Algebraic Groups I. Dynamic approach to algebraic groups

1. Subgroups associated to a 1-parameter subgroup

Let G be a smooth affine group over a field k, and $\lambda : \mathbf{G}_m \to G$ a k-homomorphism (possibly trivial, though that case is not interesting). One often calls λ a 1-*parameter* k-*subgroup* of G, even when $\ker \lambda \neq 1$. Such a homomorphism defines a left action of $\mathbf{G}_m$ on G via the functorial procedure $t.g = \lambda(t)g\lambda(t)^{-1}$ for $g \in G(R)$ and $t \in R^\times$ for any k-algebra R. In lecture we introduced the following associated subgroup functors of G: for any k-algebra R,

$$P_G(\lambda)(R) = \{g \in G(R) \,|\, \lim_{t\to 0} t.g \text{ exists }\}, \;\; U_G(\lambda)(R) = \{g \in G(R) \,|\, \lim_{t \to 0} t.g = 1\},$$

and

$$Z_G(\lambda)(R) = \{g \in G(R) \,|\, \lambda_R \text{ centralizes } g\}.$$

In the March 10 lecture it was proved that these are all represented by closed k-subgroup schemes of G, with $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$.

By a direct calculation with graded modules over the dual numbers, it is shown in Proposition 2.1.8(1) of "Pseudo-reductive groups" that when using the $\mathbf{Z}$-grading $\oplus_{n \in \mathbf{Z}} \mathfrak{g}_n$ of $\mathfrak{g} = \mathrm{Lie}(G)$ defined by the $\mathbf{G}_m$-action induced by conjugation through λ (i.e., $\mathfrak{g}_n$ is the space of $v \in \mathfrak{g}_n$ such that $\mathrm{Ad}_G(\lambda(t))(v) = t^n v$ for all $t \in \mathbf{G}_m$), we have

$$\mathrm{Lie}(Z_G(\lambda)) = \mathfrak{g}_0, \;\; \mathrm{Lie}(U_G(\lambda)) = \mathfrak{g}^+ := \bigoplus_{n>0} \mathfrak{g}_n.$$

For example, if $T \subset G$ is a split k-torus and λ is valued in T, then using the resulting T-weight space decomposition $\mathfrak{g} = \mathrm{Lie}(Z_G(T)) \oplus (\oplus_{a \in \Phi} \mathfrak{g}_a)$ (with Φ the set of nontrivial T-weights on $\mathfrak{g}$) we see that for any $n \in \mathbf{Z} - \{0\}$,

$$\mathfrak{g}_n = \bigoplus_{\langle a, \lambda \rangle = n} \mathfrak{g}_a$$

since the adjoint action of $\lambda(t)$ on $\mathfrak{g}_a$ is multiplication by $a(\lambda(t)) = t^{\langle a, \lambda\rangle}$. Hence,

$$\mathfrak{g}_0 = \mathrm{Lie}(Z_G(T)) \oplus (\oplus_{\langle a,\lambda\rangle = 0} \mathfrak{g}_a), \;\; \mathrm{Lie}(U_G(\lambda)) = \mathfrak{g}_+ = \oplus_{\langle a, \lambda\rangle > 0} \mathfrak{g}_a.$$

We write λ^{-1} to denote the reciprocal homomorphism $t \mapsto \lambda(t)^{-1} = \lambda(1/t)$. In HW10 you are led through a proof that if $G = \mathrm{GL}(V)$ then the multiplication map

$$\mu = \mu_{G,\lambda} : U_G(\lambda^{-1}) \times P_G(\lambda) \to G$$

is an open immersion, with $P_G(\lambda)$ a subgroup of "block upper-triangular matrices" and $U_G(\lambda)$ its unipotent radical (even over $\overline{k}$). We first wish to deduce the open immersion property for general G from this, which immediately implies that $U_G(\lambda)$, $P_G(\lambda)$, and $Z_G(\lambda)$ are all smooth (direct factors inherit smoothness) and that they are connected when G is connected. Likewise, it would follow that $U_G(\lambda)$ is unipotent in general since functoriality with respect to an inclusion $G \hookrightarrow \mathrm{GL}_n$ would reduce this to the settled case of GL_n. Finally, by iterating the connectedness of $Z_G(\lambda)$ several times (using λ's that generate a given torus in $G_{\overline{k}}$) it would follow that if G is connected then so is $Z_G(S)$ for any k-torus S in G.

