U_i are normalized by $G(k)$ and hence are normal in G (as $k = \overline{k}$). Thus, we could then consider $T \ltimes (U_i/U_{i-1})$ separated, moving down the composition series and inducting on $\dim U$ (the case $\dim U = 0$ being trivial). Applying these considerations to the derived series $\{\mathscr{D}^i(U)\}$ thereby reduces us to the case when U is commutative. Going a step further, if $\mathrm{char}(k) = p > 0$ then the commutative U is killed by p^N for some big N and each image p^iU is a smooth connected k-subgroup of U (in contrast with the torsion subgroups $U[p^i]$!). This is also a characteristic composition series of U, so we can get to the case when U is p-torsion.

By Exercise 2(ii) in HW9, if $\mathrm{char}(k) = 0$ then $U \simeq \mathbf{G}_a^n$ with T acting linearly. Thus, we get a weight space decomposition for the action of the k-split U and can take a flag adapted to T-eigenlines to get a T-stable flag in U. That permits us to reduce to the case $U = \mathbf{G}_a$ with a linear T-action when $\mathrm{char}(k) = 0$. Let us reach the same special case when $\mathrm{char}(k) = p > 0$. In these cases, Exercise 2(i) in HW9 (which rests on the beautiful work of Tits on the structure of unipotent smooth connected groups, presented in Appendix B of "Pseudo-reductive groups", especially Theorem B.4.3 there) implies that $U \simeq U_0 \times V$ where U_0 has trivial T action (and mysterious structure!) and V is a vector group with a *linear* T-action for some choice of isomorphism $V \simeq \mathbf{G}_a^n$. Thus, we can again reduce to the special case $U = \mathbf{G}_a$ with a linear T-action (as the case of U with trivial T-action is obvious).

Now back in the case of any characteristic, the linear T-action on $\mathbf{G}_a$ is given by some k-homomorphism $\chi : T \to \mathbf{G}_m$, and we can assume $\chi \neq 1$ (as otherwise $G = T \times U$ and we are clearly done). Thus, $T/(\ker \chi) \simeq \mathbf{G}_m$ via χ. Note that $\ker \chi \subset Z_G$, so we can easily pass to $G/(\ker \chi)$ and replace T with $T/(\ker \chi)$ without loss of generality to get to the case $G = \mathbf{G}_m \ltimes \mathbf{G}_a$ with the *standard* action $t.x = tx$ of $\mathbf{G}_m$ on $\mathbf{G}_a = \mathscr{R}_u(G)$. This is just the "$ax + b$ group" via

$$(t, x) \mapsto \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix}.$$

The semisimple point $s = (t, x) \in G(k)$ must have $t \neq 1$ (i.e., $s \notin \mathbf{G}_a(k)$), and then for $x' = x/(t-1)$ it is easy to compute

$$(1, x')g(1, -x') = (t, 0) \in T(k).$$