In general, with a general pair (G, λ), consider a k-subgroup inclusion $j : G \hookrightarrow G'$ into another smooth affine k-group (the case of interest being $G' = \mathrm{GL}(V)$), and let $\lambda' = j \circ \lambda$. By the functorial definition,

$$P_G(\lambda) = G \cap P_{G'}(\lambda'), U_G(\lambda^{\pm 1}) = G \cap U_{G'}({\lambda'}^{\pm 1}), Z_G(\lambda) = G \cap Z_{G'}(\lambda').$$

In particular, if $U_{G'}(\lambda'^{-1}) \cap P_{G'}(\lambda') = 1$ then $U_G(\lambda^{-1}) \cap P_G(\lambda) = 1$. In other words, if $\mu' = \mu_{G',\lambda'}$ is a monomorphism then so is μ. This monicity hypothesis on μ for $G' = \mathrm{GL}(V)$ (and any 1-parameter k-subgroup λ' of $\mathrm{GL}(V)$) is verified in HW10, so μ is monic in general. But is it an open immersion? If μ' is an open immersion (as is proved on HW10 for $G' = \mathrm{GL}(V)$!) then the same holds for μ by means of the following non-obvious lemma:

Lemma 1.1. *With notation as above, if μ' is monic then*

$$G \cap (U_{G'}(\lambda'^{-1}) \times P_{G'}(\lambda')) = U_G(\lambda^{-1}) \times P_G(\lambda)$$

as subfunctors of G.

Proof. Since $P_{G'}(\lambda') = U_{G'}(\lambda') \rtimes Z_{G'}(\lambda')$, by evaluating on points valued in k-algebras R we have to show that if

$$u'_- \in U_{G'}(\lambda'^{-1})(R), \ \ u'_+ \in U_{G'}(\lambda')(R), \ \ z' \in Z_{G'}(\lambda')(R)$$

and $u'_- u'_+ z' = g \in G(R)$ then that $u'_+, u'_-, z' \in G(R)$.

As usual, we can pick a finite-dimensional k-vector space V, a k-homomorphism $\rho : G' \to \mathrm{GL}(V)$, and a line L in V such that G is the scheme-theoretic stabilizer of L in G'. Let $v \in L$ be a basis element, so $\rho(g)(v) = cv$ in $V_R = R \otimes_k V$ for a unique $c \in R^\times$. Since $g = u'_- u'_+ z'$, we get

$$\rho(u'_+ z')(v) = c\rho((u'_-)^{-1})(v) \tag{1}$$

in V_R.

For any point t of $\mathbf{G}_m$ valued in an R-algebra R', the point $\lambda'(t)$ of $G'(R')$ lies in $G(R')$ and so acts on v (through ρ) by some $R'^\times$-scaling. Hence, we can replace v with $\rho(\lambda'(t)^{-1})(v)$ on both sides of (1). Now act on both sides of (1) by $\rho(\lambda'(t))$, and then commute $\rho(\lambda'(t)^{-1})$ past $\rho(z')$ (as we may, since $z' \in Z_{G'}(\lambda')(R)$) to get the identity

$$\rho((t.u'_+)z')(v) = c\rho(t.(u'_-)^{-1})(v) \tag{2}$$

as points of the affine space $\underline{V}_R$ over R covariantly associated to V_R.

Viewing the two sides of (2) as R-scheme maps $(\mathbf{G}_m)_R \to \underline{V}_R$, the left side extends to an R-map $\mathbf{P}^1_R - \{\infty\} = \mathbf{A}^1_R \to \underline{V}_R$ and the right side extends to an R-map $\mathbf{P}^1_R - \{0\} \to \underline{V}_R$. By combining these, we arrive at an R-map $\mathbf{P}^1_R \to \underline{V}_R$ from the projective line to an affine space over R. The only such map is a constant R-map to some $v_0 \in \underline{V}_R(R) = V_R$ (concretely, $R[t] \cap R[1/t] = R$ inside of $R[t, 1/t]$), so both sides of (2) are independent of t (and equal to v_0). Passing to the limit as $t \to 0$ on the left side and as $t \to \infty$ on the right side yields $\rho(z')(v) = v_0 = cv$. We have proved that z' carries v to an $R^\times$-multiple of itself. Thus, the point $z' \in G'(R)$ is an R-point of the functorial stabilizer of L inside of V. This stabilizer is exactly G, by the way we chose ρ, so z' is an R-point of $G \cap Z_{G'}(\lambda') = Z_G(\lambda)$.

Since $\rho(z')(v) = cv$, by cancellation of c on both sides of the identity (2) we get

$$\rho(t.u'_+)(v) = \rho(t.(u'_-)^{-1})(v)$$

with both sides independent of t and equal to $c^{-1}v_0 = v$. Taking $t = 1$, this says that $u'_\pm$ lies in the stabilizer G of v, so $u'_\pm$ is an R-point of $G \cap U_{G'}(\lambda'^{\pm 1}) = U_G(\lambda^{\pm 1})$, as required. ■

At the end of the March 10 lecture, we used the open immersion property for μ to prove the following crucial result:

Proposition 1.2. *Let $f : G \twoheadrightarrow G'$ be a surjective k-homomorphism between smooth connected affine k-groups, and let $\lambda : \mathbf{G}_m \to G$ be a k-homomorphism. For $\lambda' = f \circ \lambda$, the natural maps $P_G(\lambda) \to P_{G'}(\lambda')$, $U_G(\lambda) \to U_{G'}(\lambda')$, and $Z_G(\lambda) \to Z_{G'}(\lambda')$ are surjective.*

We have shown that surjective homomorphisms between smooth connected affine k-groups carry maximal k-tori onto maximal k-tori and Borel k-subgroups onto Borel k-subgroups. Another related important compatibility is the good behavior of *torus centralizers* under surjective homomorphisms. This follows from the preceding proposition:

Corollary 1.3. *Let $f : G \twoheadrightarrow G'$ be a surjective k-homomorphism between smooth connected affine k-groups. Let S be a k-torus in G, and $S' = f(S)$. Then $f(Z_G(S)) = Z_{G'}(S')$.*

This result is Corollary 2 to 11.14 in Borel's book. You may find it instructive to compare the proofs.

Proof. We may assume $k = \overline{k}$. If S_1 and S_2 are k-subtori in S such that $S_1 \cdot S_2 = S$, which is to say that the k-homomorphism $S_1 \times S_2 \to S$ is surjective, it is an exercise (do it!) to check that $Z_G(S) = Z_{Z_G(S_1)}(S_2)$. (Note that since torus centralizers in smooth affine groups are smooth, this equality may be checked by computing with geometric points.) Hence, by induction on $\dim S$, we may and do assume $S \simeq \mathbf{G}_m$.

With $S \simeq \mathbf{G}_m$, the inclusion of S into G is given by a k-homomorphism $\lambda : \mathbf{G}_m \to G$ with image S. Likewise, $\lambda' = f \circ \lambda : \mathbf{G}_m \to G'$ has image S'. Hence, $Z_G(S) = Z_G(\lambda)$ and $Z_{G'}(S') = Z_{G'}(\lambda')$. Thus, the map $Z_G(S) \to Z_{G'}(S')$ that we wish to prove is surjective is identified with the natural map $Z_G(\lambda) \to Z_{G'}(\lambda')$. By Proposition 1.2, this latter map is surjective! ■

2. Conjugacy for split tori

It is a deep fact that in smooth connected affine groups G over any field k, all maximal k-split k-tori S in G (not to be confused with k-split maximal k-tori, which may not exist!) are $G(k)$-conjugate. Their common dimension is called the *k-rank* of G; it could be considerably smaller than the common dimension of the maximal k-tori (which may be called the *geometric rank*, since it is the $\overline{k}$-rank of $G_{\overline{k}}$). The proof of this conjugacy result rests on the theory of reductive groups (and pseudo-reductive groups when k is imperfect).

The special case of PGL_2 plays a role in getting the structure theory of reductive groups off the ground, so we now give an elementary direct proof in the special case of PGL_n and GL_n:

Proposition 2.1. *Let V be a finite-dimensional vector space over a field, and $G = \mathrm{GL}(V)$ or $\mathrm{PGL}(V)$. The maximal k-split k-tori in G are $G(k)$-conjugate to each other.*

Proof. Using the quotient map $\mathrm{GL}(V) \to \mathrm{PGL}(V)$ whose kernel is $\mathbf{G}_m$ and which is surjective on k-points (!), it is easy to reduce to the case of $\mathrm{GL}(V)$ in place of $\mathrm{PGL}(V)$ (check!). By HW5, Exercise 5, such k-tori correspond precisely to commutative k-subalgebras $A \subseteq \mathrm{End}(V)$ of the form $A \simeq k^n$ with $n = \dim V$. Such a k-subalgebra amounts to a k^n-module structure on an n-dimensional vector space V, which is nothing more or less than a decomposition of V into a direct sum of lines. But any two such decompositions are clearly related via the action of $\mathrm{Aut}_k(V) = \mathrm{GL}(V)(k)$, so we are done. ■

Now we turn out attention to an "axiomatic" $G(k)$-conjugacy result. The axioms turn out to hold for all connected reductive k-groups containing a split maximal k-torus, as one shows when developing the structure theory of connected reductive groups. The verification of the axioms lies deeper in the theory (see Remark 2.4), but we note here that it rests on the dynamic method (which is why we mention the topic in this handout, to illustrate how useful the dynamic viewpoint is).

Theorem 2.2. *Let G be a smooth connected affine k-group such that for every maximal torus T in $G_{\overline{k}}$, $Z_{G_{\overline{k}}}(T) = T$ and the finite group $W_{G_{\overline{k}}}(T)$ acts transitively on the set of Borel subgroups of $G_{\overline{k}}$ containing T. Also assume that any Borel subgroup B of $G_{\overline{k}}$ satisfies $N_{G(\overline{k})}(B) = B(\overline{k})$.*

Assume that G contains a k-split maximal k-torus, and that for all such k-tori T there is a Borel k-subgroup B containing T. All such pairs (T, B) are $G(k)$-conjugate to each other.

The centralizer hypothesis on the maximal tori of $G_{\overline{k}}$ is invariant under conjugation, so by the $G(\overline{k})$-conjugacy of all maximal tori of $G_{\overline{k}}$ it suffices to check this condition for one maximal torus of $G_{\overline{k}}$. The same holds for the normalizer hypothesis on Borel subgroups.

Remark 2.3. In the homework we have seen many examples of G for which $Z_G(T) = T$ for some maximal k-torus T, such as GL_n, SL_n, Sp_{2n}, and SO_n with their "diagonal" (split) maximal k-tori. But don't forget that there are plenty of interesting nontrivial k-anisotropic connected reductive groups, such as $\mathrm{SL}(D)$ for a finite-dimensional central division algebra $D \neq k$ and $\mathrm{SO}(q)$ for an anisotropic quadratic space (V, q) over k with $\dim V \geq 3$, and in such cases there is *no nontrivial k-split torus* at all, let alone one which is maximal as a k-torus (so in such cases the proposition concerns an empty collection of k-tori).

Remark 2.4. It is a general fact that $Z_G(T) = T$ for *every* maximal torus T in any connected reductive group G, but this is not at all obvious from the definitions; it is proved as part of a general development of basic structure theory of connected reductive groups. Likewise, the general development verifies the transitivity axiom on Weyl groups in Theorem 2.2 for connected reductive groups, as well as the fact that any k-split maximal k-torus (if one exists!) in a connected reductive k-group lies in a Borel k-subgroup. Finally, the self-normalizing property of Borel subgroups is a fundamental result of Chevalley, valid for any smooth connected affine $\overline{k}$-group. It underlies the entire structure theory of connected reductive groups.

To begin the proof of Theorem 2.2, let T and T' be k-split maximal k-tori in G, and choose Borel k-subgroups $B \supset T$ and $B' \supset T'$. We have $T = Z_G(T)$ and $T' = Z_G(T')$, since such equality among k-subgroups may be checked over $\overline{k}$ (where it follows from the hypotheses). The proof goes in two steps: conjugacy over k_s, and then a Galois cohomology argument to get down to k. But we follow the usual "reduction step" style and argue in reverse, by first showing that the general case can be reduced to the separably closed case, and then handling the case $k = k_s$.

Let's first reduce to the case of maximal tori over separably closed k: we will prove that if T_{k_s} and T'_{k_s} are $G(k_s)$-conjugate then they are $G(k)$-conjugate by an element carrying B_{k_s} to B'_{k_s}. Pick $g \in G(k_s)$ such that $T'_{k_s} = gT_{k_s}g^{-1}$, so $gB_{k_s}g^{-1}$ and B'_{k_s} are Borel k_s-subgroups containing T'_{k_s}. We first seek to choose g so that also these Borel k_s-subgroups coincide.

By hypothesis, the group $W_{G_{\overline{k}}}(T'_{\overline{k}})$ acts transitively on the set of Borel $\overline{k}$-subgroups containing $T'_{\overline{k}}$. But $W_G(T')$ is a finite étale k-group, so its geometric points are defined over k_s. Thus,

$$N_G(T')(k_s)/T'(k_s) = W_G(T')(k_s) = W_G(T')(\overline{k}) = W_{G_{\overline{k}}}(T'_{\overline{k}}).$$

In other words, the group $N_G(T')(k_s) = N_{G(k_s)}(T'_{k_s})$ acts transitively on the set of Borel $\overline{k}$-subgroups of $G_{\overline{k}}$ containing $T'_{\overline{k}}$. Hence, replacing $g \in G(k_s)$ with its left-translate by some element of $N_G(T')(k_s)$ (which doesn't affect the condition that $gT_{k_s}g^{-1} = T'_{k_s}$!) brings us to the case that the Borel k_s-subgroups $gB_{k_s}g^{-1}$ and B'_{k_s} containing T'_{k_s} coincide over $\overline{k}$ and hence coincide over k_s.

Now we can carry out a Galois cohomology argument to push down the $G(k_s)$-conjugacy to $G(k)$-conjugacy. For any $\gamma \in \mathrm{Gal}(k_s/k)$ we apply γ to both sides of the equalities

$$T'_{k_s} = gT_{k_s}g^{-1}, \ \ B'_{k_s} = gB_{k_s}g^{-1}.$$

This gives

$$T'_{k_s} = \gamma(g)T_{k_s}\gamma(g)^{-1}, \ \ B'_{k_s} = \gamma(g)B_{k_s}\gamma(g)^{-1},$$

so $\gamma(g)^{-1}g$ normalizes T_{k_s} as well as B_{k_s}. By hypothesis $N_{G(\overline{k})}(B_{\overline{k}}) = B(\overline{k})$, so

$$\gamma(g)^{-1}g \in B(\overline{k}) \cap G(k_s) = B(k_s)$$

and likewise $\gamma(g)^{-1}g \in N_G(T)(k_s)$.

If we did not have available the Borel k-subgroups and only worked with the split maximal k-tori, we would only have $\gamma(g)^{-1}g \in N_G(T)(k_s)$ and then we would get hopelessly stuck due to possible obstructions in $\mathrm{H}^1(k_s/k, W_G(T))$. Now the importance of using the Borel k-subgroups emerges: $B(k_s) \cap N_G(T)(k_s) = T(k_s)$! Indeed, since $T = Z_G(T)$ (by our hypotheses) we can express this as the statement that $N_B(T)(k_s) = Z_B(T)(k_s)$, and this in turn is a special case of:

Lemma 2.5. *Let H be a connected solvable smooth affine group over a field k, and let T be a maximal k-torus in H. Then $N_H(T)(k) = Z_H(T)(k)$.*

Proof. Since $T_{\overline{k}}$ is a maximal torus in $H_{\overline{k}}$, and the problem of showing a k-point of H lies in the closed subset $Z_H(T)$ may be checked over $\overline{k}$, it is harmless to extend the ground field to $\overline{k}$ so that k is *algebraically closed*. Hence, the structure theorem for connected solvable groups becomes available: $H = T \ltimes U$ for $U = \mathscr{R}_u(H)$. To show that any $h \in H(k)$ normalizing T actually centralizes T, we may assume $h = u \in U(k)$. Hence, for any $t \in T(k)$ we have

$$utu^{-1} = t(t^{-1}ut)u^{-1}.$$

But $(t^{-1}ut)u^{-1} \in U(k)$ since U is normal in $H = T \ltimes U$, so the condition that $utu^{-1} \in T(k)$ forces it to equal t. ■

Thus, we have obtained a function $c : \gamma \mapsto \gamma(g)^{-1}g$ from $\mathrm{Gal}(k_s/k)$ to $T(k_s)$. This functor factors through the quotient $\mathrm{Gal}(K/k)$ for a finite Galois extension K/k inside of k_s such that $g \in G(K)$. It is therefore easy to check that $c \in \mathrm{Z}^1(k_s/k, T(k_s))$. Consider the cohomology class $[c] \in \mathrm{H}^1(k_s/k, T)$. Since $T \simeq \mathbf{G}_m^r$, this cohomology group vanishes by Hilbert 90. Hence, $c = \gamma(t)t^{-1}$ for some $t \in T(k_s)$. Thus, if we replace g with gt (as we may!), we get to the case when $\gamma(g) = g$ for all γ, so $g \in G(k)$. That does the job. (This idea adapts to pull down the result from $\overline{k}$ by using Hilbert 90 for the fppf topology, but we give a more hands-on procedure below to get down to k_s from $\overline{k}$.)

Now we can assume that $k = k_s$, and it remains to show:

Proposition 2.6. *If T and T' are maximal tori in a smooth connected affine group G over a separably closed field k then T and T' are $G(k)$-conjugate.*

This says that the general conjugacy result over algebraically closed fields actually holds over separably closed fields. I think it is due to Grothendieck. Regardless, the argument we give is a version of the method he used in SGA3 for smooth affine groups over any scheme (working locally for the étale topology). The idea is similar to the trick with Isom-schemes in HW4 Exercise 5.

Proof. Consider the functor I on k-algebras defined by

$$I(R) = \{g \in G(R) \,|\, T'_R = gT_Rg^{-1}\}.$$

This is a subfunctor of G, and its restriction $I_{\overline{k}}$ to $\overline{k}$-algebras is represented by a *smooth* closed subscheme of $\overline{k}$: since $T'_{\overline{k}} = g_0 T_{\overline{k}} g_0^{-1}$ for some $g_0 \in G(\overline{k})$ by the known "geometric" case over $\overline{k}$, we see that $I_{\overline{k}}(R)$ consists of points $g \in G(R)$ such that $g_0^{-1}g \in Z_{G(R)}(T_R)$. In other words, $I_{\overline{k}}$ is represented by $g_0 Z_G(T)_{\overline{k}}$. By HW8, Exercise 3, this is smooth and non-empty. Thus, if we can prove that I is represented by a closed k-subscheme of G then its $\overline{k}$-fiber represents $I_{\overline{k}}$ and hence is smooth (and non-empty)! But we know that a smooth non-empty scheme over a *separably closed* field always has a k-point, so it would follow that $I(k) \neq \emptyset$, so the desired $G(k)$-conjugacy of T and T' would follow.

It remains to prove that I is represented by a closed k-subscheme of G. We will do this by approaching tori through their torsion-levels. For each $n \geq 1$ not divisible by $\mathrm{char}(k)$, define a functor on k-algebras as follows:

$$I_n(R) = \{g \in G(R) \,|\, T'[n]_R = gT[n]_R g^{-1}\}.$$

Clearly I is a subfunctor of I_n. Since $T[n]$ and $T'[n]$ are finite étale, each is just a finite set of k-points in G (as $k = k_s$). Thus, it is rather elementary to check that I_n is represented by a closed subscheme of G (verify!). The infinite intersection $\cap_n I_n$ as subfunctors of G is likewise represented by a closed subscheme of G (form the infinite intersection of representing closed subschemes for the I_n's). Thus, we just have to check that the inclusion $I(R) \subseteq \cap_n I_n(R)$ is an equality for all k-algebras R.

Equivalently, picking a point g of $G(R)$ lying in $\cap_n I_n(R)$ and conjugating T_R by this point, we are reduced to proving that $gT_R g^{-1}$ and T'_R coincide if their n-torsion subgroups coincide for all $n \geq 1$ not divisible by $\mathrm{char}(k)$. By the same "relative schematic density" argument used in your solution to HW3 Exercise 3(iii), since the union of the $T[n](k)$ is Zariski-dense in T (why?) and likewise for T' it follows that a closed subscheme of G_R which contains all $T[n]_R$'s (resp. all $T'[n]_R$'s) must contain T_R (resp. T'_R). The automorphism of g-conjugation on G_R then implies likewise that a closed subscheme of G_R which contains every $gT[n]_R g^{-1}$ must contain $gT_R g^{-1}$. We conclude that if $gT[n]_R g^{-1} = T'[n]_R$ for all $n \geq 1$ not divisible by $\mathrm{char}(k)$ then the two closed subschemes $gT_R g^{-1}$ and T'_R of G_R each contain the other and hence are equal as such. ■